A Novel Trajectory Generation Scheme for Lunar Landing Exploration Mission

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Abstract: Landing trajectory generation scheme is one of the most important technologies for future lunar or planetary exploration mission. To achieve a precise trajectory of landing, an advanced guidance scheme is necessary. This research outlines a comparison of different methods of solution of trajectory generation scheme of lunar descent and proposes an novel solution that allows a full depiction of a descent vehicle motion from orbital states down to the final landing event. In the conventional method of solution there exist some poor assumptions such as, during descent a constant vertical gravitational acceleration is the only other force acting on the descent vehicle. This inadequate postulation limits the validity of the solutions to system with in very low altitude terminal descent area i.e., close to the lunar surface. In this research Note, an advanced descent solution is proposed where centrifugal acceleration term is retained along with the gravitational acceleration term. It allows a complete representation of the descent module motion from orbital speed conditions down to the final landing state. Mathematical derivations of new scheme are verified in terms of conventional scheme and the comparative simulation results for full integrated solution, conventional schemes and a proposed advanced scheme are demonstrated to test the performance.

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1. Introduction

Lunar landing is an exigent issue. A lot of scientists and engineers confirmed considerable interests in the past couple of decades [1-5, 12-16]. It is essential for a lunar lander to land vertically and softly on the lunar surface [6]. Gravity-turn descent is one of the solutions for this purpose and this type of descent technique which entails that the lander thrust vector is oriented opposed to the velocity vector along the complete flight path of the vehicle.[7)] Using inertial measurement unit, the information about the velocity vector can be identified to insert as an input of attitude control system that can maintain thrust vector parallel to the velocity vector instantaneously but in opposite direction as shown in Fig.1. The great benefit of using gravity-turn descent is to have guaranteed upright landing, and fuel consumption is optimal [8].

The equations of motion are solved in conventional descent solution considering a fixed thrust to mass ratio and assuming that a constant and vertical gravitational acceleration is the only other force acting on the descent vehicle ignoring the centrifugal acceleration term [7]. Also the lunar surface is imagined a plane flat surface. The conventional method of solution limits the validity to regimes where the descent vehicle velocity is very small relative to the local orbital velocity since centrifugal forces are unnoticed and therefore, it is only be used to describe terminal descent, when the vehicle has braked from orbital velocity and close to the lunar surface. Consequently the authors demonstrate an advanced method of descent solution for a spherical homogeneous lunar surface where the centrifugal forces are retained and descent can be initiated from its orbital speed condition. In this research, some logical values are examined to determine a better approximation for centrifugal acceleration term without ignoring it, but the gravity is assumed to be constant in magnitude. These assumptions are reasonable while the descent starts from vehicle's orbit [3]. The proposed advance solution over conventional descent method allows a full representation of descent module motion from orbiting condition down to final vertical landing situation. To represent the significant improvement in the new solutions, three steps are performed in this study and these are full integrated solution, conventional solution and advanced solution.

2. Fundamentals of Lunar Descent

Fundamental three dimensional equations of motion to describe the spacecraft proposition concerning a uniform sphere-shaped lunar body [9] are divided into two parts. One is the equations of spacecraft motions for dynamic states as follow:

$$\dot{u}(t) = g_l \cos\alpha - N\cos\beta \tag{1}$$

$$\dot{\alpha}(t) = \frac{1}{u}[(\frac{u^2}{y+y_l} - g_l)\sin\alpha - N\sin\beta\cos\phi] \tag{2}$$

$$\dot{\psi}(t) = \frac{1}{u\sin\alpha}[N\sin\beta\cos\phi] \tag{3}$$

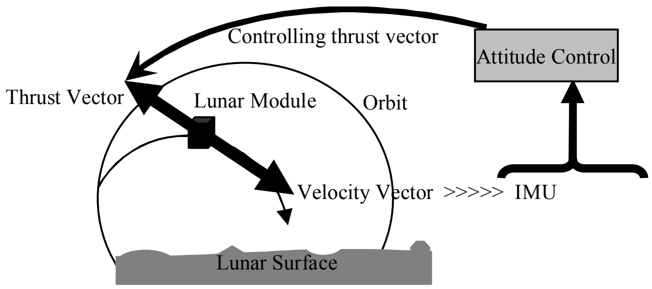

Fig. 1. Lunar descent technique

Where u is spacecraft velocity vector magnitude or spacecraft speed, g_l is lunar gravitational acceleration, N is ratio of thrust F and vehicle mass m, α is the pitch angle of the vehicle velocity vector relative to the local vertical, β is angle of thrust vector relative to reverse direction of spacecraft velocity, y is vehicle altitude from lunar surface, y_l is lunar radius, and ψ is cross range angle.

The remaining part to describe the fundamental equations of motion for kinematics states are

$$\dot{y}(t) = -u\cos\alpha \quad (4)$$

$$\dot{x}(t) = u\sin\alpha\cos\psi\frac{y_l}{y+y_l} \quad (5)$$

$$\dot{c}(t) = u\sin\alpha\sin\psi\frac{y_l}{y+y_l} \quad (6)$$

where x and c are horizontal span and cross range distance respectively.

