Potential theory method and spectral relationships of a generalized Macdonald kernel in some different domains

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Abstract: A generalized potential theory method is used to construct spectral relationships, in some different domains, for an integral equation, in three dimensional, with a generalized potential kernel. This kernel, under certain conditions, will take the generalized Macdonald function form. Also, the spectral relationships are discussed and obtained in different domains of integration. Many special cases are derived and discussed from the work.

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1. Introduction:

The mathematical formulation of physical phenomena, and contact problems in relationships the theory of elasticity, often involves singular integral equation with different kernels. In Mkhitarian and Abdou **[1,2],** using Krein's method, Mkhitarian and Abdou obtained the spectral for the integral operator containing logarithmic kernel and Carleman kernel, respectively. The monographs **[3-5]** contain many different SRs for different kinds of integral equations, in one, two and three dimensional. The generalized potential theory method is applied and used, in Abdou **[6-7],** to obtain the **SRs** for three-dimensional semi-symmetric contact problem. The Hertez contact problem contains two rigid surfaces, having two different elastic materials and occupying the contact domain $\Omega = \{(x,y,z) \in \Omega : \sqrt{x^2+y^2} \le a,\ z=0\}$ with potential kernel and generalized potential kernel, respectively. Also, in Abdou , Salama **[8],** the orthogonal polynomials method isused to obtain many different **SRs** in one, two and three dimensional, for Volterra-Fredholm integral equation of the first kind in the space $L_2(\Omega)\times C[0,T], T<1$, where the Volterra integral term is considered in time, while the Fredholm term is considered in position.

In this work, the eigenvalues and eigenfunctions of the integral equation

$$\int\int_\Omega k(x-\xi,y-\eta)\Phi(\xi,\eta)d\xi\, d\eta = f(x,y) \quad (1.1)$$

$$k(x-\xi,y-\eta) = [(x-\xi)^2+(y-\eta)^2]^{-(\mu+1/2)} (0\le\mu<1/2) \quad (1.2)$$

under the condition

$$\int\int_\Omega \Phi(x,y)dx\, dy = P\ , (P \text{ is a constant}) \quad (1.3)$$

will be obtained in some different domains of integration Ω, where

(1) $\Omega = \{(x,y,z)\in\Omega: |x|<\infty, 0\le y<\infty, |z|<\infty\}$
(2) $\Omega = \{(x,y,z)\in\Omega: |x|<\infty,\ |y|<a\ , |z|<\infty\}$
(3) $\Omega = \{(x,y,z)\in\Omega: |x|<\infty, |y|\ge a\ , |z|=0\}$
(1.4)

The physical meaning for the contact problems and the spectral relationships was explained in the work of Mkhitarian and Abdou **[9, 10].**

In order to guarantee the existence of a unique solution of Eq.(1.1) in $L_2(\Omega)$, we assumethe following conditions :

(i) The kernel $k(x-\xi,y-\eta)$ satisfies in L_2 the condition

$$\left\{\int\int_\Omega \int\int_\Omega k(x-\xi,y-\eta)\Phi(\xi,\eta)d\xi\, d\eta\, dx\, dy\right\}^{\frac{1}{2}} = A\ ,(A \text{ is a constant})$$

(ii) The given function $f(x,y)$ with its first partial derivatives are continuous.

(iii) The unknown function $\Phi(x,t)$ satisfies Lipschitz condition for $x$and y.

2. Potential theory method, see [10,11]

The integral equation (1.1), with the kernel (1.2), after using the complex Fourier integral transforms

$$\Phi_s(y) = \int_{-\infty}^{\infty}\Phi(x,y)e^{isx}dx,$$
$$f_s(y) = \int_{-\infty}^{\infty} f(x,y)e^{isx}dx \quad (2.5)$$

and the famous definition of the generalized Macdonald function Eq.(3.773.6). p.433, of Gradstein , Ryzhik **[12],** takes the form

$$\int_L \frac{K_\mu(|s||y-\eta|)\Phi_s(\eta)}{|y-\eta|^\mu} d\eta = \bar{f}(y),$$

$$\left(y \in L\,, 0 \le \mu < \frac{1}{2}, \Phi_s(y) = \Phi(y,s)\right)$$

$$\bar{f}(y) = \pi^{-\frac{1}{2}} 2^{\mu-1} \Gamma\left(\mu+\tfrac{1}{2}\right)|s|^{-\mu} f_s(y)\,, \left(f_s(y) = f(y,s)\right) \qquad (2.6)$$

where $K_\mu(|s||.|)$ is the generalized Macdonald kernel , s is the coefficients of Fourier integraland $\Gamma(.)$ is the Gamma function.

