

Fundamental contact problem and singular mixed integral equation

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Abstract:: In this work, we derived a Fredholm-Volterra integral equation (**F-VIE**) of the second kind from the plane strain problem of the bounded layer medium composed of three different materials. These different materials contain a crack on one of the interface. In addition, the existence of a unique solution of **F-VIE** is considered in the space $L_2[-1,1] \times C(0, T)$, $T < 1$. The integral equation is solved by using quadratic method to obtain SFIEs. Then, we used two direction, the first direction, is by removing the singularity and using Legendre polynomials. While the other, by using Toeplitz matrix method (**TMM**) and product Nystrom method (**PNM**). Finally, numerical examples are considered and the estimate error, in each case, is compared between the three methods.

[Mohamad Abou, Sameeha Raad, Sareefa AlHazmi. **Fundamental contact problem and singular mixed integral equation**. *Life Sci J* 2014;11(6):119-125]. (ISSN:1097-8135). <http://www.lifesciencesite.com>. 17

Keywords: Fredholm-Volterra integral equation, Cauchy kernel, series method, Toeplitz method, Nystrom method, orthogonal polynomials, linear algebraic system.

MSC: 54Bo5, 45R10.

1. Introduction

The singular **IEs** appear in many problems of mathematical physics and engineering, it considered to be of more interest than others cases of integral equations. In addition, the singular **IEs** appear in studies involving airfoil [1], fracture mechanics [2], contact radiation and molecular conduction [3], contact problems [4] and potential theory [5]. The solution of a large class of mixed boundary value problems of a great variety of contact and cracks problems in solid mechanics, physical and engineering with its numerical results can be founded in Kalandiya [6,7], Abdou [8,9], Erdogan et al. [10,11], Cuminato [12] and Theocaris et al. [13].

In this work, we considered the fundamental equations in the theory of elasticity, under certain condition. Then, using the Fourier integral transforms, we can obtain a mixed integral equation of type **F-VIE** in the space $L_2[-1,1] \times C[0, T]$, $0 \leq t \leq T < 1$. The Fredholm integral term is considered in position with Cauchy kernel (**CK**). While, the Volterra integral term is considered in time with continuous kernel. The existence of a unique solution of **F-VIE** will be discussed and obtained, under certain conditions. Then, we used quadratic method to obtain a system of Fredholm integral equations (**SFIEs**). Moreover, the existence of a unique solution of the integral system can be discussed. To discuss the solution of the system integral equations, we considered different ways. The first way is

removing the singularity and using suitable orthogonal polynomial.

Here, in the first way, we expand the solution in term of Legendre polynomials. The second way is using the two famous numerical methods **TMM** and **PNM** for solving the singular integral equations. These three methods reduce SFIEs to LAS, which can be solved numerically. Finally, numerical examples are computed and the error is compared between the three methods.

2. Formulation of the Problem:

Consider the plane strain problem for the bounded layer medium (see Fig.(2-1)) composed of three different materials. Let the medium material contains a crack on one of the interfaces. Without any loss in generality, the half-length of the crack is assumed to be unity.

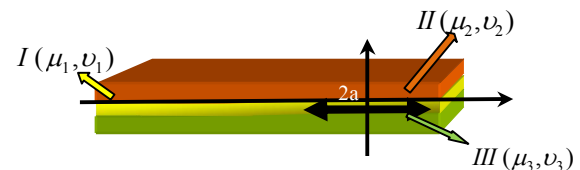


Fig. (2-1)

We will consider with the effect of the ratio of the layer thickness to the crack length on the stress, intensity factors and the strain energy release rate. For interesting the disturbed stress state, whiles is variable also with time, caused by the crack. We

assume that the overall stress distribution $\sigma_{ij}^{(0)}(x, y; t)$, in the imperfection free medium, is known. The stress state $\sigma_{ij}^{(1)}(x, y; t)$, in the cracked medium, may be expressed as

$$\sigma_{ij}^{(1)}(x, y; t) = \sigma_{ij}^{(0)}(x, y; t) + \sigma_{ij}(x, y; t), \quad i, j = x, y, z, \quad (2.1)$$

where, σ_{ij} is the disturbed state which may be obtained by using the tractions

$$\begin{aligned} P_1(x; t) &= -\sigma_{yy}^{(0)}(x, 0; t); \\ P_2(x; t) &= -\sigma_{xy}^{(0)}(x, 0; t), \\ |x| < 1, \quad t \in [0, T], \end{aligned} \quad (2.2)$$

which are the only external loads applied to the medium (the symmetry is considered with $x = 0$).

