Proper Teleparallel Homothetic vectors From Non Diagonal Tetrad Of Some Well Known Static Spacetime

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Abstract: In this paper proper teleparallel homothetic vectors for static cylindrically symmetric spacetime in context of teleparallel theory of gravitation has been investigated. For the purpose non diagonal tetrad of static cylindrically symmetric spacetime has been chosen and direct integration technique is applied to solve the teleparallel homothetic equations. It comes out that the above spacetime with non diagonal tetrad admit proper teleparallel homothetic vector for the special choice of metric functions.

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1. Introduction:

The role of symmetries in general relativity is important in understating the physical and geometrical aspects of spacetime. The interaction of matter is described in general relativity through famous Einstein's field equations. These gravitational field equations are highly non linear and therefore need some symmetry restriction on spacetime metric to be solved. Symmetries of spacetime are such a powerful tool that laws of conservation of matter in a spacetime can be studied through them [Petrov. A. Z., 1969]. In general relativity many exact solutions of field equations are obtained, the details of which can be found in [Stephani. H et al 2003]. Some of these solutions are then classified according to its Killing, homothetic and conformal vector fields [Bokhari A. H et al 1987, Feroz. T et al 2001, Shabbir G et al 2006 and 2007, Maartens R et al 1995]. The idea of Killing symmetry in teleparallel theory, is introduced by M. Sharif and M. J. Amir [Sharif M et al 2008] where they obtained teleparallel Killing vectors for Einstein Universe using a non diagonal tetrad. In [Sharif M et al 2009] the authors obtained teleparallel Killing vectors for spherically symmetric and Friedman metrics using non diagonal tetrads. B. Majeed [Majeed B 2008] has investigated teleparallel Killing vectors for static cylindrically symmetric spacetime by using a non diagonal tetrad. Later on, G. Shabbir and S. Khan extended this work for some more spacetimes using diagonal tetrads [Shabbier G and Khan S 2010-11]. The same authors also obtained proper teleparallel homothetic vector fields for some well known spacetimes using diagonal tetrad [Shabbir G and Khan S 2010-12]. Recently, S. Khan et al [Khan S et al 2013] showed that Einstein universe do not admit proper teleparallel homothetic vector fields for non diagonal tetrad. G. Shabbir et al have already obtained teleparallel proper homothetic vector fields for static cylindrically symmetric spacetime using a diagonal tetrad [Shabbir G and Khan S 2010]. Since the advantage of the choice of non diagonal tetrad over a diagonal one is discussed in detail in [Nashed G. G. L 2010], we are therefore, interested to find proper teleparallel homothetic vector fields of static cylindrically symmetric spacetime by using a non diagonal tetrad.

In [Sharif M and Amir M. J 2008] the authors defined Killing equation in teleparallel theory for the vector field X as

$$\underset{X}{L}^{T} g_{\alpha\beta} = g_{\alpha\beta,\rho}X^{\rho} + g_{\rho\beta}X^{\rho}{}_{,\alpha} + g_{\alpha\rho}X^{\rho}{}_{,\beta} + X^{\rho}(g_{\theta\beta}F^{\theta}{}_{\alpha\rho} + g_{\alpha\theta}F^{\theta}{}_{\beta\rho}) = 0, \quad (1)$$

where $\underset{X}{L}^{T}$ represents Lie derivative in teleparallel theory, a comma "," denotes partial derivative and $F^{\theta}{}_{\alpha\beta}$ are the components of torsion tensor. Torsion tensor is anti-symmetric in the lower indices. For finding proper teleparallel homothetic vectors we shall use the above definition in the extended form as:

$$\underset{X}{L}^{T} g_{\mu\nu} = 2\lambda g_{\mu\nu}, \quad \lambda \in R. \quad (2)$$