Now, Eq.(2.6) is equivalent to (1.1), where the domain of integration L is defined in theforms

$(1')\ L = \{(y,z) \in L: 0 \le y < \infty, |z| < \infty\}$,

$(2')\ L = \{(y,z) \in L: |y| < a, |z| < \infty\}$,

$(3')L = \{(y,z) \in L: |y| \ge a, |z| < \infty\}$.

Using the principal idea of the potential theory method, see **[10,11]**, we can write the integral equation (1.1) in three dimensional form

$$\int\int_\Omega \frac{\Phi(\xi,\eta)}{[(x-\xi)^2+(y-\eta)^2+z^2]^{\mu+\frac{1}{2}}} d\xi\, d\eta = U(x,y,z) \qquad (2.7)$$

The integral equation (1.1) or (2.7), is equivalent to the following **BVP**

$$\Delta U(x,y,z) + \frac{2\mu}{z}\frac{\partial U}{\partial z} = 0\,,$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\left((x,y,z) \notin \Omega\right)$$

$$U(x,y,z)|_{z=0} = f(x,y)\,,\ \left((x,y,0) \in \Omega\right)$$

$$U(x,y,z) \simeq Pr^{-1-2\mu} \to \text{finite } as\ r \to \infty\,,$$

$$r = \sqrt{x^2+y^2+z^2} \qquad (2.8)$$

After constructing the solution of the **BVP** (2.8), the solution of (1.1) is completely determined from the equivalence condition

$$-2\pi\phi(x,y) = sgn\, z\, \lim_{z\to 0} z|z|^{2\mu}\frac{\partial U(x,y,z)}{\partial z}\,, \left((x,y,o) \in \Omega\right) \qquad (2.9)$$

Using the complex Fourier integral transform, with respect to x, the **BVP** (2.8) and the integral equation (2.7), respectively, become

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2\mu}{z}\frac{\partial}{\partial z} - s^2\right)U_s(y,z) = 0$$

$$\left((y,z) \notin \mathrm{L}\right)$$

$$U_s(y,z)|_{z=0} = f_s(y)\,, \left((y,0) \in \mathrm{L}\right)$$

$$U_s(y,z) \to 0 \text{ as} y^2+z^2 \to \infty\,;\ \left(U_s(y,z) = U(y,z,s)\right) \qquad (2.10)$$

and

$$U_s(y,z) = \frac{\sqrt{\pi}|s|^\mu}{2^{\mu-1}\Gamma\left(\mu+\frac{1}{2}\right)} \int_L \frac{K_\mu\left(|s|\sqrt{(y-\eta)^2+z^2}\right)\Phi_s(\eta)}{[(y-\eta)^2+z^2]^{\frac{\mu}{2}}} d\eta \qquad (2.11)$$

Also, the equivalence condition (2.9) takes the form

$$-2\pi\phi_s(y) = sgn\, z\, \lim_{z\to 0} z|z|^{2\mu}\frac{\partial U_s(y,z)}{\partial z}\,, (y \in L) \qquad (2.12)$$

To eliminate the term $\frac{\partial U_s}{\partial z}$, from the formula (2.10), we set

$$V_s(y,z) = |z|^{-\mu}U_s(y,z) \qquad (2.13)$$

to obtain

$$\frac{\partial^2 V_s}{\partial y^2} + \frac{\partial^2 V_s}{\partial z^2} + \mu(1-\mu)\frac{V_s}{z^2} - s^2 V_s = 0\,, \left((y,z) \notin \mathrm{L}\right),$$

$$|z|^{-\mu}V_s(y,z)|_{z=0} = f_s(y)\,, \left((y,0) \in \mathrm{L}\right),$$

$$|z|^{-\mu}V_s(y,z) \to 0 \text{as} y^2+z^2 \to \infty \qquad (2.14)$$

3 Method of solution:

Here, we will discuss the solution of the BV P (2.14), when the domain of integration takes the three cases $(1')$, $(2')$ and $(3')$.

