Evolution of a Helix Curve by observing its velocity

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Abstract: In this paper, the equations of motion for a general helix curve $(\tau = \beta\kappa)$ are derived by applying the first compatibility conditions for dependent variables (time and arc length). As application of the equations of motions, mkdv equation is solved using symmetry method.

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1. Introduction

A lot of physical processes can be modelled in terms of the motion of curves, including the dynamics of vortex filaments in fluid dynamics [1], the growth of dendritic crystals in a plane [2], and more generally, the planar motion of interfaces [3]. The Subject of how space curves evolve in time is of great interest and has been investigated by many authors. Pioneering work is attributed to Hasimoto who showed in [1] the non-linear Schrӧodinger equation describing the motion of an isolated non-stretching thin vortex filament. Lamb [4] used the Hasimoto transformation to connect other motions of curves to the mKdV and sine-Gordon equations. Nakayama, et al [5] obtained the sine-Gordon equation by considering a non-local motion. Also Nakayama and Wadati [6] presented a general formulation of evolving curves in two dimensions and its connection to mKdV hierarchy. Nassar, et al [7]-[12] have studied evolution of manifolds and obtained many interesting results. R. Mukherjee and R. Balakrishnan [13] applied their method to the sine-Gordon equation and obtained links to five new classes of space curves, in addition to the two which were found by Lamb [4]. For each class, they displayed the rich variety of moving curves associated with the one-soliton, the breather, the two-soliton and the soliton-antisoliton so-lutions.

In this paper, Time evolution equations for a general helix curve are derived from applying the first compatibility conditions for dependent variables (time and arc length) as well as general helix space curve is reconstructed from its curvature.

Here, we consider the motion of curves in three-dimensional Euclidean space. Let $\mathbf{r}$ denote a point on a space curve. As usual, time is denoted by t. The conventional geometrical model is specified by velocity fields,

$$r_t = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \qquad (1)$$

Here, e_1, e_2 and e_3 are the unit tangential, normal and binormal vectors along the curve, and ω_1, ω_2 and ω_3 are the tangential, normal and binormal velocities. velocities fields are functionals of the intrinsic quantities of curves, for example, curvature, κ, torsion, τ, metric, g, etc. Time evolution equations for such quantities are derived from (1) and the geometrical relations. As applications to physics, these models are useful to describe the motion of vortex filaments in inviscid fluid, motion of fronts in viscous fingering in a Hele-Shaw cell, and kinematics of interfaces in crystal growth.

Consider a curve in 3-D represented by the parameter u i.e., $\mathbf{r} = \mathbf{r}(u)$. Let $\mathbf{r}(u,t)$ be the position vector of any point moves on the curve at the time t such that $\mathbf{r}(u,0) = \mathbf{r}(u)$. We define the metric of the curve, $g(u,t)$, and arc length $s(u,t)$, as

$$g := \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u}, \quad s(u,t) = \int_0^u \sqrt{g(v,t)}\,dv,$$

and then the unit tangent vector of the curve, $\mathbf{e}_1$ is defined by

$$\mathbf{e}_1 := \frac{\partial \mathbf{r}}{\partial s} = g^{-\frac{1}{2}} \frac{\partial \mathbf{r}}{\partial u}.$$

With this definition of the unit tangent vector one can canonically define a unit normal vector $\mathbf{e}_2$, and binormal vector $\mathbf{e}_3$, according to the well-known Serret-Frenet relations

$$\frac{\partial \mathbf{E}}{\partial s} = A\mathbf{E} \quad (2)$$

where

$$\mathbf{E} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \quad \text{and}\, A = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix},$$

κ is the curvature of the curve and τ its torsion.

In the same spirit one can also consider describing the time evolution of the curve. First we note that the FSE Eq. (2) can be written compactly as

$$\mathbf{e}_{i_s} = \overleftarrow{\omega} \times \mathbf{e}_i; \quad i = 1,2,3. \quad (3)$$

Where the Darboux vector $\overleftarrow{\omega} = \kappa \mathbf{e}_3 + \tau \mathbf{e}_1$, or simply $(\tau, 0, \kappa)$, in the $\mathbf{e}_i$, basis.