The general problem can always be expressed as the sum of a symmetric component and an anti-symmetric component. The tractions $P_i(x; t)$, ($i = 1, 2$), have the following properties

$$P_1(x; t) = P_1(-x; t), \quad P_2(x; t) = -P_2(-x; t), \quad |x| < 1, \quad t \in [0, T]. \quad (2.3)$$

The solution of the anti-symmetric problem requires only a slight modification.

Let u_i, v_i be the x, y components of the displacement vector in the i th materials and satisfy the field equations in the form

$$\mu_i \nabla^2 u_i + (\lambda_i + \mu_i) \frac{\partial}{\partial x} \left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} \right) = \rho \frac{d^2 u_i}{dt^2}, \quad (2.4)$$

$$\mu_i \nabla^2 v_i + (\lambda_i + \mu_i) \frac{\partial}{\partial y} \left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} \right) = \rho \frac{d^2 v_i}{dt^2}. \quad (2.5)$$

Then, assume the displacement functions in the following

$$\begin{aligned} u_i(x, y, z) &= U_i(x, y) + F(t), \\ v_i(x, y, z) &= V_i(x, y) + F(t), \end{aligned} \quad (2.6)$$

where $F(t)$ is known function of t .

Hence, using (2.6) in Eqs. (2.4) and (2.5), we have

$$(\lambda_i + 2\mu_i) \frac{\partial^2 U_i}{\partial x^2} + \mu_i \frac{\partial^2 U_i}{\partial y^2} + (\lambda_i + \mu_i) \frac{\partial^2 V_i}{\partial x \partial y} = 0, \quad (2.7)$$

$$(\lambda_i + 2\mu_i) \frac{\partial^2 V_i}{\partial y^2} + \mu_i \frac{\partial^2 V_i}{\partial x^2} + (\lambda_i + \mu_i) \frac{\partial^2 U_i}{\partial x \partial y} = 0, \quad (2.8)$$

and

$$\frac{d^2 F(t)}{dt^2} = \frac{\mu_i}{\rho} F(t). \quad (2.9)$$

The formula (2.9) has a solution

$$F(t) = B \exp\left(-\sqrt{\frac{\mu_i}{\rho}} t\right), \quad (F(\infty) \rightarrow 0). \quad (2.10)$$

Also, for solving the two formulas (2.8) and (2.9), we use the following Fourier integral transform

$$U_i(x, y) = \frac{2}{\pi} \int_0^\infty \phi_i(\alpha, y) \sin \alpha x \, d\alpha, \quad (i = 1, 2, 3). \quad (2.11)$$

$$V_i(x, y) = \frac{2}{\pi} \int_0^\infty \psi_i(\alpha, y) \cos \alpha x \, d\alpha. \quad (2.12)$$

Then, we have

$$-\alpha^2(\lambda_i + 2\mu_i)\phi_i + \mu_i \frac{d^2 \phi_i}{dy^2} - \alpha(\lambda_i + \mu_i) \frac{d\psi_i}{dy} = 0, \quad (2.13)$$

$$-\alpha^2(\lambda_i + 2\mu_i)\psi_i + \mu_i \frac{d^2 \psi_i}{dy^2} - \alpha(\lambda_i + \mu_i) \frac{d\phi_i}{dy} = 0, \quad (2.14)$$

After solving the system of Eqs. (2.13) - (2.14), and then using the two formulas (2.11), (2.12), we get

$$U_i(x, y) = \frac{2}{\pi} \int_0^\infty [(A_{i11} + A_{i12}y)e^{-\alpha y} + (A_{i13} + A_{i14}y)e^{\alpha y}] \sin \alpha x \, d\alpha, \quad (2.15)$$

$$V_i(x, y) = \frac{2}{\pi} \int_0^\infty \left[\left(A_{i11} + A_{i12} \left(y + \frac{k_i}{\alpha} \right) \right) e^{-\alpha y} + \left(A_{i13} + A_{i14} \left(\frac{k_i}{\alpha} - y \right) \right) e^{\alpha y} \right] \cos \alpha x \, d\alpha, \quad (2.16)$$

Where k_i have physical meaning and $k_i = 3 - 4\nu_i$ for plane strain and $k_i = \frac{3-\nu_i}{1+\nu_i}$ for generalized plane stress, ν_i are Poisson's coefficients for each materials, and $A_{ij}, j = 1, 2, 3, 4$, are functions of α which can be determined from the boundary conditions. After obtaining the values of U_i and V_i , the stresses may be evaluated by Hook's law.