2. Main Results:

Static cylindrically symmetric spacetime in its usual coordinates (t, r, θ, z) is given as [Stephani. H et al 2003]

$$ds^2 = -e^{P(r)}dt^2 + dr^2 + e^{Q(r)}d\theta^2 + e^{S(r)}dz^2, \quad (3)$$

where P, Q and S are functions of r only. We shall follow a well-known procedure given in [Majeed B 2008 and Pereira J. G et al 2001] to obtain the tetrad $H^a{}_\mu$, its inverse $H_a{}^\mu$, non zero Weitzenböck connections $W^a{}_{bc}$ and non zero torsion components $F^\theta{}_{\alpha\beta}$ for static cylindrically symmetric spacetime as

$$H^a{}_\mu = \begin{bmatrix} \sqrt{e^{P(r)}} & 0 & 0 & 0 \\ 0 & \cos\theta & -\sqrt{e^{Q(r)}}\sin\theta & 0 \\ 0 & \sin\theta & \sqrt{e^{Q(r)}}\cos\theta & 0 \\ 0 & 0 & 0 & \sqrt{e^{S(r)}} \end{bmatrix} \quad (4)$$

$$H_a{}^\mu = \begin{bmatrix} \frac{1}{\sqrt{e^{P(r)}}} & 0 & 0 & 0 \\ 0 & \cos\theta & \frac{1}{-\sqrt{e^{Q(r)}}}\sin\theta & 0 \\ 0 & \sin\theta & \frac{1}{\sqrt{e^{Q(r)}}}\cos\theta & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{e^{S(r)}}} \end{bmatrix} \quad (5)$$

$$W^0{}_{01} = \frac{P'}{2}, \quad W^2{}_{21} = \frac{Q'}{2}, \quad W^3{}_{31} = \frac{S'}{2},$$

$$W^1{}_{22} = -e^{\frac{Q}{2}}, \quad W^2{}_{12} = e^{-\frac{Q}{2}}. \quad (6)$$

Here derivative with respect to r is represented by a dash. The non zero torsion components are obtained as

$$F^0{}_{10} = \frac{P'(r)}{2}, \; F^3{}_{13} = \frac{S'(r)}{2},$$

$$F^2{}_{12} = \frac{Q'(r)}{2} - e^{-\frac{Q(r)}{2}}, \; F^0{}_{01} = -\frac{P'(r)}{2},$$

$$F^3{}_{31} = -\frac{S'(r)}{2}, \; F^2{}_{21} = -\frac{Q'(r)}{2} + e^{-\frac{Q(r)}{2}}. \quad (7)$$

X is said to be a teleparallel homothetic vector field, if it satisfies equation (2). Expanding equation (2) with the help of equations (3) and (7) we get

$$X^0_{,0} = \lambda \quad (8)$$

$$X^1_{,0} - e^{P(r)}X^0_{,1} - \frac{P'(r)}{2}e^{P(r)}X^0 = 0 \quad (9)$$

$$e^{P(r)}X^0_{,2} - e^{Q(r)}X^2_{,0} = 0 \quad (10)$$

$$e^{P(r)}X^0_{,3} - e^{S(r)}X^3_{,0} = 0 \quad (11)$$

$$e^{Q(r)}X^2_{,1} + X^1_{,2} + \frac{Q'(r)}{2}e^{Q(r)}X^2 - e^{\frac{Q(r)}{2}}X^2 = 0 \quad (12)$$

