A Four-Stage Sixth-Order RKD Method for Directly Solving Special Third-Order Ordinary Differential Equations

Mohammed Mechee [1,2], Fudziah Ismail [3], Zailan Siri [2], Norazak. Senu[3]

[1.] Department of Mathematics, Faculty of Mathematics and Computer Science, Kufa university, Najaf, Iraq.
[2.] Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia.
[3.] Department of Mathematics and Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia.
mohsabd@yahoo.com

Abstract: A new four-stage sixth order Runge-Kutta method for direct integration of special third order ordinary differential equations (ODEs) is constructed. The method is proven to be zero-stable. Stability polynomial of the method for linear special third order ODE is given. A set of test problems consisting of ordinary differential equations is tested upon. The problems are solved using the new method and numerical comparisons are made when the same problems are reduced to a first order system of ODEs and solved using the existing Runge-Kutta methods of different orders. Numerical results have clearly shown the advantage and the efficiency of the new method in terms of accuracy and computational time.

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1. Introduction

Generally speaking, a special third order differential equations (ODEs) of the form :

$$y'''(t) = f(t, y(t)), \qquad t \geq t_0, \tag{1}$$

with initial conditions

$$y(t_0) = \alpha,\ y'(t_0) = \beta,\ y''(t_0) = \gamma,$$

where $f: R \times R \rightarrow R$ which are not explicitly dependent on the first derivative $y'(x)$ and the second derivative $y''(x)$ of the solution are frequently found in many physical problems such as electromagnetic waves, thin film flow and gravity driven flow [1]. Most researchers, scientists and engineers used to solve (1) by converting the third order differential equations to a system of first order equations three times the dimension. However, it is more efficient if the problem can be directly solved using numerical methods. Such a type of work can be seen in Awoyemi and Idowu [2], Waeleh et al. [3], Zainuddin [4] and Jator [5]. All methods previously discussed are multistep methods; hence they need the starting values when used to solve ODEs (1). In this paper, we are concerned with the one-step method particularly the Runge-Kutta method of order six for directly solving third order ordinary differential equations.
Accordingly, we have developed a direct Runge-Kutta(RKD) method which can be directly used to solve (1). The advantage of the new method over multistep methods is that it is self starting. Stability polynomial of the method when applied to linear third order ODE is also presented. Numerical results on a set of problems consisting of ordinary differential equations is given and compared with the numerical results when the problems are reduced to a system of first order ODEs and solve using Runge-Kutta methods.

2. Derivation of RKD Method

The general form of RKD method with s-stage for solving initial value problem (1) can be written as

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + h^3\sum_{i=1}^{s} b_i\, k_i, \tag{2}$$

$$y'_{n+1} = y'_n + hy''_n + h^2\sum_{i=1}^{s} b'_i\, k_i, \tag{3}$$

$$y''_{n+1} = y''_n + h\sum_{i=1}^{s} b''_i\, k_i, \tag{4}$$

where

$$k_1 = f(t_n, y_n) \tag{5}$$

$$k_i = f\left(t_n + c_i h, y_n + hc_i y'_n + \frac{(c_i h)^2}{2}y''_n + h^3\sum_{j=1}^{i-1} a_{ij}k_j\right) \tag{6}$$

for i=2,3,...,s. The parameters of RKD method are $c_i, a_{ij}, b_i, b'_i, b''_i$ for $i = 1,2,\ldots,s$ and j=1,2,...,s are assumed to be real. If $a_{ij} = 0$ for i< j, it is an explicit method and otherwise implicit method. The RKD method can be expressed in Butcher notation using the table of coefficients as follows.

$$\begin{array}{c|l} C & A \\ \hline & b^T \\ & b'^T \\ & b''^T. \end{array}$$

To determine the coefficients of the RKD method, the expressions given in (2)-(6) are expanded using Taylor's series expansion. After some algebraic manipulations this expansion is equated to the true solution which are given by Taylor's series expansion. General order conditions for the RKD method can be found from the direct expansion of the local truncation error. The order conditions can be found in Mechee et al. [1] which introduced the three-stage fifth-order Runge-Kutta Method for directly solving special third-order ODEs with application to thin film flow problem.