To facilitate the simplification of mathematical operation, roll control states are held at zero ($\phi(t)=0$ and $\dot{\phi}(t)=0$). To activate the system as a plane motion, the initial states are initialized to zero ($c(0)=0$, $\psi(0)=0$, and $\dot{\psi}(0)=0$). Consequently the above governing equations are reduced to their two-dimensional form where Eqs. (1) and (4) remain same. Only the changes are observed as follow:

$$\dot{\alpha}(t) = \frac{1}{u}[(\frac{u^2}{y+y_l} - g_l)\sin\alpha - N\sin\beta] \quad (7)$$

and $$\dot{x}(t) = u\sin\alpha\frac{y_l}{y+y_l}$$

It is reasonable to assume that $y << y_l$ in order that $y_l/y+y_l \approx 1$. Then the equation for horizontal span becomes

$$\dot{x}(t) = u\sin\alpha \quad (8)$$

3. Complete Numerical Solution

To get a full integrated solution, right hand sides of the equations are reduced to function of velocity vector pitch angle α. For this purpose some reasonable assumptions are made regarding thrust to mass ratio, thrust vector angle, lunar gravitational acceleration force and lunar centrifugal force. To generate an ideal descent trajectory it is rational to assume a constant value for N i.e., F/m and g_l, and control input β is set to zero. But in the situation of constant thrust acceleration, m will not be constant and so F/m is varying. Yet, this error will be removed by the real time guidance algorithm. Therefore, using initial values for mass and gravity is a straightforward assumption for this solution.

To find the full integrated numerical solutions for speed u, time t, horizontal distance x and vertical distance y as a function of pitch angle α during power descend phase; authors has performed the mathematical derivations for simplification.

4. Conventional Descent Solution

The traditional solution for descent is obtained by assuming the lunar surface similar as a plane so that the lunar radius $y_l \rightarrow \infty$. Therefore, centrifugal acceleration term is ignored in order that $u^2/y+y_l = 0$ [7, 11]. In this perimeter, velocity vector pitch angle reduces to

$$\dot{\alpha}(t) = -\frac{g_l}{u}\sin\alpha - N\sin\beta \quad (9)$$

This reduced equation can be used to obtain a single, distinguishable differential equation with α as the self-regulating variable, such that

$$\dot{u}(\alpha) = \dot{u}(t)/\dot{\alpha}(t) = -\frac{u(g_l\cos\alpha - N)}{g_l\sin\alpha}.$$

$$\dot{u}(\alpha) = \frac{du}{d\alpha} = u_0\left[\frac{\sin\alpha_0}{\sin\alpha}\right]\left[\frac{\tan(\alpha/2)}{\tan(\alpha/2)}\right]^{\frac{N}{g_l}}\left[\frac{N}{g_l}\frac{1-\cos\alpha}{\sin(\alpha)}\right] \quad (10)$$

Now the solution for time, vertical and horizontal ranges can be acquired utilizing the value of speed u in to previously derived mathematical equations to obtain descent trajectory in terms of traditional descent solution. Therefore, acquired equations are for time,

$$\dot{t}_{go}(\alpha) = \frac{du}{d\alpha}\Big/\frac{du}{dt}$$

$$= u_0\left[\frac{\sin\alpha_0}{\sin\alpha}\right]\left[\frac{\tan(\alpha/2)}{\tan(\alpha/2)}\right]^{\frac{N}{g_l}}\left[\frac{N}{g_l}\frac{1-\cos\alpha}{\sin(\alpha)}\right]\left[\frac{1}{g_l\cos\alpha - N}\right] \quad (11)$$

vertical range, $$\dot{y}(\alpha) = \frac{dy}{dt}\Big/\frac{d\alpha}{dt}$$

$$= \frac{u^2}{g_l}\cot\alpha = u_0^2\left[\frac{\sin\alpha_0}{\sin\alpha}\right]\left[\frac{\tan(\alpha/2)}{\tan(\alpha_0/2)}\right]^{\frac{2N}{g_l}}\left[\frac{\cot\alpha}{g_l}\right] \quad (12)$$

and horizontal span,

$$\dot{x}(\alpha) = \frac{dx}{dt}\Big/\frac{d\alpha}{dt} = -\frac{u^2}{g_l} = -u_0^2\left[\frac{\sin\alpha_0}{\sin\alpha}\right]\left[\frac{\tan(\alpha/2)}{\tan(\alpha_0/2)}\right]^{\frac{2N}{g_l}}\left[\frac{1}{g_l}\right] \quad (13)$$

5. Advanced Lunar Descent SOlution

To solve those governing equations for the proposed advanced scheme, it is again necessary that the right hand sides of the equations are kept as a function of velocity vector pitch angleαconsidering further assumption for homogeneous spherical lunar surface and centrifugal acceleration term. Using initial values for mass and gravity is an uncomplicated assumption for qualitative solution. But for centrifugal acceleration term, a constant value *Γ* can be logically chosen which is defined as the ratio between centrifugal acceleration and lunar gravitational acceleration.