$$X^1_{,1} = \lambda \quad (13)$$

$$e^{S(r)}X^3_{,1} + X^1_{,3} + \frac{S'(r)}{2}e^{S(r)}X^3 = 0 \quad (14)$$

$$X^2_{,2} + e^{-\frac{Q(r)}{2}}X^1 = \lambda \quad (15)$$

$$e^{Q(r)}X^2_{,3} + e^{S(r)}X^3_{,2} = 0 \quad (16)$$

$$X^3_{,3} = \lambda \quad (17)$$

Solving equations (8)-(11) we get a system of equations as follows

$$X^0 = \lambda t + E^1(r,\theta,z),$$

$$X^1 = t e^{P(r)}E^1_r(r,\theta,z) + \frac{\lambda t^2}{4}P'(r)e^{P(r)}$$

$$+ \frac{t}{2}P'(r)e^{P(r)}E^1(r,\theta,z) + E^2(r,\theta,z),$$

$$X^2 = t e^{P(r)-Q(r)}E^1_\theta(r,\theta,z) + E^3(r,\theta,z),$$

$$X^3 = t e^{P(r)-Q(r)}E^1_z(r,\theta,z) + E^4(r,\theta,z), \quad (18)$$

where $E^1(r,\theta,z)$, $E^2(r,\theta,z)$, $E^3(r,\theta,z)$, $E^4(r,\theta,z)$ are functions of integration. In order to get a complete solution of equations (8)-(17) we will find these unknown functions with the help of equations (12)-(18). In order to write briefly we shall avoid the lengthy details of the solution. Solving equations (13) and (15) with the help of equation (18) we get $E^1(r,\theta,z) = rF^1(\theta,z) + F^2(\theta,z)$ and $Q''(r)F^1(\theta,z) = 0$, where $F^1(\theta,z)$ and $F^2(\theta,z)$ are functions of integration. In order to get a complete classification we need to solve the last equation completely, which has the possibilities $\frac{d^2}{dr^2}(e^{Q(r)/2}) = 0, \; F^1(\theta,z) \neq 0,$ $\frac{d^2}{dr^2}(e^{Q(r)/2}) = 0, \; F^1(\theta,z) = 0,$ and $\frac{d^2}{dr^2}(e^{Q(r)/2}) \neq 0, \; F^1(\theta,z) = 0.$ We shall discuss each case in turn.

Case (I)

In this case we have $\frac{d^2}{dr^2}(e^{Q(r)/2}) = 0, \quad F^1(\theta, z) \neq 0.$ Substituting these conditions in equations (12), (14), (16) and (17) respectively and solving, we reach to a contradiction that $F^1(\theta, z) = 0$. Hence this case is not possible.

Case (II)

In this case we have $\frac{d^2}{dr^2}(e^{Q(r)/2}) = 0, \quad F^1(\theta, z) = 0.$

Now $\frac{d^2}{dr^2}(e^{Q(r)/2}) = 0 \Rightarrow$ $e^{Q(r)} = (c_1 r + c_2)^2, \quad c_1, c_2 \in R \ (c_1 \neq 0).$ Substituting this information in the remaining equations and solving carefully, the metric for static cylindrically symmetric spacetime after a suitable rescaling of t and z, take the form

$$ds^2 = -dt^2 + dr^2 + (c_1 r + c_2)^2 d\theta^2 + dz^2. \tag{19}$$

Teleparallel homothetic vector fields for the spacetime (19) are obtained as

$$X^0 = \lambda t + c_5 z + c_6$$
$$X^1 = \lambda(r + c_2) + z(c_7\cos\theta + c_8\sin\theta) + (c_{13}\sin\theta - c_{14}\cos\theta)$$
$$X^2 = -\frac{z}{r + c_2}(c_7\sin\theta - c_8\cos\theta) + \frac{1}{r + c_2}(c_{13}\cos\theta + c_{14}\sin\theta) + c_{17}$$
$$X^3 = \lambda z + c_5 t - r(c_7\cos\theta + c_8\sin\theta) + c_2(-c_7\cos\theta - c_8\sin\theta) + c_{12} \tag{20}$$

For obtaining proper teleparallel homothetic vector field X, subtract the teleparallel Killing vector fields in (20), we get $X = (t, \ r + c_2, \ 0, \ z).$

Case (III)

In this case we have $\frac{d^2}{dr^2}(e^{Q(r)/2}) \neq 0, \quad F^1(\theta, z) = 0.$ Now substituting this information in the remaining equations except equation (12), the system of equations (18) takes the form

$$X^0 = \lambda t + c_5 z + c_6, \qquad X^1 = \lambda r + G^5_\theta(\theta),$$
$$X^2 = \lambda\theta - \lambda r\theta e^{\frac{Q(r)}{2}} - e^{\frac{Q(r)}{2}} G^5(\theta) + G^7(r), \quad X^3 = c_5 t + \lambda z + c_{10}, \tag{21}$$

where $G^5(\theta)$ and $G^7(r)$ are functions of integration. Now if we substitute (21) in equation (12) we get $G^5(\theta) = -c_{11}\cos\theta - c_{12}\sin\theta + c_{13}\theta + c_{14}$ and $-2\lambda e^{\frac{Q(r)}{2}} + \lambda \frac{Q'(r)}{2} e^{Q(r)} + \lambda r + c_{13} = 0 \Rightarrow$ $Q(r) = \ln r^2 \Rightarrow \frac{d^2}{dr^2}(e^{Q(r)/2}) = 0.$

Hence this case is also not possible.

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