In particular, the components of the stress vector at the interfaces and boundaries may be expressed as

$$\frac{1}{2\mu_i} \sigma_{yy}^i = \frac{2}{\pi}$$

$$\times \int_0^\infty \cos \alpha x \{ -[\alpha(A_{i11} + A_{i12}y) + 2(1 - \nu_i)A_{i12}] e^{-\alpha y} + [-\alpha(A_{i13} + A_{i14}y) + 2(1 - \nu_i)A_{i14}] e^{\alpha y} \} d\alpha, \quad (2.17)$$

and

$$\frac{1}{2\mu_i} \sigma_{xy}^i = \frac{2}{\pi}$$

$$\times \int_0^\infty \sin \alpha x \{ -[\alpha(A_{i11} + A_{i12}y) + (1 - 2\nu_i)A_{i12}] e^{-\alpha y} + [\alpha(A_{i13} + A_{i14}y) - (1 - 2\nu_i)A_{i14}] e^{\alpha y} \} d\alpha. \quad (2.18)$$

On the boundaries, the medium may have formally any one of the following four groups of homogeneous boundary conditions

$$\begin{aligned} (a) \sigma_{yy}^i &= 0 = \sigma_{xy}^i, & (b) \sigma_{xy}^i &= 0 = \nu_i, \\ (c) \sigma_{yy}^i &= 0 = u_i, & (d) u_i &= 0 = v_i, \end{aligned} \quad i = 1, 2, 3. \quad (2.19)$$

The continuity requires that on the interfaces the stress and displacement vectors in the adjacent layers be equal i.e.

$$u_{i+1} - u_i = 0, \quad v_{i+1} - v_i = 0, \quad \sigma_{yy}^{i+1} - \sigma_{yy}^i = 0, \\ \sigma_{xy}^{i+1} - \sigma_{xy}^i = 0, \quad i = 1,2,3, \quad (2.20)$$

Now, to obtain the **IE**, we first assume that at $y = 0$ the bond between the two adjacent layers is perfect except for the (symmetrically located) dislocations at $y = 0, x = y$ defined by

$$\frac{\partial}{\partial x}(u_2^+ - u_3^-) = f_1(x, t), \\ \frac{\partial}{\partial x}(v_2^+ - v_3^-) = f_2(x, t), \quad (2.21)$$

where the superscripts + and - refer to the limiting values of the displacement as y approaches zero from + and - sides respectively.

In addition to (2.21), on the interface $y = 0$, we have the following conditions

$$\sigma_{yy}^2 - \sigma_{yy}^3 = 0, \sigma_{xy}^2 - \sigma_{xy}^3 = 0, \\ (0 \leq x < \infty, y = 0). \quad (2.22)$$

After some algebraic relations, the components of the stress vector at $y = 0$ and $x > 0$ may be expressed as

$$\frac{1 + K_3}{\mu_3} \sigma_{yy}^3(x, 0, t) = \int_0^t F(\tau) f_1(x, 0, \tau) d\tau + \lim_{y \rightarrow 0^-} \frac{2}{\pi} \\ \int_0^\infty \{(a_{11}A_1(\alpha, t) + a_{12}A_2(\alpha, t))\} e^{\alpha x} \cos \alpha x \, d\alpha, \quad (2.23)$$

$$-\frac{1 + K_3}{\mu_3} \sigma_{xy}^3(x, 0, t) = \int_0^t F(\tau) f_2(x, 0, \tau) d\tau + \frac{2}{\pi} \\ \lim_{y \rightarrow 0^-} \int_0^\infty \{(a_{21}A_1(\alpha, t) + a_{22}A_2(\alpha, t))\} e^{\alpha x} \sin \alpha x \, d\alpha, \quad (2.24)$$

where A_i are the Fourier transforms of f_i defined as follows

$$A_1(\alpha, t) = \int_0^\infty f_1(z, t) \cos \alpha z \, dz, \\ A_2(\alpha, t) = \int_0^\infty f_2(z, t) \sin \alpha z \, dz, \quad (2.25)$$

The constants a_{ij} depend on the elastic properties of the materials adjacent to the crack only and are given by

$$a_{11} = -a_{22} = \frac{(1 + \lambda_2 \lambda_4)}{\lambda_4}, \\ a_{12} = -a_{21} = \frac{-(1 + 2\lambda_4 - \lambda_2 \lambda_4)}{\lambda_4},$$

$$\lambda_2 = \frac{K_2 \mu_3 - K_3 \mu_2}{\mu_2 + K_2 \mu_3}, \quad \lambda_4 = \frac{\mu_3 - K_3 \mu_2}{\mu_2 - \mu_3}, \quad (2.26)$$

where μ_i is the shear modulus and λ 's are Lamé's constants.