The dynamics of the triad can be described by defining a new vector $\overleftarrow{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ such that, similar to Eq. (3),

$$\mathbf{e}_{i_t} = \overleftarrow{\Omega} \times \mathbf{e}_i; \quad i = 1,2,3. \quad (4)$$

which can be written in matrix form as

$$\frac{\partial \mathbf{E}}{\partial t} = B\mathbf{E} \quad (5)$$

where

$$B = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix}$$

2. Reconstruction of curve from its curvature and torsion

Consider the Serret-Frenet relations

$$\frac{dE}{ds} = AE \quad (6)$$

where

$$E = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix},$$

κ is the curvature of the curve and τ its torsion.

Now, let

$$e_1(0) = e_{01}, \quad e_2(0) = e_{02}, \quad e_3(0) = e_{03}, \quad (7)$$

Basic existence and unique result for systems of linear ODEs guarantees the following fundamental theorem:

Theorem 1 *If $\kappa(s)$ and $\tau(s)$ are given smooth functions on an interval $I = (a,b)$, where $0 \in I$ and $\kappa(s) > 0$ then, given e_{01}, e_{02}, e_{03}, (6) has a unique solution on $s \in I$ satisfying (7). Moreover*

$$r(s) = r(0) + \int_0^s e_1(t)\,dt, \quad (8)$$

We have noted that knowledge of κ and τ essentially fixes a space curve and we here list some simple functions for κ and τ and the corresponding curves they generate.

(i) If $\tau(s) = \kappa(s) = 0$, the curve is an (untwisted) straight line.

(ii) If $\tau(s) = 0$, and $\kappa(s) =$ constant $\neq 0$ the curve is a circular arc.

(iii) If $\kappa(s) = 0$, but $\tau(s) \neq 0$ the curve is twisted) straight line.

(iv) If $\tau(s) =$ constant and $\kappa(s) =$ constant, the curve is a circular helix. The curve winds around a circular cylinder.

(v) If $\dfrac{\tau(s)}{\kappa(s)} =$ constant, the curve is a generalized (not necessarily circular) helix. The curve winds around a generalized circular cylinder.

We will use a powerful method called eigenvalue method to solve the homogeneous system (6)in the case (v) $\dfrac{\tau(s)}{\kappa(s)} = \beta$,i.e., we solve

$$\frac{dE}{ds} = \kappa(s) BE \quad (9)$$

with

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix},$$

The idea is to find solution of form

$$E_\lambda(s) = v e^{\lambda \phi(s)} \quad \phi(s) = \int \kappa(s)\,ds. \quad (10)$$

Now taking derivative on $E(s)$, we have

$$\frac{dE(s)}{ds} = \lambda v \kappa(s) e^{\lambda \phi(s)}$$

(11)

Put (10) and (11) into the homogeneous equation (9), we get

$$\frac{dE(s)}{ds} = \lambda v \kappa(s) e^{\lambda\phi(s)} = B\kappa(s)E(s)$$
$$= Bv\kappa(s)e^{\lambda\phi(s)}$$

So

$$Bv = \lambda v,$$

which indicates that λ must be an eigenvalue of B and v is an associate eigenvector.

We find that

$$\lambda_1 = +i\rho, \quad \lambda_2 = -i\rho, \quad \lambda_3 = 0, \quad \rho = \sqrt{1+\beta^2}$$

are the eigenvalues of B with associated eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ i\rho \\ -\beta \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -i\rho \\ -\beta \end{pmatrix}, \quad v_3 = \begin{pmatrix} \beta \\ 0 \\ 1 \end{pmatrix}$$

$E_{\lambda_1}(s), E_{\lambda_2}(s), E_{\lambda_3}(s)$ are 3 linearly independent (as vectors) solution of the homogeneous system (9).Then the general solution $E_h(s)$ of can be written as

$$E_h(s) = [c_1(v_1\cos(\rho\phi(s)) - v_2\sin(\rho\phi(s)))$$
$$c_2(v_2\cos(\rho\phi(s)) + v_2\sin(\rho\phi(s)))] + c_3 v_3 e^{\lambda_3\phi(s)}$$

which can be written in matrix form as

$$E_h(s) = \begin{pmatrix} \cos(\rho\phi) & \sin(\rho\phi) & \beta \\ -\rho\sin(\rho\phi) & \rho\cos(\rho\phi) & 0 \\ -\beta\cos(\rho\phi) & -\beta\sin(\rho\phi) & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{12}$$

If $e_{01} = (1,0,0), e_{02} = (0,1,0)$, and $e_{03} = (0,0,1)$ are the standards unit vectors then $c_1 = 1$, $c_2 = 1$, and $c_3 = 1$