$$\Gamma = \frac{u^2}{y+y_l}\Big/ g_l$$

so $$\frac{u^2}{y+y_l} - g_l = -(1-\Gamma)g_l$$

With these assumptions and making consistent with the traditional lunar descent works, [7, 10, 11] speed can be recognized by following differential equations formulating as a function of velocity vector pitch angle α.

$$\dot{u}(\alpha) = \left(\dot{u}(t)/\dot{\alpha}(t)\right) = u\left[\frac{g_l\cos\alpha - N}{-(1-\Gamma)g_l\sin\alpha}\right]$$

or $$\frac{du}{u} = \left[\frac{g_l\cos\alpha - N}{-(1-\Gamma)g_l\sin\alpha}\right]d\alpha$$

This equation can now be directly integrated to obtain the descent velocity u as a function of the velocity vector pitch
angle α as

$$u(\alpha) = u_0 e^{\int_{\alpha_0}^{\alpha}\left[\frac{g_l\cos\alpha - N}{-(1-\Gamma)g_l\sin\alpha}\right]d\alpha}$$

now $$\int_{\alpha_0}^{\alpha}\left[\frac{g_l\cos\alpha - N}{-(1-\Gamma)g_l\sin\alpha}\right]d\alpha$$

$$= \ln\left(\frac{\sin\alpha}{\sin\alpha_0}\right)^{\frac{-1}{(1-\Gamma)g_l}} + \ln\left(\frac{\tan\frac{\alpha}{2}}{\tan\frac{\alpha_0}{2}}\right)^{\frac{-N}{(1-\Gamma)g_l}}$$

therefore,

$$u(\alpha) = u_0\left(\frac{\sin\alpha}{\sin\alpha_0}\right)^{\frac{-1}{(1-\Gamma)g_l}}\left(\frac{\tan\frac{\alpha}{2}}{\tan\frac{\alpha_0}{2}}\right)^{\frac{-N}{(1-\Gamma)g_l}}$$

as $$\tan(\alpha/2) = \frac{\sin\alpha}{2\cos^2(\alpha/2)} = \frac{\sin\alpha}{1-\cos\alpha}$$

and let that $$\tau = \frac{1}{(1-\Gamma)} \ \& \ \rho = N/g_l$$

where $\rho > 0$, in order that

$$u(\alpha) = u_0\left(\frac{\sin\alpha}{\sin\alpha_0}\right)^{-\tau}\left(\frac{\frac{\sin\alpha}{1-\cos\alpha}}{\frac{\sin\alpha_0}{1-\cos\alpha_0}}\right)^{-\tau\rho}$$

where *τ=1/(1-Γ)* is a measure of the centrifugal acceleration term. Then the solution for speed currently obtains the shape

$$u(\alpha) = u_0\left(\frac{\sin\alpha}{\sin\alpha_0}\right)^{-\tau(1+\rho)}\left(\frac{1-\cos\alpha_0}{1-\cos\alpha}\right)^{-\tau\rho} \quad (14)$$

Now the time to go $t_{go}(\alpha)$, horizontal span $x(\alpha)$ and vertical range $y(\alpha)$ are resolved in an identical manner of the conventional lunar descent solution as follow

$$\dot{t}_{go}(\alpha) = 1/\dot{\alpha}(t) = \frac{-\tau u(\alpha)}{g_l\sin\alpha}$$
$$= \frac{-\tau u_0}{g_l}\frac{(1-\cos\alpha_0)^{-\tau\rho}}{(\sin\alpha_0)^{-\tau(1+\rho)}}\frac{(\sin\alpha)^{-\tau(1+\rho)-1}}{(1-\cos\alpha)^{-\tau\rho}}$$
$$= G_{t_{go}}\frac{(\sin\alpha)^{-\tau(1+\rho)-1}}{(1-\cos\alpha)^{-\tau\rho}}, \quad (20)$$

where $$G_{t_{go}} = \frac{-\tau u_0}{g_l}\frac{(1-\cos\alpha_0)^{-\tau\rho}}{(\sin\alpha_0)^{-\tau(1+\rho)}}$$

now for horizontal span,

$$\dot{x}(\alpha) = \dot{x}(t)/\dot{\alpha}(t) = \frac{-\tau u(\alpha)^2\sin(\alpha)}{g_l\sin(\alpha)}$$
$$= \frac{-\tau u_0^2}{g_l}\frac{(1-\cos\alpha_0)^{-2\tau\rho}}{(\sin\alpha_0)^{-2\tau(1+\rho)}}\frac{(\sin\alpha)^{-2\tau(1+\rho)}}{(1-\cos\alpha)^{-2\tau\rho}}$$
$$= G_x\frac{(\sin\alpha)^{-2\tau(1+\rho)}}{(1-\cos\alpha)^{-2\tau\rho}}, \quad (15)$$