Case (1′): Consider the complex plane $w = y + iz\,, i = \sqrt{-1}$and the transformation mapping

$$w = \tfrac{1}{2}\xi^2\,, \xi = u + iv \qquad (3.15)$$

Hence, we get

$$y = \tfrac{1}{2}(u^2 - v^2)\,, z = uv\,;\ (-\infty < u < \infty\,; 0 < v < \infty) \qquad (3.16)$$

Using the transformation mapping (3.15), the **BVP** (2.14), yields

$$\frac{\partial^2 W_s}{\partial u^2} + \frac{\partial^2 W_s}{\partial v^2} + \mu(1-\mu)\left(\frac{1}{u^2} + \frac{1}{v^2}\right)W_s - s^2(u^2+v^2)W_s = 0 (u > 0)$$

$$(|u|v)^{-\mu}W_s(u,v)|_{v=0} = f_s\left(\frac{u^2}{2}\right), -\infty < u < \infty$$

$$(|u|v)^{-\mu}W_s(u,v) \to 0 \quad as \quad u^2+v^2 \to \infty\,,$$

$$W_s(u,v) = V_s\left(\tfrac{1}{2}(u^2-v^2), uv\right) \qquad (3.17)$$

Also, the equivalence condition (2.12) becomes

$$-2\pi\phi_s(y) = u^{2\mu-1}\lim_{v\to 0} v^{2\mu}\frac{\partial \breve{U}(u,v)}{\partial v}\,,$$

$$\left(y = \frac{u^2}{2}\,, u > 0\right),$$

$$\breve{U}(u,v) = U_s\left(\tfrac{1}{2}(u^2-v^2), uv\right) = (|u|v)^{-\mu}W_s(u,v) \qquad (3.18)$$

To solve the **BVP** (3.17) we use the separation of variables method

$$W_s(u,v) = X(u)Y(v) \qquad (3.19)$$

Hence, we have

$$\frac{d^2X}{du^2}+\left[\frac{\mu(1-\mu)}{u^2}-s^2u^2+\lambda^2\right]X(u)=0\ ,\ (-\infty<u<\infty) \quad (3.20)$$

$$\frac{d^2Y}{dv^2}+\left[\frac{\mu(1-\mu)}{v^2}-s^2v^2+\lambda^2\right]Y(v)=0\ ,\ (0<v<\infty) \quad (3.21)$$

where λ^2 is the constant of seperation.

To solve the formula (3.20), we use the substitution

$$X(u)=t^{\frac{(2c-1)}{4}}e^{-\frac{t^2}{2}}F(t)\ ,t=|s|u^2 \quad (3.22)$$

to get (see **[13]**.P.237),

$$t\frac{d^2F}{dt^2}+(c-t)\frac{dF}{dt}-aF=0\ ,\qquad (0\le t<\infty),$$

$$a=\frac{1}{4}(4\mu+1)-\frac{\lambda^2}{4|s|}\left(c=\mu+\frac{1}{2}\right) \quad (3.23)$$

The **ODE** (3.23), has a regular point at $t=0$ and its solution can be represented in thehyper-geometric function $F(t)=\psi(a,b;t)$. Hence, the solution of (3.20) becomes

$$X(u)=\left(\sqrt{|s|}|u|\right)^{\mu}e^{-|s|\frac{u^2}{2}}\psi\left(\frac{4\mu+1}{4}-\frac{\lambda^2}{4|s|},\mu+\frac{1}{2};|s|u^2\right) \quad (3.24)$$

The function $X(u)$ of (3.24) is bounded for $u=0$ and $0\le\mu<\frac{1}{2}$, while , for $|u|\to\infty$, $X(u)$ is unbounded. Therefore, to obtain bounded solution in the hyper-geometric form$\psi(a,c;|s|u^2)$, we assume (see **[14]**)

$$\lambda_n=(4n+1+2\mu)|s|(n=0,1,2,\dots) \quad (3.25)$$

Then, we use the famous formula (see [14] p.189).

$$L_n^{\mu-\frac{1}{2}}(u)=\binom{n+\mu-\frac{1}{2}}{n}\psi\left(-n,\mu+\frac{1}{2},u\right),$$

where $L_n^{\mu-\frac{1}{2}}(u)$ is the Chebyshev-Hermit function (Laguere function), to get

$$X(u)=\left(\sqrt{|s|}|u|\right)^{\mu}e^{-\frac{|s|u^2}{2}}L_n^{\mu-\frac{1}{2}}(|s|u^2)\left(n=0,1,2,\dots,0\le\mu<\frac{1}{2}\right) \quad (3.26)$$

The second linear solution of (3.23), $F(t)=t^{\frac{1}{2}-\mu}\psi(a-c+1,2-c,t)$, is unbounded when $t\to\infty$. Hence, the bounded function (3.26), for $-\infty<u<\infty$, represents a unique solution of the **ODE** (3.20).

