Mixture of Exponentiated Frechet Distribution

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Abstract: In this paper, we will discuss the problem of estimating the parameters, reliability and failure rate functions of the finite mixture of two components from exponentiated Frechet distributions (MEFD). The maximum likelihood estimation (MLE) and Bayes methods of estimation are used. An approximation form due to Lindley (1980) is used for deriving the Bayes estimates under the squared error loss (quadratic loss) and LINEX loss functions. Through Monte Carlo simulation, the mean square errors (MSE'S) of the estimators are computed and compared between them.

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Key words: Exponentiated Frechet distribution (EFD); Maximum likelihood estimation; Bayes estimation; quadratic loss function; LINEX loss function.

1. Introduction

Mixture models play an important role in many applicable fields, such as medicine, psychology, cluster analysis, life testing and reliability analysis and so on. Mixture models have been considered extensively by many authors, for an excellent survey of estimation techniques, discussion and applications, (see Al-Hussaini (1999), Al-Hussaini et al. (2000) and Everitt and Hand (1981). Mixture distributions are obvious reason when dealing with lifetime distributions because such distributions always have several sub populations, as a unit can have more than one reason of failure. The mixture distribution from new distribution in the area which consists of two or more models from life time. However, several researchers are interested with different parameters of mixture distributions, see, Teicher (1961), Titterington *et al.* (1985), McLachlan and Basford (1988), Lindsay (1995), McCulloch and Searle (2001). The mixed Weibull distribution as a model for atmospheric data was proposed by Falls (1970), who used the method of moments for obtaining the estimators from a complete sample. The maximum likelihood estimation of parameters in mixed Weibull distribution with equal shape parameter from complete and censored Type I sample was considered by Ashour and Jones (1976). Jaheen (2005) used the maximum likelihood of mixture distribution. Nassar and Mahmoud (1985), Nassar (1988) presented statistic of characteristic this models. One of those who were interested in statistical inference about mixtures distribution parameters Rider (1961), Al-Hussaini (1999). Chen *et al.* (1989) considered the Bayes estimation for mixtures of two Weibull distribution under Type I censoring. They obtained Bayes estimate approximately for mixture distribution consisting of two models from Weibull distribution based on Type II censoring. Al-Hussaini *et al.* (2000) and Kao (1959) studied properties to mixture distribution consisting of two models from Gompertz and parameters estimate by using maximum likelihood and Lindley (1980) method for Bayes estimate John (1970) used the moment method and maximum likelihood estimate of parameters for mixture distribution consisting of two models from gamma. Abu-Zinadah (2006) presented the mixture consisting of k components from exponentiated Pareto distribution for life time distribution and found maximum likelihood estimate and Bayes estimates for parameters of mixture based on Type II censoring. Bakoban (2007) studied two parameters of mixture from exponentiated gamma distribution, reliability and failure rate function by maximum likelihood estimate and Bayes estimates by using Lindley approximately. One of the important families of distributions in lifetime tests is the exponentiated Frechet distribution (EFD) with probability density function (pdf),

$$f(x;\,\theta)=\theta x^{-2}e^{-x^{-1}}(1-e^{-x^{-1}})^{\theta-1},\quad x>0,\ \ \theta>0, \tag{1.1}$$

and the cumulative distribution function (cdf) is given by

$$F(x;\,\theta)=1-\left(1-e^{-x^{-1}}\right)^{\theta},\ x>0,\theta>0, \tag{1.2}$$

see, Nadarajah and Kotz (2003).

In this paper, the basic idea of Lindley (1980) approximate form for Bayes estimation is used in the case of mixtures of two EFD under Type II censoring. The approximate Bayes estimates are obtained and compare with their corresponding maximum likelihood estimates for different complete sample and censoring size.

2. The mixture of exponentiated Frechet distribution.

Let the probability density function (pdf) of k mixture EFD be

$$f(t;\theta_j)=\sum_{j=1}^{k}p_j f_j(t;\theta_j),\ \ t>0,\ \ \theta_j>0,\quad \sum_{j=1}^{k}p_j=1\,,\quad 0\le p_j\le 1. \tag{2.1}$$

where p_j is mixing proportions, $f_j(t;\theta_j)$ is the pdf of EFD which defined in (1.1).

The cumulative distribution function (cdf) of k mixture EFD is given by

$$F(t;\theta_j)=\sum_{j=1}^{k}p_j F_j(t;\theta_j),\ \ t>0,\ \ \theta_j>0,\quad \sum_{j=1}^{k}p_j=1,\ \ 0\le p_j\le 1, \tag{2.2}$$

where $F_j(t;\theta_j)$ is the cdf defined by (1.2).

The reliability function of k mixture distribution is given by

$$R(t;\theta_j)=\sum_{j=1}^{k}p_j R_j(t;\theta_j),\ t>0,\ \ \theta_j>0, \tag{2.3}$$

where $R_j(t;\theta_j)$ is reliability function of EFD,

$$R_j(t;\theta_j)=(1-e^{-t^{-1}})^{\theta_j}\,,\ \ t>0\,,\ \ \theta_j>0, \tag{2.4}$$

Also, the hazard rate function (HF) is

$$h(t;\theta_j)=\frac{f(t;\theta_j)}{R(t;\theta_j)}. \tag{2.5}$$

Now, when k=2, the pdf, cdf, reliability and hazard functions for finite mixture of two components from EFD, (say, MEFD), respectively, are

$$f(t;\theta_1,\theta_2)=p\theta_1 e^{-t^{-1}}t^{-2}(1-e^{-t^{-1}})^{\theta_1-1}+(1-p)\theta_2 e^{-t^{-1}}t^{-2}\left(1-e^{-t^{-1}}\right)^{\theta_2-1},\quad t>0,\theta_1,\theta_2>0, \tag{2.6}$$

$$F(t;\theta_1,\theta_2)=p\left[1-\left(1-e^{-t^{-1}}\right)^{\theta_1}\right]+(1-p)[1-\left(1-e^{-t^{-1}}\right)^{\theta_2}], \tag{2.7}$$

$$R(t;\theta_1,\theta_2)=1-F(t;\theta_1,\theta_2)$$
$$=p\left(1-e^{-t^{-1}}\right)^{\theta_1}+(1-p)\left(1-e^{-t^{-1}}\right)^{\theta_2}, \tag{2.8}$$

$$h(t;\theta_1,\theta_2)=\frac{f(t;\theta_1,\theta_2)}{R(t;\theta_1,\theta_2)}. \tag{2.9}$$

A mixture is identifiable if there exist a one-to-one corresponding between the mixing distribution and a resulting mixture. That is, there is a unique characterization of the mixture. A class D of a mixture is said to be identifiable if and only if, $\forall f(t)\in \mathfrak{D}D$ the equality of the two representations,

$$\sum_{i=1}^{c}p_i f_i(t|\theta_i)=\sum_{j=1}^{\grave{c}}\grave{p}_j f_j(t|\grave{\theta}_j), \tag{2.10}$$

implies that $c=\grave{c}$ and for all i there exists some j such as $\theta_i=\grave{\theta}_j$ and $p_i=\grave{p}_j$, see Titterington et al. (1985). In the following theorem, we provide the identifiability of a mixture of K- EF components.

Theorem

A finite of K- EF components is identifiable.