3.The Order Conditions of the Method

From Mechee et al. [1] we derived the order conditions of RKD method up to order six. In this paper using the same technique, we derive the seventh order conditions. The order conditions for four-stage sixth-order RKD method can be written as follows:

Order conditions for y

Order 3

$$\sum b_i = \frac{1}{6} \quad (7)$$

Order 4

$$\sum b_i c_i = \frac{1}{24} \quad (8)$$

Order 5

$$\sum b_i c_i^2 = \frac{1}{60} \quad (9)$$

Order 6

$$\sum b_i c_i^3 = \frac{1}{120}, \sum b_i a_{ij} = \frac{1}{720} \quad (10)$$

Order 7

$$\sum b_i c_i^4 = \frac{1}{210}, \sum b_i c_i a_{ij} = \frac{1}{1260},$$

$$\sum b_i a_{ij} c_i = \frac{1}{5040} \quad (11)$$

Order conditions for y'

Order 2

$$\sum b'_i = \frac{1}{2} \quad (12)$$

Order 3

$$\sum b'_i c_i = \frac{1}{6} \quad (13)$$

Order 4

$$\sum b'_i c_i^2 = \frac{1}{12} \quad (14)$$

Order 5

$$\sum b'_i c_i^3 = \frac{1}{20}, \sum b'_i a_{ij} = \frac{1}{120} \quad (15)$$

Order 6

$$\sum b'_i c_i^4 = \frac{1}{30}, \sum b'_i a_{ij} c_j = \frac{1}{720},$$

$$\sum b'_i c_i a_{ij} = \frac{1}{180} \quad (16)$$

Order 7

$$\sum b'_i c_i^5 = \frac{1}{42}, \sum b'_i a_{ij} c_i^2 = \frac{1}{2520},$$

$$\sum b'_i c_i^2 a_{ij} = \frac{1}{252} \quad (17)$$

Order conditions for y''

Order 1

$$\sum b''_i = 1 \quad (18)$$

Order 2

$$\sum b''_i c_i = \frac{1}{2} \quad (19)$$

Order 3

$$\sum b''_i c_i^2 = \frac{1}{3} \quad (20)$$

Order 4

$$\sum b''_i c_i^3 = \frac{1}{4}, \sum b''_i a_{ij} = \frac{1}{24} \quad (21)$$

Order 5

$$\sum b_i'' c_i^4 = \frac{1}{5}, \sum b_i'' a_{ij} c_j = \frac{1}{120},$$

$$\sum b_i'' c_j\, a_{ij} = \frac{1}{30} \qquad (22)$$

Order 6

$$\sum b_i'' c_i^5 = \frac{1}{6}, \sum b_i'' c_i^2\, a_{ij} = \frac{1}{36},$$

$$\sum b_i''\, a_{ij} c_j^2 = \frac{1}{360}, \sum b_i'' c_i\, a_{ij} c_j = \frac{1}{144} \qquad (23)$$

Order 7

$$\sum b_i'' c_i^6 = \frac{1}{7}, \sum b_i'' c_i\, a_{ij}\, c_j^2 = \frac{1}{420},$$

$$\sum b_i''\, a_{ij}\, a_{jk} = \frac{1}{252} \qquad (24)$$

All indices are from 1 to s.

4. Zero Stability of the Methods

Next, we will discuss the zero-stability of the method it is one of the criteria for the method to be convergent. Zero-stability is an important tool for proving the stability and convergence of linear multistep methods. The interested reader is referred to the textbooks by Lambert [6] and Butcher [7] in which zero-stability is discussed. Zero-stability has been discussed in Hairer et al. [8] where it is used to determine an upper bound on the order of convergence of linear multistep methods. In studying the zero stability of RKD method, we can write method (2)-(4) as follows

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ hy'_{n+1} \\ h^2 y''_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_n \\ hy'_n \\ h^2 y''_n \end{pmatrix}$$

$$\rho(\varepsilon) = |I\varepsilon - A|$$

$$\rho(\varepsilon) = \begin{vmatrix} \varepsilon - 1 & -1 & -\frac{1}{2} \\ 0 & \varepsilon - 1 & -1 \\ 0 & 0 & \varepsilon - 1 \end{vmatrix}$$

Thus the characteristic polynomial is

$$\rho(\varepsilon) = (\varepsilon - 1)^3$$

Hence, the method is zero-stable since the roots are ε =1,1,1, are less or equal to one.

5. Derivation of RKD Methods

The RKD method of s-stage and p^{th} order can be derived by solving the order conditions of the method. The system of nonlinear equations(order conditions) of the method depend on p. The existence of the solutions of this system depends on the number of coefficients of the method. Which depends on the stage of the method in addition to the number of independent order conditions of the method.