The integrals of on the right hand side of (2.24), (2.25) are uniformly convergent; as a result, certain operations such as change of order of integration are permissible. Also, note that once the dislocations $f_i(x)$ on the interface are specified the formulas (2.23), (2.24) and (2.25) give the stresses for all values of x . The crack problem under consideration $f_i(x)$ are zero for $|x| > 1$ and are unknown for $|x| < 1$. On the other hand, the stress vector on the interface $y = 0$ is unknown for $|x| > 1$ and is given by the following known functions for $|x| < 1$, i.e.

$$\sigma_{yy}^3(x, 0, t) = P_1(x, t), \\ \sigma_{xy}^3(x, 0, t) = P_2(x, t), \quad |x| < 1. \quad (2.27)$$

From the above information and the symmetric properties and in presence of time, we have

$$f_1(x, t) = f_1(-x, t), \quad f_2(x, t) = -f_2(-x, t).$$

Hence, we obtain

$$-\frac{1 + K_3}{\mu_3} P_1(x, t) = \int_0^t F(\tau) f_1(x, \tau) d\tau + \\ \lim_{y \rightarrow 0^-} \left[\frac{a_{11}}{\pi} \int_{-1}^1 f_1(z, t) dz \int_0^\infty e^{\alpha x} \cos \alpha(z - x) \, d\alpha \right. \\ \left. + \frac{a_{12}}{\pi} \int_{-1}^1 f_2(z, t) dz \int_0^\infty e^{\alpha x} \sin \alpha(z - x) \, d\alpha \right],$$

and

$$-\frac{1 + K_3}{\mu_3} P_2(x, t) = \int_0^t F(\tau) f_2(x, \tau) d\tau + \\ \lim_{y \rightarrow 0^-} \left[\frac{a_{21}}{\pi} \int_{-1}^1 f_1(z, t) dz \int_0^\infty e^{\alpha x} \sin \alpha(z - x) \, d\alpha \right. \\ \left. + \frac{a_{22}}{\pi} \int_{-1}^1 f_2(z, t) dz \int_0^\infty e^{\alpha x} \cos \alpha(z - x) \, d\alpha \right]. \quad (2.28)$$

Evaluating the infinite integrals in (2.28), passing to the Cauchy theorems in complex analysis, and adapting the coefficients with the aid of (2.26), we have

$$\frac{1 + K_3}{a_{12} \mu_3} P_1(x, t) = \gamma f_1(x, t) + \frac{1}{\pi} \int_{-1}^1 \frac{f_2(y, t)}{y - x} dy \\ + \frac{1}{a_{12}} \int_0^t F(\tau) f_1(x, \tau) d\tau,$$

and

$$\frac{1 + K_3}{a_{21}\mu_3} P_2(x, t) = \frac{1}{\pi} \int_{-1}^1 \frac{f_1(y, t)}{y - x} dy - \gamma f_2(x, t) + \frac{1}{a_{21}} \int_0^t F(\tau) f_2(x, \tau) d\tau, \quad (2.29)$$

where

$$\gamma = \frac{a_{11}}{a_{12}} = \frac{a_{22}}{a_{21}} = \frac{(\mu_3 + \mu_2 K_3) - (\mu_2 + \mu_3 K_2)}{(\mu_2 + \mu_3 K_2) + (\mu_3 + \mu_2 K_3)}. \quad (2.30)$$

The two formulas of (2.29) represent a system of mixed IEs with Cauchy kernel. For one layer.