Proof

Teicher (1961) showed that a finite mixture of K exponential components is identifiable. If $Y\sim EXP(\theta)$, by $Y=\frac{1}{T}$, follows that $T\sim EF(\theta)$, this transforrnation is bijective, so a finite mixture of $EF(\theta_j), j=1,2,\dots,K$ components is identifiable, which follows that

$$\sum_{j=1}^{k}p_i\left[1-\left(1-e^{-t^{-1}}\right)^{\theta_i}\right]=\sum_{j=1}^{\grave{k}}\grave{p}_j\left[1-\left(1-e^{-t^{-1}}\right)^{\grave{\theta}_j}\right],$$

this implies that $k=\grave{k}$ moreover, for all i, there exists some j such that $p_i=\grave{p}_j$ and $\theta_i=\grave{\theta}_j$. Therefore, a finite mixture of K- EF(θ_j), $j=1,2,\dots,k$ components is identifiable.

3. Statistical properties

3. 1. Moments

The r^{th} moment about the origin, $\mu_r'=E(T^r)$, of MEFD with the pdf (2.6) and the cdf (2.7) can be written, by

$$\grave{\mu}_r=E(T^r)=\int_0^\infty t^{r-1}[1-F(t;\theta_1,\theta_2)]dt\,,\quad r=\ 0,1,\dots, \tag{3.1}$$

$$\grave{\mu}_r=\int_0^\infty t^{r-1}\left[p\left(1-e^{-t^{-1}}\right)^{\theta_1}+(1-p)\left(1-\ e^{-t^{-1}}\right)^{\theta_2}\right]dt. \tag{3.2}$$

On setting $y=t^{-1}$, (3.2) is reduced to

$$\grave{\mu}_r=\int_0^\infty y^{-r-1}[p(1-e^{-y})^{\theta_1}+(1-p)(1-e^{-y})^{\theta_2}]dy. \tag{3.3}$$

This integral converges if $\theta_1, \theta_2 > r$. However, it is not known how (3.3) can be reduced to a closed-form (see, Nadarajah and Kotz (2006)).

3. 2. The median

The median of MEFD can't be found in an explicit form. We derive the median m by solving the given equation

$$p[1-\left(1-e^{-m^{-1}}\right)^{\theta_1}]+(1-p)\left[1-\left(1-e^{-m^{-1}}\right)^{\theta_2}\right]=0.5$$

3. 3. The mode

The mode for the MEFD can be found differentiating f(t) with respect to t, so Eq. (2.6) gives

$$\acute{f}(t)=pf_1(t)\left\{t^{-2}-2t^{-1}-(\theta_1-1)t^{-2}e^{-\frac{1}{t}}\left(1-e^{-\frac{1}{t}}\right)^{-1}\right\}+(1-p)f_2(t)\left\{t^{-2}-2t^{-1}-(\theta_2-1)t^{-2}e^{-\frac{1}{t}}\left(1-e^{-\frac{1}{t}}\right)^{-1}\right\}. \quad (3.4)$$

By equating (3.4) with zero, it cannot be found in an explicit form. We observe that the MEFD, may be unimodal (see, Fig. 1), in which the mode can be found numerically by solving (3.4). Figure 1 Shows the graphical for the probability density function (pdf) curve for MEFD with parameters $(p, \theta_1, \theta_2) = (0.3, 0.2, 0.6), (0.3, 2, 7), (0.7, 5, 0.9), (0.8, 0.7, 5)$

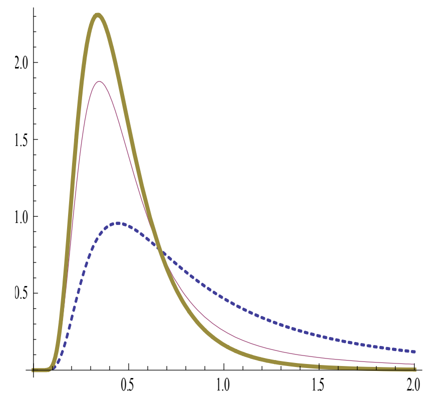

Fig. 1-b: $(p, \theta_1, \theta_2) = (0.3, 2, 7)$

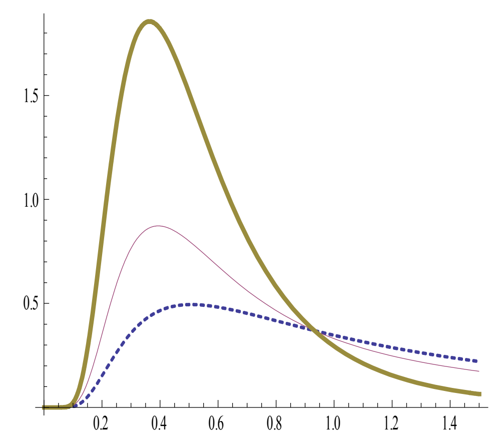

Fig.1-c: $(p, \theta_1, \theta_2) = (0.7, 0.9, 5)$

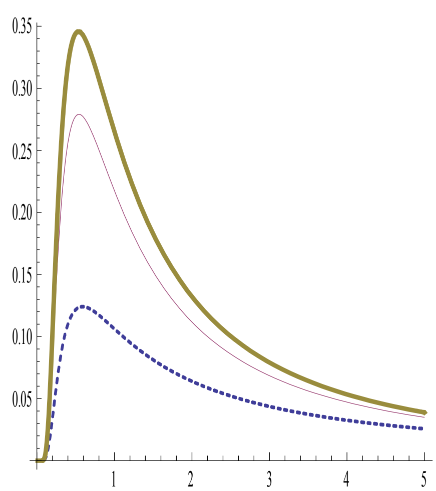

Fig.1-a: $(p, \theta_1, \theta_2) = (0.3, 0.2, 0.6)$

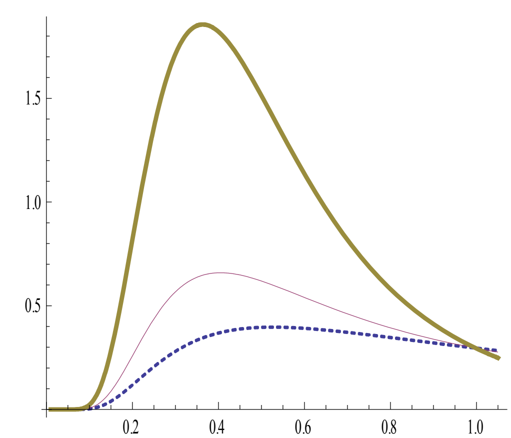

Fig.1-d: $(p, \theta_1, \theta_2) = (0.8, 0.7, 5)$

The bold curve for EF(θ_1), the regular curve for MEFD(p, θ_1, θ_2) and the dashed curve for EF(θ_2).

4. Maximum likelihood estimator

Suppose that only the r smallest observations in a random sample of n items are observed $(1 \leq r \leq n)$. That is, suppose that the data consists of the r smallest lifetimes $X_{(1)} < \cdots < X_{(r)}$ out of a random sample of n items $X_1, \dots, X_n$ (Type II censored sample). The likelihood function based on a Type II censored sample (see, Lawless (1982) and Titterington *et al.* (1985)) can be written as

$$L(\boldsymbol{\theta} | \underline{t}) = \frac{n!}{(n-r)!}[\prod_{i=1}^{r} f(t_{i:n};\, \theta_1, \theta_2)[R(t_{r:n};\, \theta_1, \theta_2)]^{n-r}, \quad (4.1)$$

where $R(t_{r:n}) = 1 - F(t_{r:n})$.