where

$$G_x = \frac{-\tau u_0^2}{g_l}\frac{(1-\cos\alpha_0)^{-2\tau\rho}}{(\sin\alpha_0)^{-2\tau(1+\rho)}}$$

and for vertical range,

$$\dot{y}(\alpha) = \dot{y}(t)/\dot{\alpha}(t) = -\frac{-\tau u^2(\alpha)\cos\alpha}{g_l\sin\alpha}$$
$$= \frac{\tau u_0^2}{g_l}\frac{(1-\cos\alpha_0)^{-2\tau\rho}}{(\sin\alpha_0)^{-2\tau(1+\rho)}}\frac{(\sin\alpha)^{-2\tau(1+\rho)}\cos\alpha}{(1-\cos\alpha)^{-2\tau\rho}\sin\alpha}$$
$$= G_y\frac{(\sin\alpha)^{-2\tau(1+\rho)-1}\cos\alpha}{(1-\cos\alpha)^{-2\tau\rho}}, \quad (16)$$

where $$G_y = \frac{\tau u_0^2}{g_l}\frac{(1-\cos\alpha_0)^{-2\tau\rho}}{(\sin\alpha_0)^{-2\tau(1+\rho)}}$$

6. Performance Test of Different Schemes

Figure 2 shows the comparison of different trajectory responses for spacecraft descent on lunar surface while the governing equations are solved by complete integration method, conventional descent illumination, and proposed advanced scheme. Conventional scheme is demonstrated with out taking any approximation regarding centrifugal acceleration term. Equations for different states are numerically integrated with the constant value for g_l, N, u_0 and α_0 where g_l = 1.623 m/sec^2, N = $4N/kg$, u_0 = 1688 m/sec and α_0 = 90 degree.

It can be realized that there is differences in responses between traditional descent solutions and numerically integrated solutions to the equations of lunar module motion. Traditional descent solutions has largest impact on the final altitude variation with respect to full integrated solutions due to the centrifugal acceleration term is ignored. Therefore it needs to perform further analysis with new advanced scheme for lunar descent and landing.

To integrate the above equations in a qualitative manner, the value for τ must be an integer. This entails τ = 1, 2, 3, 4 . . . . Instead of this solution, directly the ratio Γ, which is mentioned earlier, can be chosen some fractional values to make τ as an integer. But the authors found better results having directly the integer logical values to get a qualitative integration of these equations. Choosing a logical value directly for the τ proves more preciseness in approximation as well. The influences of differing the constant τ is demonstrated in Figures. 3(a), 3(b), 3(c) and 3(d). Unlike values (1, 2, 3, 4, 5, …) for τ are employed into Equations and these equations are numerically integrated with constant approximate values for g_l and N whereas g_l = 1.623 m/sec^2 and N = 5N/kg. Initial and final values for the velocity vector pitch angle α is taken 90[deg.] and 0[deg.] while the initial speed u_0 is considered as approximate orbital speed, 1688 m/sec. In contrast of this advanced solution, full numerical integrated resolution are performed for comparison taking same approximation for β, g_l, N, α and u_0 as it is made while no estimation are made about the centrifugal acceleration.