5.1 Derivation of four-Stage Sixth-Order RKD Method

To derive the four-stage and sixth-order RKD method, we use the algebraic conditions up to order six in the equations of order conditions in y, y' and y'' (7)-(10), (12)-(16) and (18)-(23) respectively. The resulting system of equations consists of 26 nonlinear equations with 21 unknowns variables to be solved. To get sixth-order RKD method, the simplifying assumption $b_i' = b_i''(1 - c_i)$, for i=1,…,4 is used in order to reduce the number of equations to be 22 nonlinear equations. Consequently, there is a solution with one free parameter a_{41}, however the arbitrary parameter can be chosen using minimization of the truncation error. Accordingly Dormad and Prince [9] the free parameters can be chosen by minimizing the global error of the seventh order conditions. The technique is as follows;

First: we find the error coefficients of y, y', and y'' respectively as the following:

$$\left\|\tau^{(7)}\right\|_2 = \sqrt{\sum_{j=1}^{n_0+1} \left(\tau_j^{(7)}\right)^2}, \left\|\tau'^{(7)}\right\|_2 = \sqrt{\sum_{j=1}^{n_0'+1} \left(\tau_j'^{(7)}\right)^2},$$

$$\left\|\tau''^{(7)}\right\|_2 = \sqrt{\sum_{j=1}^{n_0''+1} \left(\tau_j''^{(7)}\right)^2}. \qquad (25)$$

Second: we find the global error norm as the following:

$$\left\|\tau^{(7)}{}_g\right\|_2 = \sqrt{\sum_{j=1}^{n_0+1}\left(\tau_j^{(7)}\right)^2 + \sum_{j=1}^{n_0'+1}\left(\tau_j'^{(7)}\right)^2 + \sum_{j=1}^{n_0''+1}\left(\tau_j''^{(7)}\right)^2}\ ,(26)$$

Finally, we minimize the four truncation errors in (25)-(26) with respect to the free parameter $a_{41} = 0$ leads to the error norms for y_n, y_n' and y_n'' are given by

$$\left\|\tau^{(7)}\right\|_2 = 1.2097961976 \times 10^{-4},$$
$$\left\|\tau'^{(7)}\right\|_2 = 1.390534737 \times 10^{-4},$$
$$\left\|\tau''^{(7)}\right\|_2 = 1.150474161 \times 10^{-4},$$

respectively and the global error is

$$\left\|\tau^{(7)}{}_g\right\|_2 = 2.172736637 \times 10^{-4}$$

where $\tau^{(7)}$, $\tau'^{(7)}$ and $\tau''^{(7)}$ are error terms of the seventh-order conditions for y_n, y_n' and y_n'' respectively. The four-stage six-order RKD method is denoted by RKD6 which can be expressed in the following Butcher tableau in table1.

0	0			
0.10125422 3	0.000173018			
0.193823319	-0.014304529	0.024538837		
0.88591296 0	0	0.080405561	0.035478021	
	-0.00715226 1	0.114340881	0.057618019	0.001829997
	- 0.014304528	0.25444544 5	0.2277783711	0.032080713
	-0.0113015280	0.283111997	0.1199977573	0.281195071

Table 1: Butcher tableau for RKD6 method.

6. Absolute Stability of the method when applied to third order ODE

In studying the linear stability of the method , we apply the test equation $y''' = -\alpha^3 y$. We consider formula (2)-(4) which can be written as follows:

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + h^3\sum_{i=1}^{s} b_i(-\alpha^3 Y_i) \qquad (27)$$

$$y_{n+1}' = y_n' + hy''_n + h^2\textstyle\sum_{i=1}^{s} b_i'(-\alpha^3 Y_i) \qquad (28)$$

$$y_{n+1}'' = y''_n + h\textstyle\sum_{i=1}^{s} b_i''(-\alpha^3 Y_i), \qquad (29)$$

where

$$Y_i = y_n + c_i hy_n' + \frac{h^2}{2}c_i^2 y_n'' + h^3\sum_{i=1}^{s} a_{ij}\ (-\alpha^3 Y_i)) \qquad (30)$$

for $i = 1,2,\ldots,s$ and by multiplying equations (28) and (29) by h and h^2 respectively we obtain

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2}c_i^2 y_n'' + h^3\sum_{i=1}^{s} b_i(-\alpha^3 Y_i) \qquad (31)$$

$$hy_{n+1}' = hy_n' + h^2 y''_n + h^3\textstyle\sum_{i=1}^{s} b_i'(-\alpha^3 Y_i) \qquad (32)$$

$$h^2 y_{n+1}'' = h^2 y''_n + h^3\textstyle\sum_{i=1}^{s} b_i''(-\alpha^3 Y_i) \qquad (33)$$

where

$$Y_j = y_n + c_j hy_n' + \frac{h^2}{2}c_j^2 y_n'' + h^3\alpha\sum_{i=1}^{s} a_{ji}\ Y_i \qquad (34)$$

$j = 1,2,\ldots,s$.