The natural logarithm of the likelihood function (4.1) is given by

$$l(\boldsymbol{\theta} | \underline{t}) = \log[L(\boldsymbol{\theta} | \underline{t})] = \log\left[\frac{n!}{(n-r)!}\right] + \sum_{i=1}^{r} \log f(t_{i;n}) + (n-r)\log R(t_{r:n}). \quad (4.2)$$

Assuming that the parameters θ_1 and θ_2 are unknown and p is known, the likelihood equations are given by

$$\frac{\partial l}{\partial \theta_j} = \sum_{i=1}^{r}\left[\frac{1}{f(t_{i:n})}\frac{\partial f(t_{i:n})}{\partial \theta_j}\right] + \frac{(n-r)}{R(t_{r:n})}\frac{\partial R(t_{r:n})}{\partial \theta_j} = 0, \quad j = 1, 2. \quad (4.3)$$

From (2.8) and (2.10), respectively, we have

$$\frac{\partial f(t_{i:n})}{\partial \theta_1} = p\,\theta_1 t_{i:n}{}^{-2} e^{-t_{i:n}{}^{-1}}\left(1 - e^{-t_{i:n}{}^{-1}}\right)^{\theta_1 - 1} k_1(t_{i:n}), \quad (4.4)$$

$$\frac{\partial f(t_{i:n})}{\partial \theta_2} = (1-p)\theta_2 t_{i:n}{}^{-2} e^{-t_{i:n}{}^{-1}} . \left(1 - e^{-t_{i:n}{}^{-1}}\right)^{\theta_2 - 1} k_2(t_{i:n}), \quad (4.5)$$

where

$$\varphi(t_{i:n}) = \ln\left[1 - e^{-t_{i:n}{}^{-1}}\right],$$

$$k_j(t_{i:n}) = \varphi(t_{i:n}) + \theta_j^{-1},\ j = 1, 2, \quad (4.6)$$

$$\frac{\partial R(t_{r:n})}{\partial \theta_1} = p_1\left(1 - e^{-t_{r:n}^{-1}}\right)^{\theta_1}\varphi(t_{r:n}), \quad (4.7)$$

$$\frac{\partial R(t_{r:n})}{\partial \theta_2} = p_2\left(1 - e^{-t_{r:n}^{-1}}\right)^{\theta_2}\varphi(t_{r:n}). \quad (4.8)$$

From (4.4) and (4.7) in (4.3), we obtain

$$\frac{\partial l}{\partial \theta_1} = p\{\sum_{i=1}^{r}\zeta_1(t_{i:n})k_1(t_{i:n}) + (n-r)\zeta_1^*(t_{r:n})\varphi(t_{r:n})\} = 0. \quad (4.9)$$

Also, by substituting (4.5) and (4.8) in (4.3), we obtain

$$\frac{\partial l}{\partial \theta_2} = p_2\{\sum_{i=1}^{r}\zeta_2(t_{i:n})k_2(t_{i:n}) + (n-r)\zeta_2^*(t_{r:n})\varphi(t_{r:n})\} = 0, \quad (4.10)$$

where

$$\zeta_1(t_{i:n}) = \frac{f_1(t_{i:n})}{f(t_{i:n})}, \quad \zeta_2(t_{i:n}) = \frac{f_2(t_{i:n})}{f(t_{i:n})},$$

$$\zeta_1^*(t_{r:n}) = \frac{R_1(t_{r:n})}{R(t_{r:n})}, \quad \zeta_2^*(t_{r:n}) = \frac{R_2(t_{r:n})}{R(t_{r:n})},$$

$$f_1(t_{i:n}) = p\theta_1 t_{i:n}^{-2} e^{-t_{i:n}^{-1}}(1 - e^{-t_{i:n}^{-1}})^{\theta_1 - 1},$$

$$f_2(t_{i:n}) = (1-p)\theta_2 t_{i:n}^{-2} e^{-t_{i:n}^{-1}}\left(1 - e^{-t_{i:n}^{-1}}\right)^{\theta_2 - 1},$$

$$R_1(t_{i:n}) = p(1 - e^{-t_{i:n}^{-1}})^{\theta_1}\log\left(1 - e^{-t_{i:n}^{-1}}\right) \text{ and}$$

$$R_2(t_{i:n}) = (1-p)\left(1 - e^{-t_{i:n}^{-1}}\right)^{\theta_2}\log\left(1 - e^{-t_{i:n}^{-1}}\right). \quad (4.11)$$

The solution of the two nonlinear likelihood equations (4.9) and (4.10) yield the maximum likelihood estimate (MLE) $\widehat{\boldsymbol{\theta}} = (\hat{\theta}_{1,M}, \hat{\theta}_{2,M})$ of $\boldsymbol{\theta} = (\theta_1, \theta_2)$. The MLE's of R(t) and h(t) are respectively given by (2.8) and (2.9) after replacing θ_1 and θ_2 by their corresponding MLE's $\hat{\theta}_{1,M}$ and $\hat{\theta}_{2,M}$.

5. Bayes estimation

When the mixing proportion p is known, the parameters Θ_1, Θ_2 are assumed to be independent random variables and the joint prior density for random vector $\Theta = (\Theta_1, \Theta_2)$, is thus given by

$$g(\boldsymbol{\theta}) = g(\theta_1, \theta_2) = g(\theta_1)g(\theta_2). \quad (5.1)$$

Let Θ_j follow Gamma distribution with shape parameter β_j and scale parameter $\alpha = 1$, i.e. $G(\beta_j, 1)$, the pdf for Θ_j random variable is

$$g_j(\theta_j) = \frac{\theta_j^{\beta_j - 1} e^{-\theta_j}}{\Gamma[\beta_j]},\ \beta_j > 0, \theta_j > 0.$$

Then, the joint prior probability density function for random vector Θ is

$$g(\boldsymbol{\theta}) = \frac{1}{\Gamma[\beta_1]\Gamma[\beta_2]} \theta_1^{\beta_1 - 1} \theta_2^{\beta_2 - 1} e^{-(\theta_1+\theta_2)} , \quad \theta_j > 0, \beta_j > 0 , \; j = 1,2. \tag{5.2}$$

It is well known that the posterior density function of $\boldsymbol{\Theta}$ given the observation (data), which is denoted by $q(\boldsymbol{\theta}|\underline{t})$, is given as follows

$$q(\boldsymbol{\theta}|\underline{t}), = \frac{L(\boldsymbol{\theta}|\underline{t})g(\boldsymbol{\theta})}{\int_{\Omega} L(\boldsymbol{\theta}|\underline{t})g(\boldsymbol{\theta})d\boldsymbol{\theta}}, \tag{5.3}$$

where $L(\boldsymbol{\theta}|\underline{t})$ is given by (4.1), $g(\theta)$ by (5.2) and Ω is the region in the $\boldsymbol{\theta_1\theta_2}$ plane on which the posterior density $q(\boldsymbol{\theta}|\underline{t})$ is positive.
Then, under the squared error loss function, the Bayes estimator $\tilde{\varphi}$ of a function of the parameter $\varphi(\boldsymbol{\theta})$ is given by

$$\tilde{\varphi} = E[\varphi(\boldsymbol{\theta}|\underline{t})] = \int_{\Omega} \varphi(\boldsymbol{\theta}) q(\boldsymbol{\theta}|\underline{t}) d\boldsymbol{\theta}$$

$$= \frac{\int_{\Omega} \varphi(\boldsymbol{\theta}) L(\boldsymbol{\theta}|\underline{t}) g(\boldsymbol{\theta}) d\boldsymbol{\theta}}{\int_{\Omega} L(\boldsymbol{\theta}|\underline{t}) g(\boldsymbol{\theta}) d\boldsymbol{\theta}}. \tag{5.4}$$