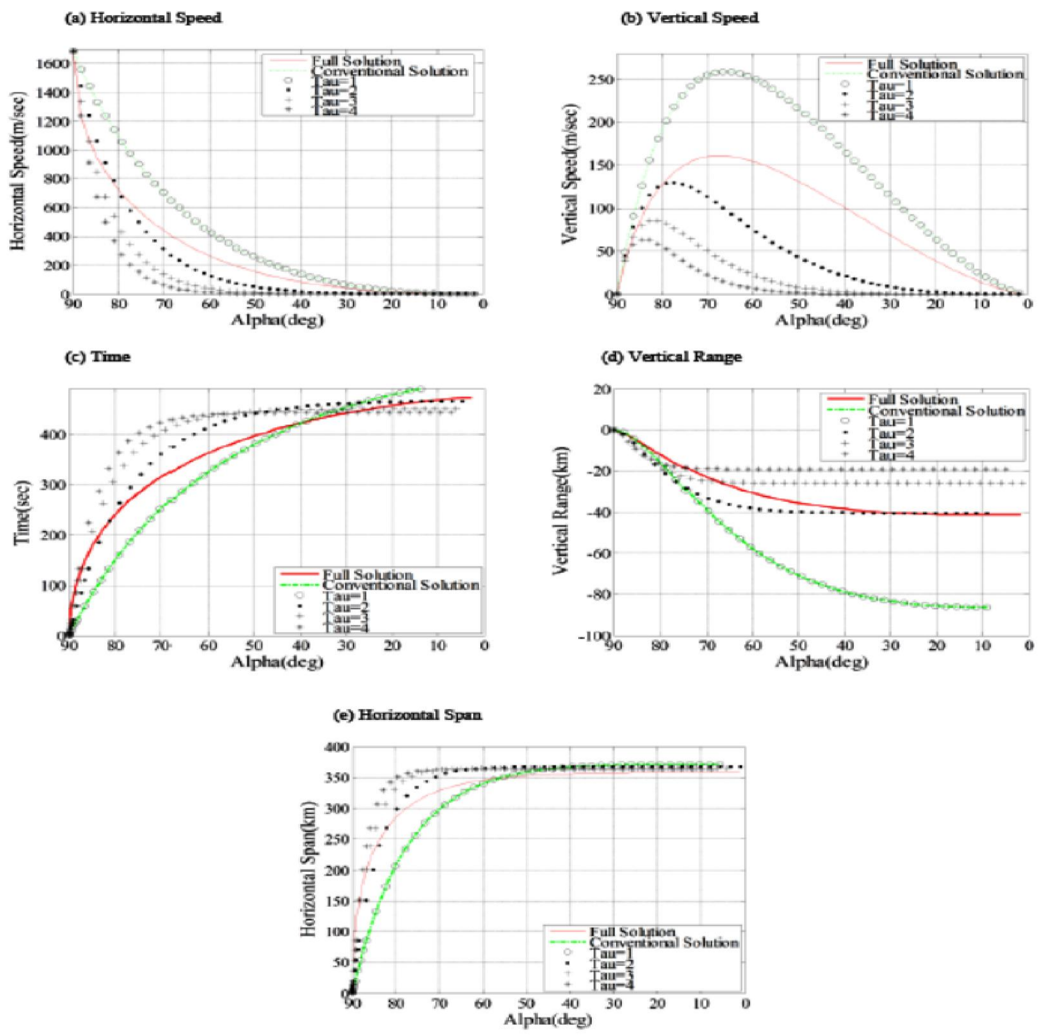

Fig. 2. Comparison of Advanced Solution to full Integrated Solution and Conventional Solution: Speed, Time, Vertical Range and Horizontal Span.

A comprehensive evaluation of this advanced solution with traditional descent solution, and a numerically integrated solution to the full equations of lunar landing mission are exposed in this investigation. It can be noted that varying τ has reasonable impact on different responses specially it directly influences the vertical range of the trajectory. From the assessment of the various values for τ, a value of $\tau = 2$ emerges to be a realistic number and improves on different responses of advanced solutions for speed, time, vertical range and horizontal span over traditional solutions. More over an accurate verification of mathematical calculation of proposed advanced scheme is observed by investigating the responses in Figs. 2. In the figures it produces exactly same results between conventional descent solution and the solution choosing a value of $\tau = 1$. On the other way it can be reproduced the same assumption for conventional descent solutions putting this value of $\tau = 1$ in to previous equations to prove in which centrifugal acceleration term is ignored.

7. Elapsed Time Analysis for Different Schemes

While the online trajectory generation is a great challenge for lunar or planetary landing, it becomes useful to compare elapsed time spent to solve the numerical calculations producing trajectory response on board. Table 1 shows a computing time performance analysis of different schemes for 15 numbers of runs. Among different responses of proposed advanced solutions, response for taking $\tau = 2$ is observed separately because this response is much attractive than the traditional scheme. Executable time is always less or same for this advanced scheme with respect to the traditional solution. Fig. 3 shows the average elapsed time comparison among different methods of descent scheme.

Table 1: Computing time performance analysis (sec.)

Computing Time Performance Analysis:			Advanced Solution			
No. of Hits	Full Sol^n	Conven-tional Solution	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$
01	3.718	0.0313	0.0000	**0.0156**	0.0156	0.0156
02	3.843	0.0000	0.1560	**0.3130**	0.0156	0.0000
03	3.671	0.0313	0.0156	**0.0000**	0.0313	0.0000
04	3.640	0.0313	0.0000	**0.0313**	0.0000	0.0000
05	3.750	0.0313	0.0000	**0.0156**	0.0156	0.0000
06	3.812	0.0313	0.0000	**0.0313**	0.0000	0.0313
07	3.703	0.0313	0.0000	**0.0156**	0.0000	0.0313
08	3.640	0.0313	0.0156	**0.0000**	0.0000	0.0313
09	3.640	0.0313	0.0000	**0.0313**	0.0000	0.0313
10	3.640	0.0313	0.0000	**0.0313**	0.0000	0.0156
11	4.593	0.2344	0.0000	**0.0156**	0.0156	0.0000
12	6.093	0.3125	0.1250	**0.0496**	0.0313	0.3125
13	5.437	0.0000	0.0313	**0.0000**	0.0313	0.0000
14	5.296	0.2500	0.0000	**0.0313**	0.0000	0.0313
15	5.828	0.0156	0.0000	**0.0000**	0.0156	0.0313
AVE	4.287	0.0729	0.0229	0.0387	0.0114	0.0354

8. Algorithm design for Reference Trajectory Generation

With the help of previously derived equations, a reference trajectory generation algorithm can be designed which will be a function of horizontal span, vertical range, speed, and velocity vector pitch angle. This type of trajectory is unspecified to any certain location on the lunar surface and it is a great benefit of this type of reference trajectory generation technique.