3. Fredholm-Volterra Integral Equation:

Consider the following F- VIE

$$\mu \phi(x, t) + \lambda \int_{-1}^1 k(x - y) \phi(y, t) dy + \lambda \int_0^t F(t - \tau) \phi(x, \tau) d\tau = f(x, t). \quad (3.1)$$

under the dynamic conditions

$$\int_{-1}^1 \phi(x, t) dx = N_1(t), \quad \int_{-1}^1 x \phi(x, t) dx = N_2(t). \quad (3.2)$$

In order to guarantee the existence of a unique solution of Eq. (3.1), under the conditions (3.2), we assume through this work the following conditions:

(1) The time function $F(t - \tau), t, \tau \in [0, T], 0 \leq t \leq T \leq 1$, is positive continuous function with its derivatives belong, to the class $C[0, T]$, i.e.

$$F(t - \tau) < A_3 \quad (A_3 \text{ is a constant}).$$

(2) The kernel of position satisfies

$$\left\{ \int_{-1}^1 \int_{-1}^1 k^2(x - y) dx dy \right\}^{\frac{1}{2}} = B < \infty, \quad (B \text{ is a constant}).$$

(3) The given function $f(x, t)$ is continuous with its derivatives with respect to the position and time in the space $L_2[-1, 1] \times C[0, T], t \in [0, T], T \leq 1$.

(4) The unknown function $\phi(x, t)$ satisfies Lipschitz condition for the first argument and Hölder condition for the second argument.

The reader can prove that: the integral operator

$$K \phi(x, t) = \int_{-1}^1 k(x - y) \phi(y, t) dy,$$

for all $t \in [0, T], T \leq 1$ is bounded and continuous in the space $L_2[-1, 1] \times C[0, T]$.

4. The system of Fredholm integral equations:

As an important way to obtain the solution $\phi(x, t)$ of the F- VIE is representing it as a SFIEs in position, see Abdou and Raad [14,15].

For this, we divide the interval $[0, T]$, $0 \leq t \leq T < 1$ as $0 = t_0 < t_1 < \dots < t_l < \dots < t_p = T$, where $t = t_l, l = 1, 2, \dots, p$, using the quadrature formula, we get

$$\mu_1 \phi_1(x) + \lambda \int_{\Omega} k(|x - y|) \phi_1(y) dy = \psi_1(x), \quad (4.1)$$

$$\text{with } \psi_1(x) = f_1(x) - \lambda \sum_{j=0}^{l-1} u_j F_{1,j} \phi_j(x),$$

$$\mu_1 = \mu + \lambda u_l F_{1,l}. \quad (4.2)$$

where we used the following notations

$$\phi_l(x) = \phi(x, t_l), \quad F_{1,j} = F(t_l, t_j),$$

$$f_l(x) = f(x, t_l), \quad l = 0, 1, 2, \dots, p$$

$$\text{and } u_j = \begin{cases} h_1/2 & j = 0, l \\ h_1 & 0 < j < l \end{cases}$$

For this, let E be the set of all continuous functions $\phi_n(x)$ in the space $L_2(\Omega)$, where $\Phi(x) = \{\phi_0(x), \phi_1(x), \dots, \phi_n(x), \dots\}$ and we can define the norm in the Banach space E by $\|\Phi\|_E = \max_n \|\phi_n(x)\|_{L_2(\Omega)}$. When the function $\psi_1(x)$ has a unique representation, the SFIEs (4.1) has a unique solution in the space E.

5. Fredholm equation:

Consider the FIE, in the interval $[-1, 1]$, takes the form:

$$\mu \phi(x) + \lambda \int_{-1}^1 \frac{\phi(y)}{y - x} dy = f(x), \quad \lambda \text{ is a constant}. \quad (5.1)$$

The sign \int denotes integration with Cauchy principal value sense. For this aim, the singularity of the integral term of Eq. (5.1) will be weakened by the following methods:

5.1 Legendre polynomials method:

In this section, we will use the removing singularity method to rewrite the integral term of (5.1) and adapting it in the form:

$$\mu \phi(x) + \lambda \int_{-1}^1 \frac{\phi(y) - \phi(x)}{y - x} dy - \lambda \phi(x) \log \frac{1 + x}{1 - x} = f(x), \quad -1 \leq x \leq 1. \quad (5.2)$$

The integral term in the right hand side of (5.2) is regular and will be evaluated. So, assume that, the unknown function can be expanded in term of Legendre polynomials form, i.e.