The ratio of the integrals (5.4) may thus be approximated by using a form due to Lindley (1980), which reduces in the case of two parameters, to the form

$$\tilde{\varphi}^* = \varphi^*(\boldsymbol{\theta}) + \frac{S}{2} + \rho_1 S_{12} + \rho_2 S_{21} + \frac{1}{2}\, [l_{30}^* v_{12} + l_{21}^* c_{12} + l_{12}^* c_{21} + l_{03}^* v_{21}], \tag{5.5}$$

where

$\boldsymbol{\theta} = (\theta_1, \theta_2)$, $S = \sum_{i=1}^{2}\sum_{j=1}^{2} \varphi_{ij}\sigma_{ij}$, $i, j = 1,2$, $\varphi_{ij} = \frac{\partial^2 \varphi}{\partial\theta_i \partial\theta_j}$, $\sigma_{ij} = (i,j)th$ element in the matrix Σ, $\Sigma = -[J(\boldsymbol{\theta})]^{-1}$, $J(\boldsymbol{\theta}) = [l_{ij}]$, $l_{ij} = \frac{\partial^2 l}{\partial\theta_j \partial\theta_i}$, $l = \ln[L(\boldsymbol{\theta}|\underline{t})]$, $L(\boldsymbol{\theta}|\underline{t})$ is as given by (4.1).

For $i \neq j$, $\quad S_{ij} = \varphi_i\sigma_{ij} + \varphi_j\sigma_{ji}$,

$$v_{ij} = (\varphi_i\sigma_{ii} + \varphi_j\sigma_{ij})\sigma_{ii},$$

$$c_{ij} = 3\varphi_i\sigma_{ii}\sigma_{ij} + \varphi_j(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2),$$

$$\varphi_i = \frac{\partial\varphi}{\partial\theta_i}, \qquad \rho_i = \frac{\partial\rho}{\partial\theta_i}, \qquad \rho = \ln[g(\boldsymbol{\theta})],$$

$g(\boldsymbol{\theta})$ is given in (5.2). Finally,

$$l_{30}^* = \frac{\partial l_{11}}{\partial\theta_1}, \quad l_{21}^* = \frac{\partial l_{12}}{\partial\theta_1}, \quad l_{12}^* = \frac{\partial l_{12}}{\partial\theta_2} \quad \text{and}$$

$$l_{03}^* = \frac{\partial l_{22}}{\partial\theta_2}$$

Now, we apply Lindley's form (5.4), we first obtain the elements σ_{ij} as follows

$$\sigma_{11} = -\frac{l_{22}}{D}, \sigma_{22} = -\frac{l_{11}}{D}, \sigma_{12} = \sigma_{21} = \frac{l_{12}}{D}, \tag{5.6}$$

where

$$D = l_{11}l_{22} - l_{12}^2, \tag{5.7}$$

$$l_{12} = l_{21} = -p(1-p)\{\textstyle\sum_{i=1}^{r} w(t_{i:n}) + (n-r)\emptyset(t_{r:n})\}, \tag{5.8}$$

$$w(t_{i:n}) = k_1(t_{i:n})k_2(t_{i:n})\zeta_1(t_{i:n})\zeta_2(t_{i:n}),$$

$$\emptyset(t_{r:n}) = \zeta_1^*(t_{r:n})\zeta_2^*(t_{r:n})\big(\gamma(t_{r:n})\big)^2,$$

$$l_{11} = p\left\{\textstyle\sum_{i=1}^{r}\left[A_1(t_{i:n}) - \big(\zeta_1(t_{i:n})k_1(t_{i:n})\big)^2\right] + (n-r)[\big(\gamma(t_{r:n})\big)^2\zeta_1^*(t_{r:n})B_1(t_{r:n})]\right\}, \tag{5.9}$$

$$l_{22} = (1-p)\left\{\textstyle\sum_{i=1}^{r}\left[A_2(t_{i:n}) - \big(\zeta_2(t_{i:n})k_2(t_{i:n})\big)^2\right] + (n-r)[\big(\gamma(t_{r:n})\big)^2\zeta_2^*(t_{r:n})B_2(t_{r:n})]\right\}, \tag{5.10}$$

$$\begin{aligned} A_1(t_{i:n}) &= \emptyset(t_{i:n})\zeta_1(t_{i:n})[\varphi(t_{i:n}) + 2\theta_1^{-1}], \\ B_1(t_{r:n}) &= 1 - \zeta_1^*(t_{r:n}), \\ A_2(t_{i:n}) &= \emptyset(t_{i:n})\zeta_2(t_{i:n})[\varphi(t_{i:n}) + 2\theta_2^{-1}], \\ B_2(t_{r:n}) &= 1 - \zeta_2^*(t_{r:n}). \end{aligned} \tag{5.11}$$

Furthermore

$$l_{30}^* = \frac{\partial l_{11}}{\partial\theta_1} = -p\{\textstyle\sum_{i=1}^{r}\frac{\partial A_1(t_i)}{\partial\theta_1} - \sum_{i=1}^{r-1}\varphi(t_i)\left[\zeta_1^*(t_i)\frac{\partial B_1(t_i)}{\partial\theta_1} + B_1(t_i)\frac{\partial\zeta_1^*(t_i)}{\partial\theta_1}\right], \tag{5.12}$$

$$l_{03}^* = \frac{\partial l_{22}}{\partial\theta_2} = -(1-p)\{\textstyle\sum_{i=1}^{r}\frac{\partial A_2(t_i)}{\partial\theta_2} - \sum_{i=1}^{r-1}\varphi(t_i)\left[\zeta_2^*(t_i)\frac{\partial B_2(t_i)}{\partial\theta_2} + B_2(t_i)\frac{\partial\zeta_2^*(t_i)}{\partial\theta_2}\right], \tag{5.13}$$

$$l_{21}^* = \frac{\partial\, l_{21}}{\partial\theta_1} = -p(1-p)\{\textstyle\sum_{i=1}^{r}\frac{\partial w(t_i)}{\partial\theta_1} - \sum_{i=1}^{r-1}\varphi^2(t_i)\frac{\partial}{\partial\theta_1}\emptyset(t_i)\}. \tag{5.14}$$