We can write equations (31)-(33) in the following matrix form:

$$z_{n+1} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} z_n + \alpha h^3 \begin{pmatrix} b_1 & b_2 & b_3 \\ b_1' & b_2' & b_3' \\ b_1'' & b_2'' & b_3'' \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

where

$$z_n = \begin{pmatrix} y_n \\ hy_n' \\ h^2 y_n'' \end{pmatrix}$$

and equation (34) can be written in the following form :

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ . \\ Y_s \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & c_2 & \frac{c_2^3}{2} \\ \vdots & \vdots & \vdots \\ . & . & . \\ 1 & c_s & \frac{c_s^3}{2} \end{pmatrix} z_n + H \begin{pmatrix} 0 & \ldots & 0 \\ a_{21} & \ldots & 0 \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{s1} & \ldots & a_{ss} \end{pmatrix}\begin{pmatrix} Y_1 \\ \vdots \\ Y_s \end{pmatrix},$$

where

$$H = (\alpha h)^3.$$

Hence

$$z_{n+1} = D(H) z_n$$

where

$$D(H) = \begin{pmatrix} 1+Hb^TN^{-1}e & 1+Hb^TN^{-1}e & \frac{1}{2}+Hb^TN^{-1}e \\ Hb'^TN^{-1}e & 1+Hb'^TN^{-1}e & 1+Hb'^TN^{-1}e \\ Hb''^TN^{-1}e & Hb''^TN^{-1}e & 1+Hb''^TN^{-1}e \end{pmatrix} \quad (35)$$

$$e = (1\,1 \dots 1)^T, c = (0\, c_2 \dots c_s)^T,$$

$$d = \left(0\, \tfrac{c_2^2}{2}\, \tfrac{c_3^2}{2} \dots \tfrac{c_s^2}{2}\right)^T$$

and $N^{-1} = I - HA$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ \vdots & \vdots & \ddots \\ a_{s1} & a_{s2} & a_{ss} \end{pmatrix},$$

$$B = \begin{pmatrix} b_1 & \dots & b_s \\ b_1' & \dots & b_s' \\ b_1'' & \dots & b_s'' \end{pmatrix},$$

and

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & c_2 & \frac{c_2^2}{2} \\ \vdots & \vdots & \vdots \\ 1 & c_s & \frac{c_s^2}{2} \end{pmatrix}.$$

The stability function associated with this

method is given by

$$\varphi(\vartheta, H) = |\vartheta I - D(H)|$$

where D(H) defined in (35) is a stability matrix and its characteristic equation can be written as

$$P(\vartheta, H) = p_0(H)\vartheta^3 + p_1(H)\vartheta^2 + p_2(H)\vartheta + p_3(H).$$

7. Numerical Results

In this section a set of third order ordinary differential equations are solved using RKD6 method of order six and numerical results are compared with the existing RK methods of the orders four, five and six.

The following notations are used in Figures (1)-(4):

- **h** : Stepsize used.
- **RKD6**: The Direct Rung-Kutta method of order six derived in section 6.
- **RK4** : Existing Runge-Kutta method order four as given in [6]
- **RK5** : Existing Runge-Kutta method five as given in Dormand [9].
- **RK6** : Existing Runge-Kutta method six as given in [7].
- **Total time** : The total time in second to solve the problems.
- **MAX ERROR**: Max | y(x)-y_n| Absolute value of the true solution minus the computed solution.

