Dynamics Of Population Growth Model With Fractional Temporal Evolution

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Abstract: This paper studies the population model that is governed by a nonlinear fractional evolution equation. The fractional derivative is considered in modified Riemann-Liouville derivative sense. The first integral method is applied to carry out the integration. The model is studied in (2+1) - dimensions. Exact analytical solution is obtained.

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1. Introduction

The dynamics of population growth is one of the most important areas of research in the field of mathematical biology. In fact the research in this area is being conducted for the past few decades. There are several papers where analytical and numerical results are reported. Additionally, there are other forms of mathematical analysis that are seen in several other papers in the context of population growth models [1-20].

It is about time to focus the issue of population growth in a slightly generalized tone. So, the burning question is "What happens when the evolution term is made fractional?" This question will be answered in this paper after considering the model equation in (2+1)-dimensions with temporal evolution term being fractional.

Jumarie [8] presented a modification of the Riemann-Liouville definition which appears to provide a framework for a fractional calculus. This modification was successfully applied in the probability calculus, fractional Laplace problem, exact solutions of the nonlinear fractional differential equation and many other types of linear and nonlinear fractional differential equations. Lu [9] applied the modified Riemann-Liouville derivative with properties and first integral method to obtain exact solutions of some fractional nonlinear evolution equations. In this present paper, we apply the first integral method to study the nonlinear time fractional biological population model by using the first integral method.

The paper is arranged as follows. In Section 2, we describe briefly the modified Riemann-Liouville derivative with properties and first integral method. In Section 3, we apply this method to the nonlinear time fractional biological population model.

2. Jumarie's Modified Riemann-Liouville Derivative And First Integral Method

Jumarie's modified Riemann-Liouville derivative of order is defined by the following expression [8]

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} [f(\xi) - f(0)] d\xi \tag{1}$$

If $\alpha < 0$,

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} [f(\xi) - f(0)] d\xi \tag{2}$$

If $0 < \alpha < 1$,

$$D_x^{\alpha} f(x) = (f^{(n)}(x))^{(\alpha - n)} \tag{3}$$

If $n \le \alpha \le n+1,\ n \ge 1$, where $f : R \to R$ is a continuous function. Some properties of the fractional modified Riemann-Liouville derivative were summarized and three useful formulas of them are

$$D_x^{\alpha} x^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, \gamma > 0, \tag{4}$$

$$D_x^{\alpha} (u(x)v(x)) = v(x) D_x^{\alpha} u(x) + u(x) D_x^{\alpha} v(x), \tag{5}$$

$$D_x^{\alpha} [f(u(x))] = f_u'(x) D_x^{\alpha} u(x) = D_x^{\alpha} f(u)(u_x')^{\alpha} \tag{6}$$

which are direct consequences of the equality $d^{\alpha}x(t)=\Gamma(1+\alpha)dx(t)$.

The main steps of the first integral method [9] are summarized as follows.

Step-I: We first consider a general form of the time fractional differential equation

$$P(u, D_t^{\alpha}u, D_t^{2\alpha}u, u_{xx}, ...) \quad (7)$$

To find the exact solution of Eq. (7) we introduce the variable transformation

$$u(x,t)=U(\xi),\ \ \xi=lx-\frac{\lambda}{\Gamma(1+\alpha)}t^{\alpha}, \quad (8)$$

where l and λ are constants to be determined later.

Using Eq. (8) we can write Eq. (7) in the folowing nonlinear ordinary differentia equation (ODF):

$$Q\left(U(\xi), \frac{dU(\xi)}{d\xi}, \frac{d^2U(\xi)}{d\xi^2}, ...\right)=0, \quad (9)$$

where $U(\xi)$ is an unkown function, Q is a a polynomial in the variable $U(\xi)$ and its derivative. If all terms contain derivatives, then Eq. (9) is integrated where integration constants are considered zeros.

Step-II: We assume that Eq. (9) has a soution in the form

$$U(\xi)=X(\xi), \quad (10)$$

and introduce a new independent variabe $Y(\xi)=\frac{dX(\xi)}{d\xi}$ which leads to a new system of

$$\frac{dX(\xi)}{d\xi}=Y(\xi), \quad (11)$$

$$\frac{dY(\xi)}{d\xi}=G(X(\xi), Y(\xi)),$$

Step-III: By using the Division Theorem for two variabes in the complex domain C which is based on the Hilbert-Nullstensatz Theorem [12], we can obtain one first integral to Eq. (11) which can reduce Eq. (9) to a first-order integrabe ordinary differential equation. An exact solution to Eq. (7) is then obtained by solving this equation directly.

Division Theorem: Suppose that $P(\omega,v)$ and $Q(\omega,v)$ are poynomial in $C[\omega,v]$; and $P(\omega,v)$ is irreducibe in $C[\omega,v]$. If $Q(\omega,v)$ vanishes at all zero points of $P(\omega,v)$, then there exixts a poynomial $G(\omega,v)$ in $C[\omega,v]$ such that

$$Q(\omega,v)=P(\omega,v)G(\omega,v)$$

3. Application To Population Growth Model

We consider the nonlinear time fractional biolgical population model [8, 13] :

$$D_t^{\alpha}u=(u^2)_{xx}+(u^2)_{yy}+h(u^2-r),\ 0<\alpha\le 1, \quad (12)$$

where h, r are constant.