Similarly

$$l_{12}^{*}=\frac{\partial\, l_{12}}{\partial\theta_2}=$$
$$-p(1-p)\{\sum_{i=1}^{r}\frac{\partial w(t_i)}{\partial\theta_2}-\sum_{i=1}^{r-1}\varphi^2(t_i)\frac{\partial}{\partial\theta_2}\emptyset(t_i)\},$$
$$j\neq s\,,\ j,s=1,2, \quad (5.15)$$

where

$$\frac{\partial A_j(t_i)}{\partial\theta_j}=\theta_j^{-2}\frac{\partial}{\partial\theta_j}\zeta_j(t_i)-2\theta_j^{-3}\zeta_j(t_i)+$$
$$p_s\zeta_s(t_i)\{k_j^2(t_i)\frac{\partial}{\partial\theta_j}\zeta_j(t_i)+2\zeta_j(t_i)k_j(t_i)\frac{\partial k_j(t_i)}{\partial\theta_j}\},$$

$$\frac{\partial\zeta_j(t_i)}{\partial\theta_j}=\frac{f(t)(\partial f_j(t_i)/\partial\theta_j)-f_j(t_i)(\partial f(t_i)/\partial\theta_j)}{(f(t))^2},$$

$$\frac{\partial\zeta_j^*(t_i)}{\partial\theta_j}=\varphi(t_i)\zeta_j^*(t_i)\{1+p_j\zeta_j^*(t_i)\},$$

$$\frac{\partial B_j(t_i)}{\partial\theta_j}=k_j(t_i)\zeta_j(t_i)+p_j\varphi(t_i)\tau_j(t_i)\zeta_j^*(t_i)$$
$$=\tau_j(t_i)\{k_j(t_i)+p_j\varphi(t_i)\zeta_j^*(t_i)\},$$

$$\tau_j(t_{r:n})=\frac{f_j(t_{r:n})}{R(t_{r:n})},\ p_1=p,\ \ p_2=1-p.$$

In Bayesian estimation, we consider two types of loss functions. The first is the squared error loss function (quadratic loss) which is classified as a symmetric function and associates equal importance to the losses for overestimation and underestimation of equal magnitude. The second is the LINEX (linear exponential) loss function which is asymmetric, see Varian (1975). These loss functions were widely used by several authors; among of them Rojo (1987), Basu and Ebrahimi (1991), Pandey (1997), Soliman (2000), Shawky and Bakoban (2009).

The quadratic loss function for Bayes estimate of a parameter β say, is the posterior mean assuming that exists, denoted by β_s . The LINEX loss function may be expressed as

$$L(\Delta)\propto e^{c\Delta}-c\Delta-1,\qquad c\neq 0, \qquad (5.16)$$

where $\Delta=\hat{\beta}-\beta$. The sign and magnitude of the shape parameter c reflects the direction and degree of asymmetry respectively. If $c>0$, the overestimation is more serious than underestimation, and vice- versa. For c closed to zero, the LINEX loss is approximately squared error loss and therefore almost symmetric.

The posterior expectation of the LINEX loss function equation (5.16) is

$$E_\beta\left[L(\hat{\beta}-\beta)\right]\propto\exp\left(c\hat{\beta}\right)E_\beta[\exp(-c\beta)]-$$
$$c\left(\hat{\beta}-\ E_\beta(\beta)\right)-1, \qquad (5.17)$$

where $E_\beta(.)$ denoting posterior expectation with respect to the posterior density of β.

By a result of Zellner (1986), the (unique) Bayes estimator of β, denoted by $\hat{\beta}_L$ under the LINEX loss is the value $\hat{\beta}$ which minimizes (5.16), is given by

$$\hat{\beta}_L=\frac{1}{c}\ln\left[E_\beta[\exp(-c\beta)]\right], \qquad (5.18)$$

provided that the expectation $E_\beta[\exp(-c\beta)]$ exists and is finite, see Calabria and Pulcini (1996).

5.1 Bayes estimation under quadratic loss function

5.1.1 Estimation of two parameters

The two parameters θ_1,θ_2 can be estimated by using Lindley's approximation from (5.5) as follows:

(i) Bayes estimation of parameter θ_1

Put $\theta_1=\varphi^*(\theta)$ in (5.5) for values $i,j=1,2$. Then

$$\varphi_1^*=\frac{\partial\varphi^*}{\partial\theta_1}=1,\ \varphi_2^*=\frac{\partial\varphi^*}{\partial\theta_2}=0,\ \varphi_{11}^*=\frac{\partial^2\varphi^*}{\partial\theta_1^2}=0,$$
$$\varphi_{22}^*=\frac{\partial^2\varphi^*}{\partial\theta_2^2}=0,\qquad \varphi_{21}^*=\varphi_{12}^*=0,$$

$$S=\varphi_{11}^*\sigma_{11}+\varphi_{12}^*\sigma_{12}+\varphi_{21}^*\sigma_{21}+\varphi_{22}^*\sigma_{22}=0,$$

$$S_{12}=\varphi_1^*\sigma_{11}+\varphi_2^*\sigma_{12}\ \ =\sigma_{11}\,,$$

$$S_{21}=\sigma_{12},\ \ v_{12}=\sigma_{11}^2,\ \ \ v_{21}=\sigma_{12}\sigma_{22},$$

$$c_{12}=3\sigma_{11}\sigma_{12},\ c_{21}=\sigma_{22}\sigma_{11}+2\sigma_{21}^2.$$

By using the above functions, (5.12) – (5.15) and (5.5), yields the Bayes estimator under squared error loss function, $\hat{\theta}_{1,s}$, of θ_1.

(ii) Bayes estimation of parameter θ_2

Put $\theta_2=\varphi^*(\theta)$ in (5.5) for values $i,j=1,2$, then

$$\varphi_1^*=\frac{\partial\varphi^*}{\partial\theta_1}=0,\ \ \varphi_2^*=\frac{\partial\varphi^*}{\partial\theta_2}=1,\ \ \varphi_{11}^*=0,$$

$$\varphi_{22}^*=0,\ \varphi_{12}^*=\varphi_{21}^*=0,$$

$$S=0,\ \ S_{12}=\sigma_{21}\,,\ S_{21}=\sigma_{22},\ \ v_{12}=\sigma_{12}\sigma_{11},$$

$v_{21} = \sigma_{22}^2, c_{12} = \sigma_{11}\sigma_{22} + 2\sigma_{12}^2,\ \ c_{21} = 3\sigma_{22}\sigma_{21}.$

By using the pervious functions, (5.12) – (5.15) and (5.5), yield the Bayes estimator under squared error loss function, $\hat{\theta}_{2,s}$, of θ_2.

5.1.2 Bayes estimation of reliability function

Put $\varphi^*(\theta) = R(t)$ in (5.5) for values $i, j = 1,2$, where $R(t)$ defined by (2.8), then

$$\varphi_1^* = p_1 R_1(t)\varphi(t),\ \ \varphi_2^* = p_2 R_2(t)\varphi(t)\,,\ \ \varphi_{11}^* = p_1 R_1(t)\varphi^2(t), \tag{5.19}$$

$$\varphi_{22}^* = p_2 R_2(t)\varphi^2(t),\ \ \varphi_{21}^* = \varphi_{12}^* = 0,$$

$$S = \varphi_{11}^*\sigma_{11} + \varphi_{22}^*\sigma_{22}\,, \tag{5.20}$$

$$S_{12} = \varphi_1^*\sigma_{11} + \varphi_2^*\sigma_{21} = [p_1 R_1(t)\varphi(t)]\sigma_{11} + [p_2 R_2(t)\varphi(t)]\sigma_{21},$$

$$S_{21} = [p_2 R_2(t)\varphi(t)]\sigma_{22} + [p_1 R_1(t)\varphi(t)]\sigma_{12},$$