Hence

$$r(s) = r(0) + \left(\int_0^s \cos(\rho\phi)dt, \int_0^s \sin(\rho\phi)dt, \beta s \right). \tag{13}$$

3. Equations of motion

It is important to notice that u and t are independent but s and t are not independent. As a consequence, wile u and t derivatives commute, s and t derivatives in general do not commute;

Lemma 1

$$\frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}\right) - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}\right) = \frac{g_t}{2g}\frac{\partial}{\partial s} \tag{14}$$

Proof.

$$\frac{\partial}{\partial u}\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial u}\right)$$
$$g^{\frac{1}{2}}\frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t}\left(g^{\frac{1}{2}}\frac{\partial}{\partial s}\right)$$
$$\frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}\right) + \frac{g_t}{2g}\frac{\partial}{\partial s}$$

Applying the first compatibility condition ((14)) to the matrix E and vector r respectively, yields the following equations:

$$\begin{aligned} &\frac{\partial A}{\partial t} = \frac{\partial B}{\partial s} - [A,B] - \frac{g_t}{2g}A \\ &e_{1t} + \frac{g_t}{2g}e_1 = (\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3)_s \end{aligned} \tag{15}$$

Written explicitly, Eq. (15) reads

$$\left.\begin{aligned} &\frac{\kappa g_t}{2g} + \kappa_t - \Omega_{3s} = \tau\Omega_2; \\ &\frac{\tau g_t}{2g} + \tau_t - \Omega_{1s} = -\kappa\Omega_2, \\ &\Omega_{2s} = -\kappa\Omega_1 + \tau\Omega_3 \\ &\frac{g_t}{2g} = (\omega_{1s} - \kappa\omega_2) \\ &\Omega_3 = \omega_{2s} + \kappa\omega_1 - \tau\omega_3 \\ &\Omega_2 = -(\omega_{3s} + \tau\omega_2) \end{aligned}\right\} \tag{16}$$

From the above equation

$$\Omega_1 = \frac{1}{\kappa}(\tau\Omega_3 - \Omega_{2s}) = \frac{\tau}{\kappa}(\omega_{2s} + \kappa\omega_1 - \tau\omega_3) + \frac{1}{\kappa}(\omega_{3s} + \tau\omega_2)_s \tag{17}$$

Theorem 2 *If the dynamics of the curve* $r(u,t)$, *is given by*

$$r_t = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$$

Then the motion of the curve is described by

$$\frac{g_t}{2g} = (\omega_{1s} - \kappa\omega_2)$$
$$\kappa_t = (\omega_{2s} + \kappa\omega_1 - \tau\omega_3)_s - \tau(\omega_{3s} + \tau\omega_2) - \kappa(\omega_{1s} - \kappa\omega_2)$$
$$\tau_t = [\frac{\tau}{\kappa}(\omega_{2s} + \kappa\omega_1 - \tau\omega_3) + \frac{1}{\kappa}(\omega_{3s} + \tau\omega_2)_s]_s + \kappa(\omega_{3s} + \tau\omega_2) - \tau(\omega_{1s} - \kappa\omega_2) \tag{18}$$

For a given g and ω_i, $i = 1,2,3,$ the motion of the curve is determined from these equations.

4. Helix Case

Here we restrict ourselves to arc length parameterized general helix curves. That is $\tau = \beta\kappa$. which implies that the motion of the curve is described by

$$\left.\begin{aligned} &\frac{\kappa g_t}{2g} + \kappa_t - \Omega_{3_s} = \beta\kappa\Omega_2; \\ &\frac{\beta\kappa g_t}{2g} + \beta\kappa_t - \Omega_{1_s} = -\kappa\Omega_2, \\ &\Omega_{2_s} = -\kappa\Omega_1 + \beta\kappa\Omega_3 \\ &\frac{g_t}{2g} = (\omega_{1s} - \kappa\omega_2) \\ &\Omega_3 = \omega_{2s} + \kappa\omega_1 - \beta\kappa\omega_3 \\ &\Omega_2 = -(\omega_{3s} + \beta\kappa\omega_2) \end{aligned}\right\} \tag{19}$$