Also, when $0<v<\infty$, we can obtain the unique solution of the ODE (3.21) in the form

$$Y(v)=\left(\sqrt{|s|}|v|\right)^{\mu}e^{-\frac{|s|v^2}{2}}T\left(n+\mu+\frac{1}{2},\mu+\frac{1}{2};|s|v^2\right)$$

$$\left(0<v<\infty,0\le\mu<\frac{1}{2},n=0,1,2,\dots\right) \quad (3.27)$$

where $T(\propto,\beta;\gamma)$ is the Tricome function. The relation between Tricome function and thehyper-geometric function is (see [13] p.245)

$$T(\propto,\beta;\gamma)=\frac{\Gamma(1-c)}{\Gamma(a-c+1)}\psi(a,c;v)+\frac{\Gamma(c-1)}{\Gamma(a)}v^{1-c}\psi(a-c+1,2-c;v)$$

$$(c\ne n) \quad (3.28)$$

Now, using (3.26),(3.28) in the fourth formula of (3.17) we completely determine the potential function $W_s(u,v)$, then using the result in the second formula of (3.18), we obtain

$$\breve{U}(u,v)=|s|^{\mu}e^{\left(-\frac{|s|}{2}(u^2+v^2)\right)}L_n^{\mu-\frac{1}{2}}(|s|u^2)T\left(n+\mu+\frac{1}{2},\mu+\frac{1}{2};|s|v^2\right)$$

$$\left(-\infty<u<\infty,\ 0<v<\infty,|\mu|<\frac{1}{2},n=0,1,2,\dots\right) \quad (3.29)$$

Also, if we use (3.29) in the first formula of (3.18), we have

$$\Phi_s(y)=\binom{2n+2\mu+1}{\pi}2^{\mu-\frac{3}{2}}\frac{\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{|s|}}{\Gamma\left(n+\mu+\frac{1}{2}\right)}y^{\mu-\frac{1}{2}}e^{-|s|y}L_n^{\mu-\frac{1}{2}}(2|s|y)(0<y<\infty) \quad (3.30)$$

With the aid of (3.29) and (3.16), when $z=0$, the second formula of Eq.(2.10) or (3.17),becomes

$$f_s(y)=U_s(y,0)=\frac{\Gamma\left(\frac{1}{2}-\mu\right)}{n!}|s|^{\mu}e^{-|s|y}L_n^{\mu-\frac{1}{2}}(2|s|y) \quad (3.31)$$

Using the two formulas (3.30) and (3.31) in (2.6), we have the following spectral relationships with the generalized Macdonald kernel

$$\int_0^{\infty}\frac{K_{\mu}(|y-\eta|)e^{-\eta}}{|y-\eta|^{\eta}\eta^{\frac{1}{2}-\mu}}L_n^{\mu-\frac{1}{2}}(2\eta)d\eta=\omega_n e^{-y}L_n^{\mu-\frac{1}{2}}(2y)$$

$$\omega_n=\frac{\sqrt{\pi}}{\sqrt{2}n!}\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(n+\frac{1}{2}-\mu\right)\left(|\mu|<\frac{1}{2},n\ge 0\right) \quad (3.32)$$

For obtaining the **SRs** when y < 0, we let, in (3.29), $u=0,y=-\frac{v^2}{2}$,hence, we can obtain

$$\int_0^{\infty}\frac{K_{\mu}(|y-\eta|)}{|y-\eta|^{\mu}}\frac{L_n^{\mu-\frac{1}{2}}(2\eta)}{e^{\eta}\eta^{\frac{1}{2}-\mu}}d\eta=\frac{\sqrt{\pi}\Gamma^2\left(n+\mu+\frac{1}{2}\right)}{\sqrt{2}\Gamma\left(\mu+\frac{1}{2}\right)}e^{y}T\left(n+\mu+\frac{1}{2},\mu+\frac{1}{2},-2y\right) \quad (3.33)$$

Case(2′): To obtain the solution of the **BVP** (3.16) we use the elliptic coordinates **[14]**

$$\breve{\omega}=y+iz=a\cosh\xi\ ,\qquad \xi=u+iv\ ,\qquad i=\sqrt{-1}$$

i.e.