$$v_{ij} = \left(\varphi_i^*\sigma_{ii} + \varphi_j^*\sigma_{ij}\right)\sigma_{ii},$$

$$v_{12} = [p_1 R_1(t)\varphi(t)]\sigma_{11}^2 + [p_2 R_2(t)\varphi(t)]\sigma_{12}\sigma_{11},$$

$$v_{21} = [p_2 R_2(t)\varphi(t)]\sigma_{22}^2 + [p_1 R_1(t)\varphi(t)]\sigma_{21}\sigma_{22},$$

$$c_{ij} = 3\varphi_i^*\sigma_{ii}\sigma_{ij} + \varphi_j^*(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2)$$

$$c_{12} = 3[p_1 R_1(t)\varphi(t)]\sigma_{11}\sigma_{12} + [p_2 R_2(t)\varphi(t)](\sigma_{11}\sigma_{21} + 2\sigma_{12}^2),$$

$$c_{21} = 3[p_2 R_2(t)\varphi(t)]\sigma_{22}\sigma_{21} + [p_1 R_1(t)\varphi(t)](\sigma_{22}\sigma_{11} + 2\sigma_{21}^2)$$

Substituting (5.19), (5.20) and (5.12) – (5.15) in (5.5), yield the Bayes estimator under squared error loss function, $\hat{R}_s$, of $R(t)$.

5.1.3 Bayes estimation of failure rate function

Put $\varphi^*(\theta) = h(t)$ in (5.5) for values $i, j = 1,2$, where $h(t)$ defined by (2.9), then

$$\varphi_j^*(t) = \frac{p_j}{(R(t))^2}\{R(t)f_j(t)k_j(t) - f(t)R_j(t)\varphi(t)\}, \tag{5.21}$$

$$\varphi_{jj}^* = \frac{p_j}{(R(t))^4}[E_1 - E_2], \tag{5.22}$$

where

$$E_1 = \left(R(t)\right)^2[R(t)k_j(t)f_j(t)k_j(t) - R(t)f_j(t)\theta_j^{-2} + p_j f_j(t)k_j(t)R_j(t)\varphi(t) - \left(R(t)\right)^2\varphi(t)[p_j f_j(t)k_j(t) + R_j(t)\varphi(t)], \tag{5.23}$$

$$E_2 = 2p_j R(t)R_j(t)\varphi(t)\{R(t)f_j(t)k_j(t) - f(t)R_j(t)\varphi(t)\}, \tag{5.24}$$

$f_j(t)$, $f(t), R_j(t), R(t), k_j(t)$ are defined in (2.2), (2.1), (2.6), (2.10) and (5.6), for values $i, j = 1,2$, we get

$$\varphi_{ij}^* = \frac{[E_1^* - E_2^*]}{(R(t))^4}, \tag{5.25}$$

$$E_1^* = \frac{1}{(R(t))^2}[\{p_i p_j f_j(t)k_j(t)R_i(t)\varphi(t) - p_i p_j f_i(t)k_i(t)R_j(t)\varphi(t)\}, \tag{5.26}$$

$$E_2^* = -\frac{2}{(R(t))^3}\{p_j f_j(t)k_j(t)R(t) - p_j R_j(t)f(t)\varphi(t)\}p_i R_i(t)\varphi(t)]. \tag{5.27}$$

Substituting (5.21) - (5.27) and (5.12) – (5.15) in (5.5), yield the Bayes estimator under squared error loss function, $\hat{h}_s$, of $h(t)$.

5.2 Bayes estimation under LINEX loss function

On the basis of the LINEX loss function (5.18), the Bayes estimate of a function $q = q(\theta_1, \theta_2)$, where θ_1, θ_2 unknown, as follows

$$\hat{q}_L = -\frac{1}{c}\ln\left[E\left(e^{-qc}|\underline{t}\right)\right],\ \ c \neq 0, \tag{5.28}$$

where

$$E\left(e^{-qc}|\underline{t}\right) = \frac{\int_\Omega e^{-cq}L(\boldsymbol{\theta}|\underline{t})g(\boldsymbol{\theta})d\boldsymbol{\theta}}{\int_\Omega L(\boldsymbol{\theta}|\underline{t})g(\boldsymbol{\theta})d\boldsymbol{\theta}}. \tag{5.29}$$

Let $\varphi^*(\theta) = e^{-cq(\theta)}$, so we can use Lindley's approximation for finding the estimators of unknown parameters, as follows

5.2.1 Estimation of two parameters

(i) Bayes estimate of the parameter $\boldsymbol{\theta_1}$

Put $\varphi^*(\boldsymbol{\theta}) = e^{-c\theta_1}$ in (5.5), we get

$$\varphi_1^* = \frac{\partial\varphi^*}{\partial\theta_1} = -ce^{-c\theta_1},\ \ \varphi_2^* = \frac{\partial\varphi^*}{\partial\theta_2} = 0, \varphi_{11}^* = c^2 e^{-c\theta_1},\ \ \varphi_{22}^* = 0,\ \text{ for values } i, j = 1,2.$$

$\varphi_{ij}^* = \frac{\partial^2\varphi^*}{\partial\theta_i\partial\theta_j} = 0,\ S = 0,\ S_{ij} = \varphi_i^*\sigma_{ii} + \varphi_j^*\sigma_{ji},$

$S_{12} = -ce^{-c\theta_1}\sigma_{11},\ S_{21} = -ce^{-c\theta_1}\sigma_{12}.$

Then

$v_{ij} = (\varphi_i^*\sigma_{ii} + \varphi_j^*\sigma_{ij})\sigma_{ii},$

$v_{12} = -\sigma_{11}^2 ce^{-c\theta_1}\ ,\ v_{21} = -\sigma_{12}\sigma_{22}ce^{-c\theta_1}$

$c_{ij} = 3\varphi_i^*\sigma_{ii}\sigma_{ij} + \varphi_j^*(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2),$

$c_{12} = -3\sigma_{11}\sigma_{12}c\, e^{-c\theta_1},$

$c_{21} = -(\sigma_{22}\sigma_{11} + 2\sigma_{21}^2)ce^{-c\theta_1}.$

Substituting (5.12) – (5.15) in (5.5) then into (5.28), yield the Bayes estimator under LINEX loss function, $\hat{\theta}_{1,L}$ of θ_1.

(ii) Bayes estimation of parameter $\boldsymbol{\theta_2}$

Put $\varphi^*(\boldsymbol{\theta}) = e^{-c\theta_2}$ in (5.5), we get

$\varphi_1^* = \frac{\partial\varphi^*}{\partial\theta_1} = 0, \varphi_2^* = \frac{\partial\varphi^*}{\partial\theta_2} = -ce^{-c\theta_2}, \varphi_{11}^* = \frac{\partial^2\varphi^*}{\partial\theta_1^2} = 0, \varphi_{22}^* = \frac{\partial^2\varphi^*}{\partial\theta_2^2} = c^2e^{-c\theta_2}$, for values $i,j = 1,2$, $\varphi_{ij}^* = \frac{\partial^2\varphi^*}{\partial\theta_1\partial\theta_2} = 0\ \ S = 0,\ S_{ij} = \varphi_i^*\sigma_{ii} + \varphi_j^*\sigma_{ji},$

$S_{12} = -ce^{-c\theta_2}\sigma_{21},\ S_{21} = -ce^{-c\theta_2}\sigma_{22}.$

Then

$v_{ij} = (\varphi_i^*\sigma_{ii} + \varphi_j^*\sigma_{jj})\sigma_{ii},$

$v_{12} = -ce^{-c\theta_2}\sigma_{12}\sigma_{11},\ \ v_{21} = -ce^{-c\theta_2}\sigma_{22}^2.$

$c_{ij} = 3\varphi_i^*\sigma_{ii}\sigma_{ij} + \varphi_j^*(\sigma_{ii}\sigma_{jj} + 2\sigma_{22}^2),$

$c_{12} = -ce^{-c\theta_2}\ (\sigma_{11}\sigma_{22} + 2\sigma_{12}^2),$

$c_{21} = -3ce^{-c\theta_2}\sigma_{11}\sigma_{12}.$

Substituting (5.12) – (5.15) in (5.5) then into (5.28), yield the Bayes estimator under LINEX loss function, $\hat{\theta}_{2,L}$ of θ_2.