If we set $\kappa = \phi_s$ and $\Omega_3 = \phi_t$ then from the compatibility condition (14), (19)leads to $\Omega_2 = 0$ and $\Omega_1 = \beta\Omega_3$. which implies that the motion of the helix curve is described by

$$\left.\begin{aligned} &\phi_t = \Omega_3 = \omega_{2s} + \phi_s\omega_1 - \beta\phi_s\omega_3, \\ &\frac{g_t}{2g} = (\omega_{1s} - \phi_s\omega_2) \\ &\omega_{3s} = -\beta\phi_s\omega_2 \end{aligned}\right\} or$$
$$\left.\begin{aligned} &\frac{\kappa g_t}{2g} + \kappa_t = (\omega_{2s} + \kappa\omega_1 - \beta\kappa\omega_3)_s, \\ &\frac{g_t}{2g} = (\omega_{1s} - \kappa\omega_2) \\ &\omega_{3s} = -\beta\kappa\omega_2 \end{aligned}\right\} \tag{20}$$

which leads to

$$\boxed{\kappa_t = \omega_{2ss} + \kappa_s\omega_1 - \beta\kappa_s\omega_3 + \rho^2\kappa^2\omega_2} \tag{21}$$

To reduce it to the two-dimensional case, set

$$\beta = 0, \quad \omega_3 = 0,$$

in which implies (20)is reduce to

$$\boxed{\kappa_t = \omega_{2ss} + \kappa_s\omega_1 + \kappa^2\omega_2} \tag{22}$$

We can deal with the motion of helix curves ($\tau = \beta\kappa$) in a different way from that in the previous sections. In terms of the components of the tangent, normal and bi-normal vectors $e_1 e_2, e_3$ of helix curve of the curve is expressed as (see (12))

$$e_1 = \begin{pmatrix} \cos\rho\phi \\ \sin\rho\phi \\ \beta \end{pmatrix}, \quad e_2 = \begin{pmatrix} -\rho\sin\rho\phi \\ \rho\cos\rho\phi \\ 0 \end{pmatrix},$$
$$e_3 = \begin{pmatrix} -\beta\cos\rho\phi \\ -\beta\sin\rho\phi \\ 1 \end{pmatrix}$$

where $\phi = \int_0^s \kappa ds, \quad \rho = \sqrt{1+\beta^2}$

Applying the first compatibility condition ((14)) to the vector r, yields the following equations

$$\begin{pmatrix} \cos\rho\phi \\ \sin\rho\phi \\ \beta \end{pmatrix}_t + \frac{g_t}{2g}\begin{pmatrix} \cos\rho\phi \\ \sin\rho\phi \\ \beta \end{pmatrix} = \begin{pmatrix} \omega_1\cos\rho\phi - \omega_2\rho\sin\rho\phi - \omega_3\beta\cos\rho\phi \\ \omega_1\sin\rho\phi + \omega_2\rho\cos\rho\phi - \omega_3\beta\sin\rho\phi \\ \omega_1\beta + \omega_3 \end{pmatrix}_s \tag{23}$$

This set of equations is essentially equivalent to (20).

The equation of motion for three-dimensional curves is represented by components explicitly in (23). To reduce it to the two-dimensional case, set

$$\beta = 0, \quad \omega_3 = 0,$$

in which implies (23)is reduce to

$$\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}_t + \frac{g_t}{2g}\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \begin{pmatrix} \omega_1\cos\phi - \omega_2\sin\phi \\ \omega_1\sin\phi + \omega_2\cos\phi \end{pmatrix}_s \tag{24}$$

which is equivalent to (21).

5. Application mkdv equation

In the above equation (21)if we take $\omega_1 = -a_1\kappa, \quad \omega_2 = -a_2\kappa_s, \quad \omega_3 = -a_3\kappa$ we get the known mkdv equation

$$\Delta = \kappa_t + \bar{a}_1\kappa\kappa_s + a_2\kappa_{sss} + \bar{a}_3\kappa^2\kappa_s = 0, \qquad (25)$$

where $\bar{a}_1 = a_1 - \beta a_3$ and $\bar{a}_3 = \rho^2 a_2$.