$y = a\cosh u\cos v, z = a\sinh u\sin v, (0 \le u < \infty, -\pi < v < \pi)$ (3.34)

The conformal mapping (3.34) transforms the complex plane $\breve{\omega}$ of the domain $L = \{|y| < a, |z| < \infty\}$ into the domain $\Pi = \{0 \le u < \infty, -\pi < v < \pi\}$. Taking the transformation mapping (3.34), into account, the solution of the BVP (2.14),can be written in the form

$V_s(y,z) = V_s(a\cosh u\cos v, a\sinh u\sin v) = \breve{V}_s(u,v), \overline{V}_s(u,v) = F(u)G(v)$ (3.35)

Substituting from (3.35) into the first formula of (2.14), after certain elementary manipulation, we have the following two **ODEs**

$F''(u) - [\alpha + 2q\cosh 2u - \mu(1-\mu)\sinh^{-2} u]F = 0, \left(0 \le u < \infty, q = \frac{a^2 s^2}{4}\right),$ (3.36)

$G''(v) + [\alpha + 2q\cos 2v + \mu(1-\mu)\sin^{-2} v]G = 0 \quad (-\pi < v < \infty)$ (3.37)

where $\propto$ is the constant of seperation.

The formula (3.37) reduces to (3.36), if we assume $v = iu, i = \sqrt{-1}$; hence, we can limitourselves to (3.37). For this aim, we assume $G(v) = \sqrt{|\sin v|}H(v)$, to get

$\frac{d^2H}{dv^2} + \cot v\frac{dH}{dv} + [\lambda + 4\gamma\sin^2 v - v^2\sin^{-2} v]H = 0, \left(\lambda = \alpha - \frac{1}{4} + 2q, \gamma = -q, v = \frac{1}{2} - \mu\right)$ (3. 38)

After obtaining the solution of (3.38), the general solution of (3.37), for $m \ge 0, 0 < v < \pi$, can be written in the forms (see**[10])**

$$G_n^v(v) = \sqrt{\sin v}\,P_{S_{n-v}}^v(\cos v, \theta) = 2^v\frac{(\sin v)^{\frac{1}{2}-v}\Gamma(1-2v)}{\Gamma(1-v)} \times \sum_{r=0}^{\infty}\frac{(-1)^{r-m}(2r)!}{\Gamma(2r+1-2v)}b_{2m-v,r-m}^v(\theta)C_{2r}^{\frac{1}{2}-v}(\cos v) \quad (n = 2m) \quad (3.39)$$

and

$$G_n^v(v) = 2^v\frac{(\sin v)^{\frac{1}{2}-v}\Gamma(1-2v)}{\Gamma(1-v)}\sum_{r=0}^{\infty}\frac{(-1)^{r-m}(2r+1)!}{\Gamma(2r+2-2v)} \times b_{2m+1-v,r-m}^v(\theta)C_{2r+1}^{\frac{1}{2}-v}(\cos v), (n = 2m+1) \quad (3.40)$$

where $G_n^v(x)$ is the Gegenbaur polynomial, and the recurrence coefficients $b_{\int,r}^v(\theta)$ can bedetermined by using the following orthogonal relation [14],

$$\int_0^\pi [G_n^v(v)]^2 dv = h_n = \begin{cases} \frac{n!}{\Gamma(n+1-2v).\left(n+\frac{1}{2}-v\right)} & (n \ge 1) \\ \{\Gamma(2-2v)\}^{-1} & (n = 0) \end{cases} \quad (3.41)$$

Using the two formulas (3.39) and (3.40) in (3.41), we obtain

$$\sum_{r=0}^{\infty} A_{n.r}^v\left[b_{n-v,r-\left[\frac{n}{2}\right]}(\theta)\right]^2 = h_n;\ \left[\frac{n}{2}\right] = \begin{cases}\frac{n}{2} & n \text{ even} \\ \frac{1}{2}(n-1) & n \text{ odd}\end{cases} \quad (3.42)$$

Where

$$A_{n.r}^v = \begin{cases} (2r)!\left[\Gamma(2r+1-2v).\left(2r+\frac{1}{2}-v\right)\right]^{-1} \\ \text{as} \quad (n = 2m, m \ge 0) \\ (2r+1)!\left[\Gamma(2r+2-2v).\left(2r+\frac{3}{2}-v\right)\right]^{-1} \\ as\ (n = 2m+1, m \ge 0)\end{cases} \quad (3.43)$$

Since the unique solution of the **ODE** (3.36) is bounded in the interval $u \in [0,\infty)$ andvanishes at infinity, then it can be assumed in the form

$F_n^v(u) = \sqrt{\sinh u}\,S_{n-v}^{v(3)}(\cosh u, \theta)(0 \le u < \infty)$ (3.44)

where $S_{n-v}^{v(3)}(\cosh u, \theta)$ are spheroidal wave equations of the third kind, see **[10,14]**.

By following the same way of [10] p.90-91, we can arrive to the spectral relationships.