5.2.2 Bayes estimation of reliability function

Put $\varphi^*(\boldsymbol{\theta}) = e^{-cR(t)}$ in (5.5) for values $i,j = 1,2$, where $R(t)$ defined by (2.8), then

$\varphi_j^* = \frac{\partial\varphi^*}{\partial\theta_j} = -cp_j\, e^{-cR(t)}\, R_j(t)\varphi(t),$

$\varphi_{ij}^* = \frac{\partial^2\varphi^*}{\partial\theta_i\partial\theta_j} = c^2p_ip_jR_j(t)R_i(t)e^{-cR(t)}(\varphi(t))^2,$

$\varphi_{jj}^* = \frac{\partial^2\varphi^*}{\partial\theta_j^2} = -cp_j(\varphi(t))^2.$
$.R_j(t)e^{-cR(t)}\{p_jR_j(t) + 1\},$ (5.30)

$S_{ij} = \varphi_1^*\sigma_{11} + \varphi_2^*\sigma_{21},$

$S_{12} = -ce^{-cR(t)}\varphi(t)[p_1R_1(t)\sigma_{11} + p_2R_2(t)\sigma_{21}]\,,$

$S_{21} = -ce^{-cR(t)}\varphi(t)[p_2R_2(t)\sigma_{22} + p_1R_1(t)\sigma_{12}].$

Then

$v_{12} = -ce^{-cR(t)}\varphi(t)[p_1R_1(t)\sigma_{11}^2 + p_2R_2(t)\sigma_{12}\sigma_{11}],$

$v_{21} = -ce^{-cR(t)}\varphi(t)[p_2R_2(t)\sigma_{22}^2 + p_1R_1(t)\sigma_{21}\sigma_{22}],$

$c_{ij} = 3\varphi_i^*\sigma_{ii}\sigma_{ij} + \varphi_j^*(\sigma_{ii}\sigma_{jj} + 2\sigma_{22}^2),$

$c_{12} = -ce^{-cR(t)}\varphi(t)[3\sigma_{11}\sigma_{12}p_1R_1(t) + p_2R_2(t)[\sigma_{11}\sigma_{22} + 2\sigma_{12}^2],$

$c_{21} = -ce^{-cR(t)}\varphi(t)[3\sigma_{22}\sigma_{21}p_2R_2(t) + (\sigma_{22}\sigma_{11} + 2\sigma_{21}^2)p_1R_1(t)].$

Substituting (5.30) and (5.12) – (5.15) in (5.5) then into (5.28), yield the Bayes estimator under LINEX loss function, $\hat{R}_L$ of $R(t)$.

5.2.3 Bayes estimation of failure rate function

Put $\varphi^*(\boldsymbol{\theta}) = e^{-ch(t)}$ in (6.3) for values $i,j = 1,2$, where $h(t)$ defined by (2.11), then

$\varphi_j^* = \frac{\partial\varphi^*}{\partial\theta_j} = -\frac{cp_j}{(R(t))^2}e^{-ch(t)}\,\delta_j,$ (5.31)

where
$\delta_j = \{R(t)f_j(t)k_j(t) - f(t)R_j(t)\varphi(t)\},$

$\varphi_{jj}^* = -\frac{cp_j}{(R(t))^4}e^{-ch(t)}\{(R(t))^2[p_jf_j(t)R(t)\left(k_j(t)\right)^2 - R(t)f_j(t)\theta_j^{-2}\} + p_jk_j(t)f_j(t)R_j(t)\varphi(t)] - \frac{cp_j\delta_j^2}{(R(t))^2} - 2\,\delta_jp_jR_j(t)\varphi(t)\},$ (5.32)

$\varphi^*_{ij} = -cp_ip_j \frac{e^{-ch(t)}}{(R(t))^4}\{(R(t)^2[\frac{-c\delta_i\delta_j}{(R(t))^2} + f_j(t)k_j(t)R_i(t)\varphi(t) - \varphi(t)R_j(t)f_i(t)k_i(t)] -2\delta_jR(t)R_i(t)\varphi(t)\},$

$S_{ij} = \varphi^*_i\sigma_{ii} + \varphi^*_j\sigma_{ji},$

$$S_{12} = -\frac{ce^{-ch(t)}}{(R(t))^2}\{p_1\delta_1\sigma_{11} + p_2\delta_2\sigma_{21}\}$$

$S_{21} = -\frac{ce^{-ch(t)}}{(R(t))^2}\{p_2\delta_2\sigma_{22} + p_1\delta_1\sigma_{12}\},$

$v_{ij} = (\varphi^*_i\sigma_{ii} + \varphi^*_j\sigma_{ij})\sigma_{ii},$

$v_{12} = -\frac{ce^{-ch(t)}}{(R(t))^2}\{p_1\delta_1\sigma^2_{11} + p_2\delta_2\sigma_{12}\sigma_{11}\},$

$v_{12} = -\frac{ce^{-ch(t)}}{(R(t))^2}\{p_2\delta_2\sigma^2_{22} + p_1\delta_1\sigma_{21}\sigma_{22}\},$

$c_{ij} = 3\varphi^*_i\sigma_{ii}\sigma_{ij} + \varphi^*_j(\sigma_{ii}\sigma_{jj} + 2\sigma^2_{22}),$

$c_{12} = -\frac{ce^{-ch(t)}}{(R(t))^2}\{3p_1\delta_1\sigma_{11}\sigma_{12} + p_2\delta_2(\sigma_{11}\sigma_{22} + 2\sigma^2_{12})\},$

$c_{21} = -\frac{ce^{-ch(t)}}{(R(t))^2}\{3p_2\delta_2\sigma_{22}\sigma_{21} + p_1\delta_1(\sigma_{22}\sigma_{11} + 2\sigma^2_{21})\}.$

Substituting (5.31) and (5.12) – (5.15) in (5.5) then into (5.28), yield the Bayes estimator under LINEX loss function, $\hat{h}_L$ of $h(t)$.

6. Simulation Study

We obtained, in the above sections, Bayesian and non-Bayesian estimates of the vector parameters $\boldsymbol{\theta} = (\theta_1, \theta_2)$, reliability function $R(t)$ and failure rate function $h(t)$ of the MEFD. We can obtain Bayes estimation by using quadratic and LINEX loss functions. The MLE's are obtained as well. In order to assess the statistical performances of these estimates, a simulation study is performed for samples of different sizes and censoring percentages of 80% and 100% (complete sample case). We can use the mean square errors (MSE's) and biases to compare between these estimators.