$$\phi(x) = \sum_{j=0}^{\infty} a_j P_j(x), \quad (5.3)$$

Here, a_j are constants and $P_j(x)$ are the Legendre polynomials. Substituting from (5.3) in (5.2), we get

$$f(x) = \left(\mu - \lambda \log \frac{1+x}{1-x} \right) \sum_{j=0}^{\infty} a_j P_j(x) + \lambda \sum_{j=0}^{\infty} a_j \int_{-1}^1 \frac{P_j(y) - P_j(x)}{y-x} dy. \quad (5.4)$$

The value of a_0 in (5.4) can be obtained. After using the following orthogonal relation of the Legendre polynomial and the Rodriguez formula of the Legendre polynomial $P_j(x)$ of degree j ,

$$P_j(x) = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} b_k x^{j-2k},$$

$$b_k = \frac{(-1)^k (2j-2k)!}{2^j k! (j-k)! (j-2k)!}, \quad (5.5)$$

we have $\gamma_{j,k,l} = \frac{1 - (-1)^{j-l}}{j-2k-l}$, and

$$a_0 = \frac{1}{2} \int_{-1}^1 \phi(x) dx, \text{ we obtain}$$

$$\left(\mu - \lambda \log \frac{1+x}{1-x} \right) \sum_{j=0}^{\infty} a_j P_j(x) + \lambda \sum_{j=1}^{\infty} \sum_{k=0}^{\lfloor \frac{j-1}{2} \rfloor} \sum_{\ell=0}^{j-2k-1} a_j b_k \gamma_{j,k,\ell} x^\ell = f(x). \quad (5.6)$$

Multiply both sides of (5.6) by x^{i-1} for $i = 1, 2, \dots, N$ and then integrating the result over the interval

$$[-1, 1], \left(\log \frac{1+x}{1-x} \cong 2 \sum_{\alpha=1,3,\dots}^{\infty} \frac{x^\alpha}{\alpha}, |x| < 1 \right), \text{ and}$$

using the formula, see [16]

$$\int_{-1}^1 x^i P_j(x) dx = 2 \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \frac{b_k \delta_{i,j}}{(j-2k+i+1)},$$

$$\delta_{i,j} = \begin{cases} 1 & i+j \text{ even} \\ 0 & \text{otherwise} \end{cases} \quad (5.7)$$

The formula (5.6), after truncating the infinite series to the first N terms yields

$$\sum_{j=1}^N D_{ij} a_j = d_i, \quad (1 \leq i \leq N) \quad (5.8)$$

where

$$D_{ij} = \begin{cases} \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} 2b_k \left[\frac{\mu \delta_{i-1,j}}{(j-2k+i)} \right. \\ \left. - 2\lambda \sum_{\alpha=1}^{\infty} \frac{\delta_{i-1,j-\alpha}}{(j-2k+i-\alpha)} \right] \end{cases}$$

$$-2\lambda \sum_{\alpha=1}^{\infty} \frac{\delta_{i,j}}{(j-2k+i-1)(2\alpha-1)} \left. + \lambda \sum_{k=0}^{\lfloor \frac{j-1}{2} \rfloor} \sum_{\ell=0}^{j-2k-1} b_k \frac{2 \delta_{i,j-1} [1 - (-1)^{j-1}]}{(j-2k-1)(1+i)} \right\},$$

$$d_i = \int_{-1}^1 x^{i-1} f(x) dx + 2a_0 \left[\frac{2\lambda \delta_i}{i+1} - \frac{\delta_{i-1}}{i} \right]. \quad (5.9)$$

5.2. The Toeplitz matrix method

Consider the system of Fredholm integral equations (4.1), which can be reduced, by using the Toeplitz matrix method, (see Abdou, et al., [17,18]), to a system of linear algebraic equations:

$$\mu \phi(mh) + \lambda \sum_{n=-N}^N G_{n-m} \phi(nh) = \psi(mh). \quad (5.10)$$

where

$$G_{n-m} = A_n(mh) + B_{n-1}(mh),$$

$$A_n(x) = \frac{[(nh+h)I(x) - J(x)]}{h},$$

$$B_n(x) = \frac{[J(x) - nh I(x)]}{h},$$

$$I(x) = \int_{nh}^{nh+h} k(|x-y|) dy,$$

$$J(x) = \int_{nh}^{nh} y k(|x-y|) dy, \quad (5.11)$$

the solution of the formula (5.10) will be in the form

$$\phi(mh) = [\mu I - \lambda YG_{n-m}]^{-1} \psi(mh),$$

$$|\mu I - \lambda G_{n-m}| \neq 0, \quad (5.12).$$

I is the identity matrix, G and the Toeplitz matrix, it's elements are given by

$$G_{n-m} = (n+1-m) \ln|n-m+1| - 2(n-m) \ln|n-m| + (n-1-m) \ln|n-m-1|. \quad (5.13)$$

The algebraic system in (5.12), has a unique solution in Banach space ℓ^∞ .