$\int_{-a}^{a}\frac{K_\mu(|s||y-u|)}{|y-u|^\mu}(a^2-u^2)^{-\frac{v}{2}}P_{S_{n-v}}^v(u,\theta)du = \delta_n^v(a^2-y^2)^{\frac{v}{2}}P_{S_{n-v}}^v(y,\theta)$ (3.45)

where $P_{S_\int}^v(u,\theta)$ represents the solution of (3.38) and given by

$P_{S_\int}^v(u,\theta) = \sum_{r=-\infty}^{\infty}(-1)^r b_{\int,r}^v(\gamma)P_{\int+2r}^v(u)\ (\theta = -q, \int = n-v)$ (3.46)

$\int$is the characteristic index and $P_m^n(u)$ is the Legendre polynomial of the first kind of order m. Also,

$$\delta_n^v = (-1)^{\left[\frac{n}{2}\right]}\beta_n\sin(\pi\mu).a^{2v}q^{\frac{n+\mu}{2}}|2s|^{-\mu}2^{1-n} \times B_n^v(\theta)D_n^v(\theta)[E_n^v(\theta)]^{-1}$$

$$B_n^v(\theta) = (-1)^{\left[\frac{n}{2}\right]}\sum_{r=-\left[\frac{n}{2}\right]}^{\infty} b_{n-v,r}^v(\theta)K_{n+2r+\mu}^v(2\sqrt{q}),$$

$$D_n^v(\theta) = \sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r b_{n-v,-r}^v(\theta)}{r!\,\Gamma(n+\mu+1-r)}\left[\sum_{r=0}^{\infty}\frac{(-1)^r b_{n-v}^v(\theta)}{r!\,\Gamma(1-n-\mu-r)}\right]^{-1}$$

$$E_n^v(\theta) = \frac{1}{n!}\sum_{r=-\left[\frac{n}{2}\right]}^{\infty} \frac{(-1)^r (n+2r)!\, b_{n-v,r}^v(\theta)}{\Gamma(n+2\mu+2r)}$$

$$+\frac{1}{\Gamma(n+2\mu)}\sum_{r=-\infty}^{-\left[\frac{n}{2}\right]-1} (-1)^r b_{n-v,r}^{-v}(\theta);$$

$$\beta_n = \begin{cases} 1 & (n=2m) \\ -1 & (n=2m+1) \end{cases} \quad (3.47)$$

The famous coefficients $K_m^{\int}(\theta)$ of spheriodal wave function of the first kind $S_m^{v(1)}(u,\theta), m \geq 0$is given by Bateman , Ergelyi **[13]** p.175.The **SRs** for $|y| > a$ can be discussed from the transformation mapping(3.34) by putting $v = 0$ to get $y > a$, and $v = \pi$ to have $y < -a$. Therefore, we have

$$\int_{-a}^{a} \frac{K_\mu(|s||y-u|)}{|y-u|^\mu}(a^2-u^2)^{-\frac{v}{2}} P_{s_{n-v}}^v(u,\gamma)du =$$
$$\mu_n^v sgn\, y(y^2-a^2)^{\frac{v}{2}} S_{s_{n-v}}^{v(s)}(|y|,\gamma)(|y|>a). \quad (3.48)$$

Where

$$\mu_n^v = (-1)^{1+\left[\frac{n}{2}\right]}\beta_n^* \sin(\pi\mu).q^{\frac{n+\mu}{2}}|2s|^{-\mu}2^{1-n}D_n^v(\gamma)[E_n^v(\gamma)]^{-1}$$

$$\beta_n^* = \begin{cases} \sqrt[4]{\pi^2 q}e^{-\frac{i\pi v}{2}} & (n=2m) \\ \sqrt[4]{\pi^2 q}e^{\frac{i\pi(1-v)}{2}} & (n=2m+1) \end{cases} \quad (3.49)$$