According to Lie's algorithm [15]-[17], the infinitesimal generator of the maximal symmetry group admitted by (25) is given by

$$X = \xi^i(t,s,\kappa)\partial_t + \eta(t,s,\kappa)\partial_\kappa \qquad (26)$$

if and only if the invariance condition of (25) is

$$X^{[3]}(\Delta)_{|(25)} = 0 \qquad (27)$$

where

$$X^{[3]} = X + \zeta_i \frac{\partial}{\partial\kappa_i} + \zeta_{i_1 i_2}\frac{\partial}{\partial\kappa_{i_1 i_2}} + \zeta_{i_1 i_2 i_3}\frac{\partial}{\partial\kappa_{i_1 i_2 i_3}}, \qquad (28)$$

is the prolongation of the vector field (26). The variables ζ' s are given by the formulae :

$$\begin{aligned}
\zeta_i &= D_i(\eta - \xi^j\kappa_j) + \xi^j\kappa_j,\\
\zeta_{ij} &= D_i D_j(\eta - \xi^j\kappa_j) + \xi^j\kappa_{\ell j},\\
\zeta_{i_1..i_3} &= D_{i_1}..D_{i_3}(\eta - \xi^j\kappa_j) + \xi^j u^\alpha_{j i_1..i_3},
\end{aligned} \qquad (29)$$

Executing the Lie's algorithm, we obtain the Lie point symmetries of (25) given by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial s},$$
$$X_3 = -3t\frac{\partial}{\partial t} + \left(\frac{\bar{a}_1 t}{2\bar{a}_3} - s\right)\frac{\partial}{\partial s} + \left(\frac{\bar{a}_1}{2\bar{a}_3} + \kappa\right)\frac{\partial}{\partial\kappa}. \qquad (30)$$

We look for solutions invariant under the linear combination $X_1 + \lambda X_2$ where $X_1 + \lambda X_2$ is a constant. Solving the characteristic system for the invariants of the linear combination, we obtain

$$\xi = s - \lambda t, \quad \kappa(t,s) = \upsilon(\xi), \qquad (31)$$

where υ is an arbitrary function of ξ . The substitution of (31) into (25) yields

$$\begin{aligned}-\lambda\upsilon'(\xi) + \bar{a}_1\upsilon(\xi)\upsilon'(\xi) + \bar{a}_3\upsilon^2(\xi)\upsilon'(\xi)\\ + a_2\upsilon'''(\xi) = 0\end{aligned} \qquad (32)$$

where a prime denotes differentiation with respect to ξ and λ represents the wave speed. Integration of (31) once leads to

$$-\lambda\upsilon + \frac{\bar{a}_1}{2}\upsilon^2 + \frac{\bar{a}_3}{3}\upsilon^3 + a_2\upsilon'' = k_1. \qquad (33)$$

Multiplying both sides of (33) by υ' and integrating once more we obtain

$$-\frac{\lambda}{2}\upsilon^2 + \frac{\bar{a}_1}{6}\upsilon^3 + \frac{\bar{a}_3}{12}\upsilon^4 + \frac{a_2}{2}(\upsilon')^2 = k_1\upsilon + k_2. \qquad (34)$$

Finally, from (34) we have

$$\frac{\sqrt{a_2}\,d\upsilon}{\sqrt{2k_1\upsilon + 2k_2 + \lambda\upsilon^2 - \frac{\bar{a}_1}{3}\upsilon^3 - \frac{\bar{a}_3}{6}\upsilon^4}} = d\xi \qquad (35)$$

where k_1 and k_2 are arbitrary constants. Integrating both sides of this last equation we obtain the general traveling wave solution to (25) in implicit form

$$\int\frac{\sqrt{a_2}\,d\upsilon}{\sqrt{2k_1\upsilon + 2k_2 + \lambda\upsilon^2 - \frac{\bar{a}_1}{3}\upsilon^3 - \frac{\bar{a}_3}{6}\upsilon^4}} = \xi + k_3 = s - \lambda t + k_3 \qquad (36)$$

5.1 First particular case

Setting $\bar{a}_1 = a_1 - \beta a_3 = 0$ in (25)we obtain the following KdV of higher order

$$\kappa_t + a_2\kappa_{sss} + \bar{a}_3\kappa^2\kappa_s = 0, \qquad (37)$$

If we assume that $\upsilon \to 0, \quad \upsilon' \to 0, \quad \upsilon'' \to 0,$ when $\xi \to \infty,$ in the analysis presented in the previous section, then (36) reduces to

$$\sqrt{\frac{a_2}{\lambda}}\int\frac{d\upsilon}{\upsilon\sqrt{1 - \frac{\bar{a}_3}{6\lambda}\upsilon^2}} = \xi + k_3 \qquad (38)$$

for an arbitrary constant k_3 .