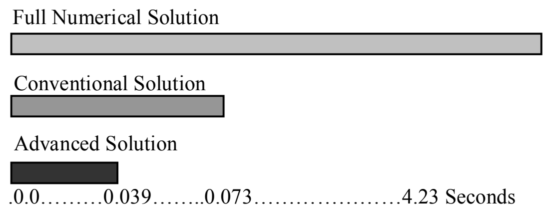

Fig. 3 Average time line comparison

The proposed advanced solutions for speed, vertical range, and horizontal span equations of can be derived. Assuming $\tau = 2$ and integrating the equations, the new form for time is shown by

$$t(\alpha) = G_{tgo}[\frac{1}{4\rho(\rho^2-1)}(1-\cos(\alpha))^{2\rho}$$
$$(2\rho^2(\csc\alpha)^2 + 2\rho\cot\alpha\csc\alpha - 1)(\sin\alpha)^{-2\rho}] + C_t$$

where $C_t = t_0 - t(\alpha_0)$

and t_0 is the initial time. The solution for vertical range is

$$y(\alpha)=G_y[((1-\cos\alpha)^{4\rho}((32\rho^2-8)\csc^4\alpha-2\rho(16\rho^2+3\cos 2\alpha-7)\cot\alpha\csc^3\alpha+24\rho^2\csc^2\alpha+3)\sin^{-4\rho}\alpha)/(32(4\rho^4-5\rho^2+1))]+C_y$$

where $C_y=y_0-y(\alpha_0)$ and y_0 is the initial altitude. Let,

$$Y(\alpha,\rho)=\frac{(32\rho^2-8)+2\rho(16\rho^2+3\cos 2\alpha-7)\cos\alpha}{32(4\rho^4-5\rho^2+1)}+\frac{-24\rho^2(\sin\alpha)^2+3(\sin\alpha)^4}{32(4\rho^4-5\rho^2+1)}$$

and $\frac{2u(\alpha)^2}{g_l}=\frac{2u_0^2}{g_l}\frac{(1-\cos\alpha_0)^{-4\rho}}{(\sin\alpha_0)^{-4(1+\rho)}}\frac{(\sin\alpha)^{-4\rho-4}}{(1-\cos\alpha)^{-4\rho}}$

so that

$$y(\alpha)=y_0+\frac{2u(\alpha)^2}{g_l}Y(\alpha,\rho)-\frac{2u_0^2}{g_l}Y(\alpha_0,\rho) \quad (17)$$

Where the equation for speed considering $\tau=2$ is

$$u(\alpha)=u_0\left(\frac{\sin\alpha}{\sin\alpha_0}\right)^{-2(1+\rho)}\left(\frac{1-\cos\alpha_0}{1-\cos\alpha}\right)^{-2\rho} \quad (18)$$

The solution for horizontal span is

$$x(\alpha)=G_x[((1-\cos\alpha)^{4\rho}(128\rho^3+24\rho\cos 2\alpha-32\rho+(96\rho^2-9)\cos\alpha+3\cos 3\alpha)\sin^{-4\rho-3}\alpha)/(2(256\rho^4-160\rho^2+9))]+C_x$$

where $C_x=x_0-x(\alpha_0)$ and x_0 is the initial horizontal distance from landing spot. let

$$X(\alpha,\rho)=-\frac{\sin\alpha(128\rho^3+24\rho\cos 2\alpha-32\rho+(96\rho^2-9)\cos\alpha+3\cos 3\alpha)}{2(256\rho^4-160\rho^2+9)}$$

so that

$$x(\alpha)=x_0+\frac{2u(\alpha)^2}{g_l}X(\alpha,\rho)-\frac{2u_0^2}{g_l}X(\alpha_0,\rho) \quad (19)$$

To develop an algorithm of generating reference trajectory it is comprised of n steps where thrust to mass ratio, N is unchanged over each step and the combination of all the steps results in a collective horizontal span and vertical range. Different steps are joined by the continuity conditions that the speed and velocity vector pitch angle be continuous over each link. Let the initial horizontal span and vertical range of the complete trajectory are x_0 and y_0 and terminal values these parameters are x_t and y_t. Now the complete displacements in horizontal and vertical direction are characterized by

$$\delta x=x_t-x_0=\sum_{i=1}^{n}\delta x_i \quad (20)$$

and

$$\delta y=y_t-y_0=\sum_{i=1}^{n}\delta y_i \quad (21)$$

The ending speed and the ending velocity vector pitch angle for the total trajectory are also associated to steps by the reality that final velocity vector pitch angle, α_t, is the same as the n^{th} vector pitch angle, α_n and the ending speed. u_t is the same as the n^{th} speed, u_n. The individual step's change in horizontal span and vertical range are shown by

$$\delta x_i=\frac{2u_i^2}{g_l}X(\alpha_i,\rho_i)-\frac{2u_{i-1}^2}{g_l}X(\alpha_{i-1},\rho_i)$$