As an important case, the **SRs** when $\mu = 0$, can be obtained, afterobtaining the solution of the BVP of (2.10), in the form

$$U_s(y,z) = F_e k_n(u,-q)ce_n(v,-q) \quad (3.50)$$

where $ce_n(x,-q)$ and $F_e k_n(y,-q)$ are the Mathieu functions of order n of the second andthird kind, respectively.Then, using (3.48), in (2.11), when $\mu = 0$, and introducing the results in (2.10), to have

$$\frac{1}{\pi}\int_{-a}^{a}\frac{K_0(|s||y-u|)}{\sqrt{a^2-u^2}}ce_n[\cos^{-1}u,-q]du =$$
$$\begin{cases} -\frac{F_e k_n(0,-q)}{F_e k'_n(0,-q)}ce_n[\cos^{-1}y,-q] & (|y|<a, n\geq 0) \\ v_n \breve{\gamma}_n(y) F_e k_n\left[\ln\frac{|y|+\sqrt{a^2-y^2}}{a},-q\right] & |y|>a \end{cases}$$
$$(3.51)$$

where, $K_0(|s||y-u|)$ is the Macdonald function, and $v_n = -\pi ce_n(0,-q)[F_e k'_n(0,-q)]^{-1}$,

$$\breve{\gamma}_n(y) = \begin{cases} 1 & n=2m \\ sgn\, y & n=2m+1 \end{cases} \quad \left(m>0, q=\frac{a^2s^2}{4}\right)$$
$$(3.52)$$

Case(3′): Here the transformation (3.34) transforms the region L into a region $\Pi = \{-\infty < u < \infty, 0 \leq v \leq \pi\}$, where the point $v = 0$ in Π is equivalent to $y \geq a$ in L, and ,$v = \pi$ is corresponding to $y \leq -a$. Hence, in this case, the solution, of the BVP (2.14) is equivalentto the solution of the ODE's (3.36) and (3.37), where $-\infty < u < \infty, 0 \leq v \leq \pi$. Thesolution of Eq.(3.36) in the interval $-\infty < u < \infty$, is the same solution of Eq.(3.37) inthe interval $0 < v < \pi$, after replacing α by $-\alpha$. Also the formula (3.37) will satisfy theperiodic condition $G(\pi - v) = G(v)$. The general solution of (3.37), after using conditions $G_{\gamma,v}^{\pm}(\pi - v) = \pm G_{\gamma,v}^{\pm}(v), 0 < v < \pi$, for even and odd respectively, and the famous relation,see [14],

$$\sum_{r=-\infty}^{\infty}\frac{\Gamma(\gamma+v+2r+1).\Gamma(\gamma-v+1)}{\Gamma(\gamma-v+2r+1).\Gamma(\gamma+v+1)}.\frac{[b_{\gamma,r}^v(\theta)]^2}{2(\gamma+2r)+1} =$$
$$\frac{1}{2\gamma+1}\left(b_{\gamma,0}^v(\theta) = 0\right) \quad (3.53)$$

takes the form

$$G_{\gamma,v}^{\pm} =$$
$$\sqrt{\sin v}\Big\{P_{s_\gamma}^v(\cos v,\theta) \pm$$
$$\frac{2}{\pi}Re\begin{Bmatrix}\tan(\pi\delta_1)\\ \cot(\pi\delta_1)\end{Bmatrix}.Q_{s_\gamma}^v(\cos v,\theta)\Big\},\Big(\gamma = -\frac{1}{2}+iT, T>$$
$$0; v = \frac{1}{2}-\mu;\ \delta_1 = \frac{1}{2}(\mu - iT)\Big) \quad (3.54)$$

Also, the solution of the **ODE** (3.36), becomes

$$F_\gamma^v(u) = \sqrt{|\sinh u|}|\tanh u|^{-v}(\cosh u)^{-\frac{1}{2}\int_\gamma^v(\theta)}$$
$$\times\sum_{r=-\infty}^{\infty}(-1)^r b_{\gamma,r}^v(\theta)K_{iT+2r}\left(2\sqrt{q}\cosh u\right)$$
$$(3.55)$$

$$\int_\gamma^v(\theta) = (\pi^2 q)^{-\frac{1}{4}}e^{\frac{\pi}{4}(2T-3i)}.\left[\sum_{r=-\infty}^{\infty}(-1)^r b_{\gamma,r}^v(\theta)\right](-\infty < u < \infty; T>0)$$

The general solution of BVP (2.10), yields

$$U_s(y,z) = (|\sinh u|\sin v)^{-\mu}F_\gamma^v(u)G_{\gamma,v}^{\pm}(v)(-\infty < u < \infty, 0 \leq v \leq \pi) \quad (3.56)$$

where $G_{\gamma,v}^{\pm}(v)$ and$F_\gamma^v(u)$ are completely determined by (3.54) and (3.55), respectively.