5.3. The product Nystrom method:

The integral equation (4.1) can be reduced by the product Nyström method (see 19-20) to a system of linear algebraic equations

$$\mu \phi(ih) - \lambda \sum_{j=0}^N w_{i,j} \bar{k}(|x_i - y_j|) \phi(jh) = \psi(ih), \quad (5.14)$$

where

$$w_{i,0} = \frac{h[2\chi_0(z) - 3\chi_1(z) + \chi_2(z)]}{2} = \frac{1}{2}$$

$$[\ln|2-z| - \ln|z|] \{-z-4+(z-2)(z-1)\}, z=i$$

$$w_{i,2j+1} = h[2\chi_1(z) - \chi_2(z)] = \{2(1-z)$$

$$+z(z-2)\ln|2-z| - \ln|z|\}, z=i-2j$$

$$w_{i,2j} = h[\chi_2(z) - \chi_1(z) + 2\chi_0(z-2) - 3\chi_1(z-2) + \chi_2(z-2)]/2$$

$$= \frac{1}{2}\{4(z-2) + (z-3)(z-4)\ln|4-z| + 6(z-2)\ln|2-z| - z(z-1)\ln|z|\},$$

$$z = i - 2j + 2$$

$$w_{i,N} = \frac{h[\chi_2(z) - \chi_1(z)]}{2} = \frac{1}{2}\{2z + z(z-1)\ln|2-z| - \ln|z|\}, z = i - N + 2. \quad (5.15)$$

which has the solution

$$\Phi = [\mu I - \lambda W]^{-1}\Psi, |\mu I - \lambda W| \neq 0. \quad (5.16)$$

The algebraic system in (5.21), has a unique solution in Banach space ℓ^∞ .

6. Numerical Results:

Consider the Fredholm - Volterra integral equation

$$\mu \phi(x, t) + \lambda \int_{-1}^1 \frac{\phi(y, t)}{(x-y)} dy + \lambda \int_0^t F(t, \tau) \phi(x, \tau) d\tau = f(x, t). \quad (6.1)$$

We solved this equation numerically, by maple 8 program, at the times $t = 0.001, 0.05$ and 0.009 , with $\lambda = .7857$ and 3.5045 , the parameter $\mu = 1$. The exact solution is $\phi(x, t) = x^2t^2$.

1- When $F(t, \tau) = t^2 \tau^2$

Case 1-1: $\lambda = 0.7857, T = 0.001$

x	Exact	Error Top.	Er. Nys.	Error Leg.
-1	1.0E-06	3.7613E-08	5.0E-15	7.2098E-09
-0.6	3.6E-07	4.0734E-09	1.0E-15	1.6285E-08
-0.2	4.0E-08	2.8388E-09	3.1E-16	1.3292E-08
0.2	4.0E-08	2.8388E-09	2.9E-16	1.3292E-08
0.6	3.6E-07	4.0734E-09	6.0E-16	1.6285E-08
1	1.0E-06	3.7613E-08	5.5E-15	7.2098E-09

Case 1-2: $\lambda = 3.5045, T = 0.001$

x	Exact	Error Top.	Er. Nys.	Error Leg.
-1	1.0E-06	3.9742E-08	5.0E-15	7.2098E-09
-0.6	3.6E-07	4.1843E-09	1.0E-15	1.6285E-08
-0.2	4.0E-08	2.9128E-09	3.1E-16	1.3292E-08
0.2	4.0E-08	2.9128E-09	2.9E-16	1.3292E-08
0.6	3.6E-07	4.1843E-09	6.0E-16	1.6285E-08
1	1.0E-06	3.9742E-08	5.5E-15	7.2098E-09

Case 1-3: $\lambda = 0.7857, T = 0.9$

x	Exact	Error Top.	Er. Nys.	Error Leg.
-1	0.81	0.0292925	8.336E-04	9.652E-03
-0.6	0.2916	0.0034167	1.387E-04	1.604E-02
-0.2	0.0324	0.0022912	6.034E-06	8.268E-03
0.2	0.0324	0.0022912	6.034E-06	8.268E-03
0.6	0.2916	0.0034167	1.387E-04	1.604E-02
1	0.81	0.0292925	8.336E-04	9.652E-03