$$\delta y_i=-\frac{2u_i^2}{g_l}Y(\alpha_i,\rho_i)+\frac{2u_{i-1}^2}{g_l}Y(\alpha_{i-1},\rho_i)$$

where α_{i-1} and α_i are the velocity vector pitch angle at the beginning and at the end of i^{th} step. Moreover, the speed at the end of i^{th} step is known by

$$u_i=\frac{u_{i-1}(1-\cos\alpha_{i-1})^{-2\rho_i}}{(\sin\alpha_{i-1})^{-2(1+\rho_i)}}\frac{(\sin\alpha_i)^{-2(1+\rho_i)}}{(1-\cos\alpha_i)^{-2\rho_i}} \quad (28)$$

where u_{i-1} and u_i are the speed at the beginning and at the end of i^{th} step and

$$\rho_i=\frac{N_i}{g_{l_i}}$$

where $N_i=F/m_i$ and ρ_i is constant over the step.

Actually the objective is stipulate the traveling distance both in vertical and horizontal direction by evaluating the velocity vector pitch angle and speed at the beginning and end of the total trajectory ($\alpha_0,\alpha_t,u_0,u_t$). This would result in a preferred total horizontal span and vertical range given a set of values for n values of N_i. By these identities and n steps, there exist $n+2$ equations and $3n+4$ variables ($\delta x,\delta y,\alpha_0,u_0,n\times\alpha_i,n\times u_i,n\times N_i$).

If a single step is considered to describe the total trajectory, there exist only three equations where n = 1 and where i = 1) with seven variables ($\alpha_0,\alpha_1,u_0,u_1,\delta x,\delta y,N$). Given the wish to identify six of those parameters ($\alpha_0,\alpha_1,u_0,u_1,\delta x,\delta y$), the difficulty is rapidly over-constrained. This problem can be overcome to bring under constraints by accumulating more steps to the solution.

For example, if two steps are considered with the same objective (identify $\alpha_0,\alpha_t,u_0,u_t,\delta x,\delta y$), the problems become constrained. There exist four equations where n = 2 and two from other equation with i = 1 and i = 2) with ten variables ($\alpha_0,\alpha_1,\alpha_2,u_0,u_1,u_2,\delta x,\delta y,N_1,N_2$). The objective is to figure out α_1,u_1,N_1 and N_2 given the other six predefined variables ($\alpha_0, \alpha_2, u_0, u_2, \delta x, \delta y$).

It is easy to compute u_1 if α_1, N_1 and N_2 are already known together with α_0 and u_0 or α_2 and u_2:

$$u_1 = \frac{u_0(1-\cos\alpha_0)^{-2\rho_1}}{(\sin\alpha_0)^{-2(1+\rho_1)}}\frac{(\sin\alpha_1)^{-2(1+\rho_1)}}{(1-\cos\alpha_1)^{-2\rho_1}} = \frac{u_2(1-\cos\alpha_2)^{-2\rho_2}}{(\sin\alpha_2)^{-2(1+\rho_2)}}\frac{(\sin\alpha_1)^{-2(1+\rho_2)}}{(1-\cos\alpha_1)^{-2\rho_2}} \quad (22)$$

This parity can be used to compute u_1. First,

$$u_2 = \frac{u_0(1-\cos\alpha_0)^{-2\rho_1}}{(\sin\alpha_0)^{-2(1+\rho_1)}}\frac{(\sin\alpha_1)^{-2\rho_1+2\rho_2}}{(1-\cos\alpha_1)^{-2\rho_1+2\rho_2}}\frac{(\sin\alpha_2)^{-2(1+\rho_2)}}{(1-\cos\alpha_2)^{-2\rho_2}} \quad (23)$$

or,

$$\frac{u_2}{u_0}\frac{(1-\cos\alpha_2)^{-2\rho_2}}{(\sin\alpha_2)^{-2(1+\rho_2)}}\frac{(\sin\alpha_0)^{-2(1+\rho_1)}}{(1-\cos\alpha_0)^{-2\rho_1}} = \frac{(\sin\alpha_1)^{-2\rho_1+2\rho_2}}{(1-\cos\alpha_1)^{-2\rho_1+2\rho_2}} = \frac{(1-\cos\alpha_1)^{2\rho_1-2\rho_2}}{(\sin\alpha_1)^{2\rho_1-2\rho_2}}$$

as we know $\tan(\alpha_1/2) = \dfrac{1-\cos\alpha_1}{\sin\alpha_1}$

$$\frac{u_2}{u_0}\frac{(1-\cos\alpha_2)^{-2\rho_2}}{(\sin\alpha_2)^{-2(1+\rho_2)}}\frac{(\sin\alpha_0)^{-2(1+\rho_1)}}{(1-\cos\alpha_0)^{-2\rho_1}} = \left(\tan(\alpha_1/2)\right)^{2\rho_1-2\rho_2}$$

therefore,

$$\alpha_1 = 2\left(\tan^{-1}\left(\frac{u_2}{u_0}\frac{(1-\cos\alpha_2)^{-2\rho_2}}{(\sin\alpha_2)^{-2(1+\rho_2)}}\frac{(\sin\alpha_0)^{-2(1+\rho_1)}}{(1-\cos\alpha_0)^{-2\rho_1}}\right)^{\frac{1}{2\rho_1-2\rho_2}}\right) \quad (24)$$