After using (2.10), then (2.9), we can arrive to the following **SRs**:

$$\int_a^\infty\left\{\frac{K_\mu(|s||y-u|)}{|y-u|^\mu} \pm \frac{K_\mu(|s||y+u|)}{|y+u|^\mu}\right\}(u^2-a^2)^{-\frac{v}{2}}S_\gamma^{v(3)}(u,\theta)du =$$
$$\lambda_{\gamma,v}^{\pm}(a^2-y^2)^{\frac{v}{2}}S_\gamma^{v(3)}(u,\theta)(y>a). \quad (3.57)$$

$$\lambda_{\gamma,v}^{\pm} = \pi^{\frac{3}{2}}a^{2v}(2|s|)^{-\mu}K_{\gamma,v}^{\pm}(\theta)\left[\int_{\gamma,v}^{\pm}(\theta)\right]^{-1},$$

$K^{\pm}_{\gamma,v}(\theta)$

$= [\Gamma(1-v)]^{-1} \sum_{r=-\infty}^{\infty} Re[b^{v}_{\gamma,v}(\theta)]$

$\pm \pi^{-1} \sin(\pi\mu)\, \Gamma(v) \left\{ Re \begin{bmatrix} \tan(\pi\delta_1) \\ \cot(\pi\delta_1) \end{bmatrix} . \sum_{r=0}^{\infty} (-1)^r b^{v}_{\gamma,r}(\theta) \right.$

$$\mathcal{J}^{\pm}_{\gamma,v}(\theta) = \pm \left\{ Re \begin{bmatrix} \tan(\pi\delta_1) \\ \cot(\pi\delta_1) \end{bmatrix} . \sum_{r=0}^{\infty} (-1)^r b^{v}_{\gamma,r}(\theta)\, \Gamma(1 - 2\delta_1 + 2r) \left(\Gamma(2\bar{\delta} + 2r) \right)^{-1} \right\}$$

$$\delta_1 = \frac{1}{2}(\mu - iT)\,, T > 0\,, \gamma = -\frac{1}{2} + ir\,, v = \frac{1}{2} - \mu$$

The spheroidal wave function, for $0 < y < a$, we have $0 < v < \frac{\pi}{2}$, is defined in the form (see [4] p.173-175),

$$S^{v(3)}_{\gamma}(\cosh u\,, \theta) = [\pi \sinh(\pi T)]^{-1} e^{\pi(2T + i\mu)} \times \left\{ \sin(2\pi\bar{\delta}_1)\, K^{v}_{\gamma}(\theta) Q^{v}_{s-\gamma-1} - i \sin(2\pi\delta_1) \bar{K}^{v}_{\gamma}(\theta) Q^{v}_{s_\gamma}\, (\cosh u\,, \theta) \right\},$$

$$(-\infty < u < \infty),$$

$$K^{v}_{\gamma}(\theta) = 2^{-1} \left(\frac{q}{4}\right)^{\frac{\gamma}{2}} \Gamma(2\bar{\delta}_1) e^{-\frac{3\pi}{4}(2T+i)} \sum_{r=-\infty}^{\infty} \left[(-1)^r b^{v}_{\gamma,v}(\theta)\right]^{-1} \times L^{v}_{\gamma}(\theta) \left[L^{v}_{\gamma}(\theta)\right]^{-1}, \quad (3.58)$$

$L^{v}_{\gamma}(\theta) = \sum_{r=0}^{\infty} b^{v}_{\gamma_1,r}(\theta)$, $\left(\gamma = -\frac{1}{2} + ir\,, r > 0, \gamma_1 = \right.$ *−γ−1 , v=12−μ ,δ1=12μ−iT.*

Then, the **SRs**, in this case, take the form

$\int_a^\infty \left\{ \frac{K_\mu(|s||y-u|)}{|y-u|^\mu} \pm \right.$ *Kμsy+uy+uμu2−a2−v2Sγv3u,θdu=hγ,v±a2−y2v 2Hγ,v±*cos*−1ya0<y<a,* (3.59)

$h^{\pm}_{\gamma,v} =$
$\pm 2^{-\frac{1}{2}} \pi^{\frac{3}{2}} a^{2v} |s|^{-\mu} \times$

$$\left\{ Re \left[\begin{bmatrix} \tan(\pi\delta_1) \\ \cot(\pi\delta_1) \end{bmatrix} . \sum_{r=0}^{\infty} (-1)^r b^{v}_{\gamma,r}(\theta)\, \Gamma(1 - 2\delta_1 + 2r) \left(\Gamma(2\bar{\delta} + 2r) \right)^{-1} \right\}^{-1},$$

$$H^{\pm}_{\gamma,v}(v) = (\sin v)^{-\frac{1}{2}} G^{\pm}_{\gamma,v}(v)$$

Where $G^{\pm}_{\gamma,v}(v)$ are given by (3.54).

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