Uunder the traveling wave transormation

$$u(x,y,t)=U(\xi),\quad \xi=x+y-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} \quad (13)$$

we obtain the foowing ODF

$$-\lambda U'=2(U^2)''+h(U^2-r), \quad (14)$$

or

$$4UU''+4(U')^2+\lambda U'+h(U^2-r)=0, \quad (15)$$

where the primes denote derivative with respect to ξ.

Using (10) and (11). Eq. (15) is equvalent to the two-dimensional autonomous system

$$\frac{dX}{d\xi}=Y$$

$$\frac{dX}{d\xi}=\frac{h(r-X^2)-4Y^2-\lambda Y}{4X} \quad (16)$$

Making the following transformation

$$d\eta=\frac{d\xi}{4X}, \quad (17)$$

Then system (16) becomes

$$\frac{dX}{d\eta}=4XY,$$

$$\frac{dY}{d\eta}=h(r-X^2)-4Y^2-\lambda Y. \quad (18)$$

Now, we applying the Division Theorem to seek the first integral to system (18). Suppose that $X=X(\eta), Y=Y(\eta)$ are the nontrivial solution to (18), and $Q(x,y)=\sum_{i=0}^{N}a_i(X)Y^i$ is an irreducible polynomial in complex domain C such that where $a_i(X)(i=0,1,...,N)$ are polynomial of X and $a_N(X)\neq 0$ and . Eq. (19) is called the first integral to system (18). Note that $\frac{dQ}{d\eta}$ is a polynomial in X and Y and $q[X(\eta), Y(\eta)]=0$

$\frac{dQ}{d\eta}|_{(18)}=0$ implies . According to the Division Theorem, there exists a polynomial $g(X)+h(X)Y$ in the complex domain C such that

$$\frac{dQ}{d\xi}=\frac{dQ}{dX}.\frac{dX}{d\eta}+\frac{dQ}{dY}.\frac{dY}{d\eta}$$

$$=(g(X)+h(X)Y)\sum_{i=0}^{N}a_i(X)Y^i \quad (20)$$

Suppose that $N=1$ in (19). By comparing with the coffecients of Y^i $(i=2,1,0)$ on both sides of (20), we have

$$4Xa_1'(X)=h(X)a_1(X)+4a_1(X) \quad (21)$$

$$4Xa_0'(X)=(g(X)+\lambda)a_1(X)+h(X)a_0(X) \quad (22)$$

$$ha_1(X)(r-X^2)=g(X)a_0(X). \quad (23)$$

Since $a_i(X)(i=0,1)$ are polynomials, then from (21) we deduce $a_1(X)$ that is constant and $h(X)=-4$. For simplicity $a_1(X)=1$. Balancing the degree of $g(X)$ and $h(X)$, we conclude that deg ($g(X)$)=deg ($a_0(X)$)=1 only. Suppose that

$$g(X)=A_0+A_1X,a_0(X)=B_0+B_1X,\ A_1\neq 0,\ B_1\neq 0 \quad (24)$$

where A_0,A_1,B_0,B_1 are all constants to be determined.

Substituting (24) int (22), we obtain

$$A_0=4B_0-c,\ A_1=8B_1 \quad (25)$$

Substituting $a_0(X),a_1(X)$ and $g(X)$ into (23) and setting all the coefficients of power X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it we obtain

$$B_0=\frac{1}{4}\sqrt{-2hr},\ B_1=\frac{1}{4}\sqrt{-2h},\ \lambda=3\sqrt{-2hr} \quad (26)$$

and

$$B_0=-\frac{1}{4}\sqrt{-2hr},B_1=-\frac{1}{4}\sqrt{-2h},\lambda=-3\sqrt{-2hr} \quad (27)$$

where h and r are arbitrary constants.

Using the conditions (26) and (27) in Eq. (19), we obtain

$$Y\pm\frac{1}{4}\sqrt{-2h}(\sqrt{r}+X)=0, \quad (28)$$

Combining this first integral with system (18), the second order differential Eq. (15) can be reduced to

$$U'(\xi)=\mp\frac{1}{4}\sqrt{-2h}(\sqrt{r}+U(\xi)). \quad (29)$$

Solving Eq. (29), we have

$$U(\xi)=-\sqrt{r}+e^{\mp\frac{1}{4}\sqrt{-2h}(\xi+\xi_0)}. \quad (30)$$

where ξ_0 is arbitrary constant.

Thus, we have an exact peaked wave solution of nonlinear time fractional biological population model in the following form

$$u(x,y,t)=-\sqrt{r}+e^{\mp\frac{1}{4}\sqrt{-2h}\left(x+y\mp\frac{3\sqrt{-2hr}}{\Gamma(1+\alpha)}t^{\alpha}+\xi_0\right)} \quad (31)$$

The domain restrictions or constraint conditions for (31) are $r>0$ and $h<0$.

4 Conclusions

In this paper, we extended the first integral method to construct the exact solution of the population growth model with fractional temporal evolution. The result shows that this method is efficient in finding the exact soliton solutions of some nonlinear fractional differential equations. We predict that the first integral method can be extended to solve many systems of nonlinear fractional PDEs in mathematical and physical sciences. These results will be reported in future.

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