Problems of ODEs

Problem 1(Homogenous linear)

$y'''(t) = -y(t) \qquad 0 < t < b$

Initial conditions

$y(0) = 1, y'(0) = -1, y''(0) = 1,$

Exact solution $y(t) = e^{-t}, b = 1.$

Problem 2(Non homogenous linear)

$y'''(t) = -e^{-t} \qquad 0 < t < b$

Initial conditions

$y(0) = 1, y'(0) = -1, y''(0) = 1,$

Exact solution $y(t) = e^{-t}, b = 1.$

Problem 3(Homogenous non linear)

$y'''(t) = \frac{3}{8y^5(t)}, \qquad 0 < t < b$

Initial conditions

$y(0) = 1, y'(0) = \frac{1}{2}, y''(0) = -\frac{1}{4},$

Exact solution $y(t) = \sqrt{1+t}, b = \pi.$

Problem 4(Non homogenous linear)

$y'''(t) = -6y^4, \qquad 0 < t < b$

Initial conditions

$y(0) = 1, y'(0) = -1, y''(0) = 2,$

Exact solution $y(t) = \frac{1}{1+t}, b = \pi.$

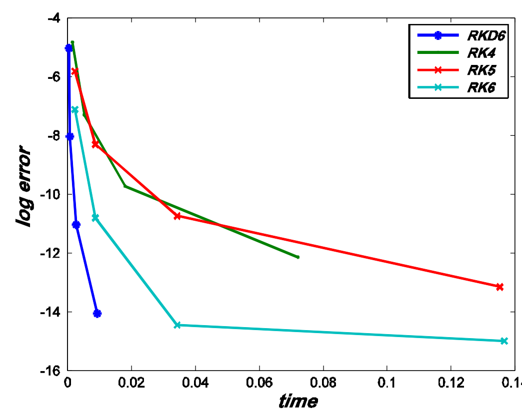

Figure 1: log Max Errors versus computational time for Problem1.

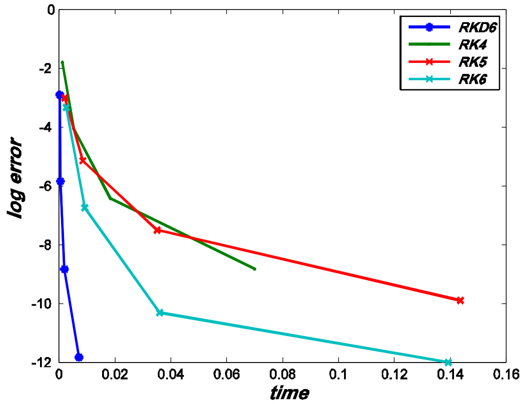

Figure 2: log Max Errors versus computational time for Problem 2.

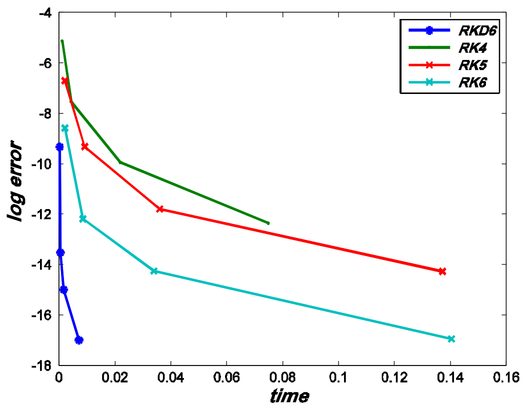

Figure 3: log Max Errors versus computational time for Problem3.

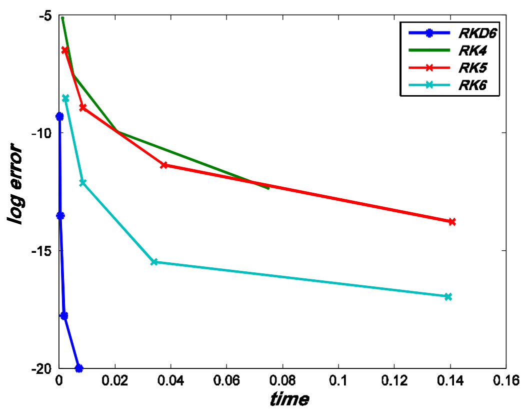

Figure 4: log Max Errors versus computational time for Problem4.

8. Discussion and Conclusion

In this paper, we derived RKD method of four stage, sixth order. The zero-stability of the method is proven. Stability polynomial of the method when applied to linear third order ODE is also given. We used the method for solving special third order ODEs. Numerical results show that the RKD method is more accurate and requires less computational time compared to the existing RK methods when used to solve special third order ordinary differential equations.

Corresponding Author:
Mohammed Mechee
Department of Mathematics,
Faculty of Mathematics and Computer Science, Kufa university, Najaf, Iraq
E-mail: mohsabd@yahoo.com

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