The following algorithm will be used to generate the random samples and then calculate the estimators:

1. For given values of the prior parameters β_1 and β_2 one generate a random values for θ_1 and θ_2 from the gamma distributions $G(\beta_j, 1)$ for $j = 1,2$.
2. Using θ_1 and θ_2, obtained in step (1), one generate random samples of different sizes n= 30, 40 and 55 from MEFD as given by (2.6). The computations are carried out for such sample sizes and censored samples of sizes r= 24, 32, 44, respectively.
3. The MLE's $\hat{\boldsymbol{\theta}}_M = (\hat{\theta}_{1,M}, \hat{\theta}_{2,M})$ of the vector parameters $\boldsymbol{\theta} = (\theta_1, \theta_2)$ are obtained by solving (4.9) and (4.10) iteratively. The estimators $\hat{R}_M(t_0)$ and $\hat{h}_M(t_0)$ of the functions R(t) and h(t) are computed at some values t_0.
4. The Bayes estimate relative to squared error loss, $\hat{\boldsymbol{\theta}}_s = (\hat{\theta}_{1,s}, \hat{\theta}_{2,s})$, $\hat{R}_s$ and $\hat{h}_s$ are computed, using (5.5) together with the appropriate changes according to subsections (5.1.1), (5.1.2) and (5.1.3). Also, the Bayes estimates relative to LINEX loss, $\hat{\boldsymbol{\theta}}_L = (\hat{\theta}_{1,L}, \hat{\theta}_{2,L})$, $\hat{R}_L$ and $\hat{h}_L$ are computed, using (5.5) together with the appropriate changes according to subsections (5.2.1), (5.2.2) and (5.2.3).
5. The above steps (2-4) are repeated 1000 times, the biases and MSE are computed for different sample sizes n and censoring sizes r. In all above cases the prior parameters $\beta_1 = 2$, $\beta_2 = 1.5$ which yield the generated values as $\theta_1 = 1.8107$, $\theta_2 = 0.3841$ are preparing two real values. The true values of R(t) and h(t) when $t = t_0 = 0.5$, are computed to be $R(0.5) = 0.8571$ and $h(0.5) = 0.6409$.

The biases (first entries) and MSE's (second entries) are displayed in Tables 1-4. The computations are achieved under complete and censored samples.

Tables 1, 2, 3 and 4 contain estimated biases and MSE's of MLE's, Bayes estimators under squared error and LINEX loss functions of $\theta_1, \theta_2, R(t)$ and $h(t)$ respectively.

Table 1: The biases (first entries) and MSE's (second entries) of different estimators for shape parameter θ_1.

n	r	$\hat{\theta}_{1,M}$	$\hat{\theta}_{1,s}$	$\hat{\theta}_{1,L}$: $c = -1.5$
30	24	-1.22920	-0.80448	-0.81072
		1.55369	0.65683	0.65527
	30	-1.13009	-0.80617	-0.81007
		1.30913	0.65623	0.65205
40	32	-1.24623	-0.80729	-0.80764
		1.28505	0.65172	0.64953
	40	-1.14649	-0.80843	-0.80545
		1.23431	0.65167	0.64874
55	44	-1.25186	-0.22501	-0.78152
		1.22229	0.65063	0.61077
	55	-1.15190	-0.22542	-0.71635
		1.22200	0.65052	0.60967

Table 3: The biases (first entries) and MSE's (second entries) of different estimators for $R(t)$.

n	r	$\hat{R}_M(t)$	$\hat{R}_s(t)$	$\hat{R}_L(t)$: $c = -1.5$
30	24	-0.74826	0.03773	-0.10730
		0.20518	0.00278	0.00151
	30	-0.68702	0.03784	0.04017
		0.19668	0.00277	0.00161
40	32	-0.66089	0.03846	0.03465
		0.19620	0.00258	0.00161
	40	-0.68722	0.03877	0.04157
		0.17008	0.00181	0.00159
55	44	-0.66041	0.03729	0.03916
		0.16987	0.00173	0.00153
	55	-0.70191	0.03657	0.04235
		0.10884	0.00157	0.00153

Table 2: The biases (first entries) and MSE's (second entries) of different estimators for shape parameter θ_2.

n	r	$\hat{\theta}_{2,M}$	$\hat{\theta}_{2,s}$	$\hat{\theta}_{2,L}$: $c = -1.5$
30	24	0.29671	0.11685	0.09629
		0.40751	0.01139	0.19271
	30	0.29652	0.11599	0.11587
		0.35568	0.01138	0.01343
40	32	0.18038	0.11599	0.11266
		0.29327	0.01138	0.01269
	40	0.28011	0.11594	0.11155
		0.26450	0.01136	0.01258
55	44	0.17475	0.11592	0.11154
		0.11995	0.01135	0.01143
	55	0.27469	0.11591	0.11154
		0.07974	0.01034	0.01087

Table 4: The biases (first entries) and MSE's (second entries) of different estimators for $h(t)$.

n	r	$\hat{h}_M(t)$	$\hat{h}_s(t)$	$\hat{h}_L(t)$: $c = -1.5$
30	24	-0.36986	-0.73716	0.08041
		0.27103	0.94341	0.13736
	30	-0.42676	-1.11052	1.13388
		0.21414	0.83321	0.06679
40	32	-0.45463	-0.87382	1.38651
		0.18626	0.76356	0.06485
	40	-0.42445	-0.65665	1.16384
		0.18544	0.43119	0.06115
55	44	-0.45226	-0.55043	0.76678
		0.18463	0.12280	0.05137
	55	-0.43373	-0.44261	0.78023
		0.10716	0.12197	0.05020

6. Concluding Remarks

Based on results which obtained in Tables 1-4, we compared between maximum likelihood estimators, Bayes estimators under quadratic loss function and *LINEX* loss function for parameters, reliability and failure rate function for mixture exponentiated Frechet distribution with two components of *EFD* in case of complete and type II censoring samples. The Bayes estimators are derived in approximate forms by using Lindley's method.

Our observations about the results are stated in the following points:

1. Tables 1 and 3 show that the *MLE*'s are the best estimates as compared with the biases of estimates under squared error or *LINEX* loss functions. This is true for both complete and censored samples. It is immediate to note that *MSE*'s decrease as sample size increases. On the other hand, the Bayes estimates under the *LINEX* loss function have the smallest *MES*'s as compared with the other estimates in both complete and censored samples
2. In Table 2, the Bayes estimates under quadratic loss function have the smallest estimated *MSE*'s as compared with the estimates of the other methods for complete and censored samples. On the other hand, the Bayes estimates under the *LINEX* loss function have the best biases as compared with the others estimates. Also, we note that *MSE*'s usually decrease as a sample size increases.
3. In Table 4, the Bayes estimates under *LINEX* loss function have the smallest estimated *MSE*'s as compared with the other estimates. On the other hand, the Bayes estimates under the squared error loss function have the best biases as compared with the other methods for complete and censored samples. In general, we note that *MSE*'s usually decrease as a sample size increases.

From the previous observations, the estimations of a finite mixture of two *EF* components data is possible and flexible using Bayes approach, especially using asymmetric loss function such as LINEX function, which is the most appropriate for all parameters as shown throughout this article.

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