Case 1-4: $\lambda = 3.5045, T = 0.9$

x	Exact	Err. Top.	Er. Nys.	Error Leg.
-1	0.81	0.030933	9.915E-04	1.5760E-02
-0.6	0.2916	0.003508	1.648E-04	1.7163E-02
-0.2	0.0324	0.002350	7.153E-06	7.2824E-03
0.2	0.0324	0.002350	7.153E-06	7.2824E-03
0.6	0.2916	0.003508	1.648E-04	1.7163E-02
1	0.81	0.030933	9.915E-04	1.5760E-02

2- When $F(t, \tau) = t \tau^2$

Case 2-1: $\lambda = 0.7857, T = 0.001$

x	Exact	Error Top.	Er. Nys.	Error Leg.
-1	1.0E-06	3.7613E-08	2.0E-15	7.2284E-08
-0.6	3.6E-07	4.0734E-09	7.0E-16	3.0903E-08
-0.2	4.0E-08	2.8388E-09	3.4E-16	4.7030E-10
0.2	4.0E-08	2.8388E-09	2.7E-16	4.7030E-10
0.6	3.6E-07	4.0734E-09	7.0E-16	3.0903E-08
1	1.0E-06	3.7613E-08	6.7E-15	7.2284E-08

Case 2-2: $\lambda = 3.5045, T = 0.001$

x	Exact	Error Top.	Er. Nys.	Error Leg.
-1	1.0E-06	3.9742E-08	2.0E-15	7.22837E-08
-0.6	3.6E-07	4.1843E-09	7.0E-16	3.09031E-08
-0.2	4.0E-08	2.9128E-09	3.4E-16	4.70301E-10
0.2	4.0E-08	2.9128E-09	2.7E-16	4.70301E-10
0.6	3.6E-07	4.1843E-09	7.0E-16	3.09031E-08
1	1.0E-06	3.9742E-08	6.7E-15	7.22837E-08

Case 2-3: $\lambda = 0.7857, T = 0.9$

x	Exact	Error Top.	Er. Nys.	Error Leg.
-1	0.81	0.02921016	8.7916E-04	0.05465367
-0.6	0.2916	0.00344625	1.4198E-04	0.02431504
-0.2	0.0324	0.00229461	5.8936E-06	0.00100934
0.2	0.0324	0.00229461	5.8936E-06	0.00100934
0.6	0.2916	0.00344624	1.4198E-04	0.02431504
1	0.81	0.02921017	8.7917E-04	0.05465367

Case 2-4: $\lambda = 3.5045, T = 0.09$

x	Exact	Error Top.	Er. Nys.	Error Leg.
-1	0.81	0.03085053	1.0458E-03	0.05276037
-0.6	0.2916	0.00353868	1.6866E-04	0.02396688
-0.2	0.0324	0.00235362	6.9856E-06	0.00131471
0.2	0.0324	0.00235362	6.9855E-06	0.00131471
0.6	0.2916	0.00353868	1.6866E-04	0.02396688
1	0.81	0.03085053	1.0458E-03	0.05276037

In the previous tables, we used shorthand words, Error Leg. (the error when we find an approximate solution by using Legendre polynomials), Er. Nys. (the error when we used Product Nystrom method), Error Top. (the error when we used Toeplitz matrix method).

7. Conclusion:

1. The solution $\phi(x, t)$ is symmetric with respect to x .
2. The error takes maximum values at the ends when $x = -1, 1$, while it is minimum at the middle when $x = 0$.
3. The approximation function has the best values by using Product Nystrom method, then by Toeplitz matrix, and then by Legendre method.
4. For $\lambda = 0.7857$, the error is smaller than the error when $\lambda = 3.5045$.
5. The error is increasing by increasing the time.
6. By Legendre polynomials method, the maximum value of the error is 0.05465367, when $\lambda = 0.7857$, $T = 0.9$, $F(t, \tau) = t^2 \tau^2$.
7. By Toeplitz matrix method, the maximum value of the error is 0.03093284, when $\lambda = 3.5045$, $T = 0.9$, $F(t, \tau) = t \tau^2$.
8. By product Nystrom method, the maximum value of the error is 1.0458E-03, when $\lambda = 3.5045$, $T = 0.9$, $F(t, \tau) = t \tau^2$.

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