From here the two steps trajectory can be evaluated for horizontal span and vertical range:

$$\delta x = \frac{2u_1^2}{g_l}X(\alpha_1,\rho_1) - \frac{2u_0^2}{g_l}X(\alpha_0,\rho_1) + \frac{2u_2^2}{g_l}X(\alpha_2,\rho_2) - \frac{2u_1^2}{g_l}X(\alpha_1,\rho_2) \quad (25)$$

$$\delta y = -\frac{2u_1^2}{g_l}Y(\alpha_1,\rho_1) + \frac{2u_0^2}{g_l}Y(\alpha_0,\rho_1) - \frac{2u_2^2}{g_l}Y(\alpha_2,\rho_2) + \frac{2u_1^2}{g_l}Y(\alpha_1,\rho_2) \quad (26)$$

The solution is to specify α_0, α_2, u_0 and u_2 and search for reasonable values N_1 and N_2 to generate flight path space. An expected trajectory can be chosen from available options.

A sample reference trajectory space is shown in fig. 4. Boundary condition for velocity vector pitch angles were set at 89[deg] and 0.1[deg]. The initial and final speeds were set at 1688 m/sec. and 8 m/sec. The gravity is considered as constant in this analysis where the values for thrust to mass ratios, N_1 and N_2 are varied from 0.1 N/kg to 10 N/kg with 0.25 N/kg increments. Fig 6 shows the impact of varying the thrust acceleration and it appears to have set of curves whereas each curve is created varying N_1 and N_2 simultaneously. Three dimensional vies are demonstrated in Figs. 5 and 6.

Conclusion

The conventional lunar descent and landing problem has been advanced to allow an accurate representation of lunar descent from orbital condition. Finding a reasonable assumption for lunar surface and centrifugal acceleration, it significantly advanced the sphere of validity of the traditional gravity-turn solution from low velocity terminal descent to a complete descent from orbital situation. The accessibility of the descent velocities, time, vertical range and horizontal span as a function of the velocity vector pitch angle could be utilized to lessen the computational trouble on real-time lunar descent guidance scheme for future landing mission.

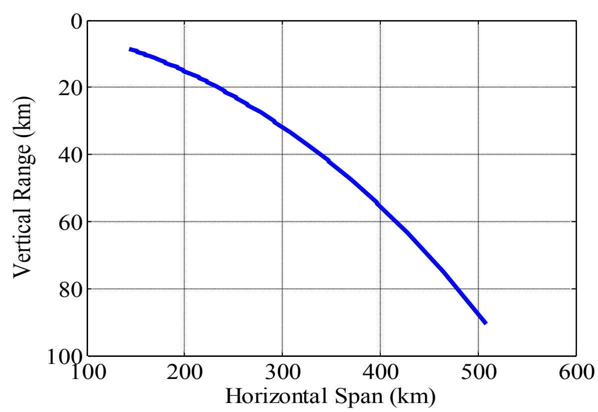

Fig. 4: Sample Trajectory Space Varying N

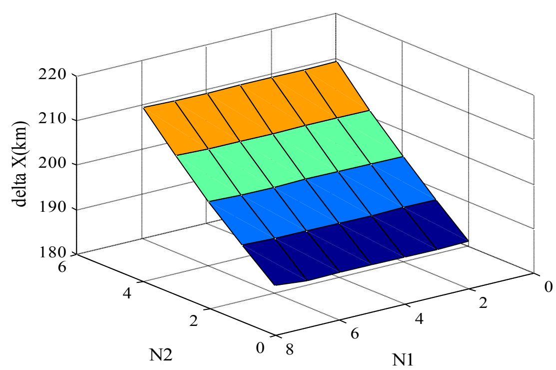

Fig. 5: 3D View of Horizontal Span Sample Space

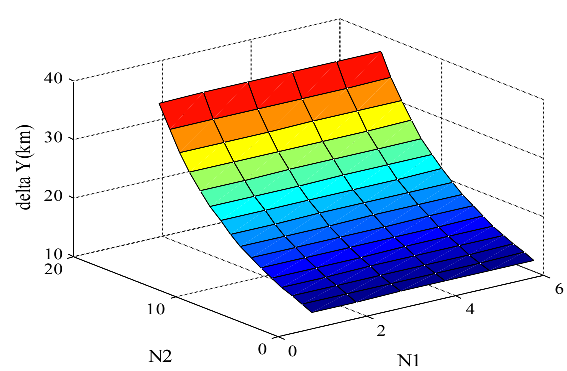

Fig. 6: 3D View of Vertical Range Sample Space

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1/2/2014