With the substitution $\upsilon = \sqrt{\frac{6\lambda}{\bar{a}_3}}\,sech\,\omega$ finally we obtain the following soliton solution to (38)

$$\kappa(t,s) = \sqrt{\frac{6\lambda}{\bar{a}_3}}\,sech\left[\sqrt{\frac{\lambda}{a_2}}(s - \lambda t + k_4)\right], \qquad (39)$$

where k_4 is a constant of integration.

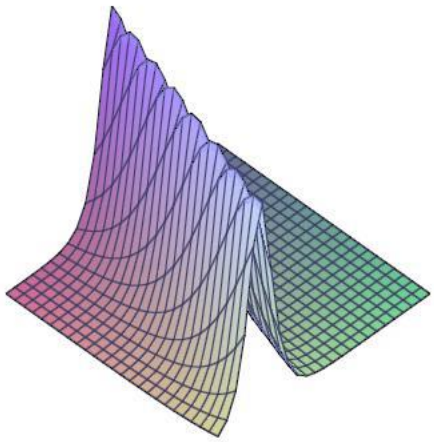


Figure 1: Surface corresponding to (39)

$a_2 = \lambda = 2, \sigma^2 = 6, s \in [-10,10], t \in [0,0.3]$

5.2 Second particular case

If we set $\beta = 0$ in the above case then the equation (37) reduces to

$$\kappa_t + a_2 \kappa_{sss} + \kappa^2 \kappa_s = 0, \qquad (40)$$

and (39) reduces to

$$\kappa(t,s) = \sqrt{6\lambda}\, sech\left[\sqrt{\frac{\lambda}{a_2}}(s - \lambda t + k_4)\right], \qquad (41)$$

If we set $a_2 = \lambda = 6, \quad k_4 = 0$ then

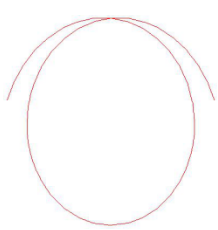
$$\kappa(t,s) = 6sech[(s-6t)], \qquad (42)$$

In this case we can construct the curve from its curvature

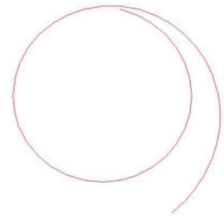
$$r(s) = \left(\int_0^s \cos(\phi(s))ds, \int_0^s \sin(\phi(s))ds\right) \qquad (43)$$

where

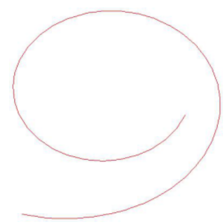
$$\phi = \int 6sech(s-6t)ds = 6arctan(sinh(s-6t)$$



[Curve corespponding to (43) at $t = 0$]



[Curve coresppanding to (43) at $t = 0.5$]



[Curve coresppanding to (43) at $t = 1$]

Figure 2: Curves of second case with $s \in [-0.8, 0.8$

5.3 Third particular case

If we set $\rho = 0, a_2 = 0,$ then the equation under study (25) reduces to

$$\kappa_t + \bar{a}_1 \kappa \kappa_s + a_2 \kappa_{sss} = 0, \qquad (44)$$

which is the generalized KDV equation. Proceeding as in the first case,then (36) reduces to

$$\sqrt{\frac{a_2}{\lambda}} \int \frac{d\upsilon}{\upsilon\sqrt{1 - \frac{\bar{a}_1}{3\lambda}\upsilon}} = \xi + k_3 \qquad (45)$$

Using substitution $\upsilon = \frac{3\lambda}{\bar{a}_1} sech\omega$ finally we obtain the following soliton solution to (45)

$$\kappa(t,s) = \frac{3\lambda}{\bar{a}_1} sech^2\left[\sqrt{\frac{\lambda}{4a_2}}(s - \lambda t + k_4)\right], \qquad (46)$$

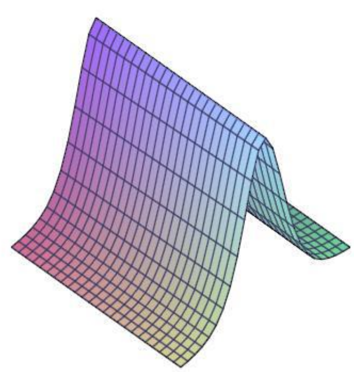


Figure 3: Surface corresponding to (46)

$\lambda = 4a_2 = 1, \bar{a}_1 = 3, s \in [-\pi, \pi], t \in [0, 0.05]$

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