Krein's method *and mixed* integral equation of Volterra – Fredholm *type*

R. T. Matoog

Department of Mathematics, Faculty of Applied Sciences, Umm Al-Qura University, Makkah, Saudi Arabia, P.O. Box 171
r.matoog@yahoo.com

Abstract: Here, the existence of a unique solution of Volterra – Fredholm integral equation (**V-FIE**) of the first kind is considered in the space $L_2[-1,1]\times C(0,T)$, T< 1. The Fredholm integral term is considered in position with discontinuous kernel, while the Volterra integral term is considered in time with continuous kernel. Using a numerical, we have system of Fredholm integral equations (**SFIEs**) of the first find. Then, using Krein's method, the solution of **SFIEs** is obtained in the form of spectral relationships (**SRs**). Finally, many special cases in fluid mechanics and contact problems are discussed.
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1. Introduction:

Singular integral equations of the first kind have received considerable interest in the mathematical literatures, because of their many field of applications in different areas of sciences, for example see **[1- 4]**. The solution of these **IEs** can be obtained analytically using one of the following methods: Cauchy method **[5]**, potential theory method [6], orthogonal polynomials method **[7]**, integral transformation methods **[4-7]** and Krein's method **[8]**. **Mkhitarian and Abdou**, **[9]** discussed some different methods for solving the **FIE** of the first kind with logarithmic kernel.

In this work, we consider the **V-FIE** of the first kind

$$\int_0^t\int_{-a}^{a}F(t,\tau)k\left(\frac{x-y}{\lambda}\right)\varphi(y,\tau)\,dy\,d\tau=\pi f(x,t)=\pi\left[r(t)-f_*(x)\right]$$

$$\left((x,y)\in[-1,1],\ (t,\tau)\in[0,T],\ T\le 1;\ \ \lambda\in(0,\infty)\right) \quad (1.1)$$

$$k(z)=\int_0^\infty\frac{L(v)\cos v z}{v}\,dv,$$

$$L(v)=\frac{m+v}{1+v},\quad m\ge 1, \quad (1.2)$$

under the condition

$$\int_{-a}^{a}\varphi(x,t)\,dx=P(t) \qquad (1.3)$$

The function $L(v)$ is continuous and positive for $\mathbf{v}\in(0,\infty)$ and satisfies the following asymptotic equalities

$$L(v)=m-(m-1)v+O\left(v^3\right),\quad \mathbf{v}\to 0$$

$$L(v)=1-\frac{m-1}{v}+O\left(\frac{1}{v^3}\right)\quad (v\to\infty,\quad m\ge 1) \qquad (1.4)$$

The **V-FIE** (1.1), under the condition (1.2), can be investigated from the contact problem of a rigid surface (G,υ) having an elastic material occupying the domain $[-a,a]$, where $f_*(x)$ is describing the surface base of a stamp. This stamp is impressed into an elastic layer surface by a variable known force $P(\mathbf{t})$, $t\in[0,T]$, $T<1$, whose eccentricity of application $e(t)$, that case a rigid displacement $\gamma(t)$. Here, $\mathbf{G}$ is called the displacement magnitude and υ is Poisson's coefficient.

In order to guarantee the existence of unique solution of (2.1), we assume, for the two constants E and D, the following conditions:

(i) The kernel of position satisfies

$$\left\{\int_{-a}^{a}\int_{-a}^{a}k^2\left(\frac{x-y}{\lambda}\right)dx\,dy\right\}^{\frac{1}{2}}=E,$$

(ii) The positive continuous kernel, which represents the resistance force of the material, $F(t,\tau)\in C([0,T]\times[0,T])$ and satisfies $F(t,\tau)<D$,

(iii) The continuous function of time $\gamma(t)\in C[0,T]$, while the position function $f_*(x) \in L_2[-a, a]$ and $f(x, t) \in L_2[-a, a] \times C[0, T]$. The norm of $f(x, t)$ is

$$\| f \|_{L_2 \times C} = \max_t \int_0^t \left(\int_{-a}^{a} f^2(x, \tau)\, dx \right)^{\frac{1}{2}} d\tau$$

.

(iv) The unknown potential function $\varphi(x, t)$ satisfies Hölder condition with respect to time and Lipschitz condition with respect to position.

In this work, we use a numerical method to transform the **V-FIE** into linear **SFIEs** of the first kind. Then, using Krein's method, the solution of **SFIEs** can be obtained in the form of spectral relationships (**SRs**) of **CPs**. Many special cases are derived and discussed from the work. Moreover, some applications in contact problems and fluid mechanics are considered.

2. System of FIEs. If we divide the interval [0,T], $0 \le t \le T \le 1$ as $0 \le t_0 < t_1 < \cdots < t_N = T$, when $t = t_k,\ k = 0, 1, 2, \ldots, \ell$. The **V-FIE** (1.1) takes the form, see **[2]**

$$\int_0^t \int_{-a}^{a} F(t, \tau) k(x, y) \varphi(y, \tau) dy d\tau = \sum_{j=0}^{k} u_j F_{j,k} \int_{-a}^{a} k(x, y) \phi_j(y) dy = \pi f_k(x). \quad (2.1)$$

In (2.1) we neglect the error term, $O\left(\hbar_\ell^{p+1}\right)$ where $\hbar_\ell = \max h_j$, $h_j = t_{j+1} - t_j$. The constant u_j defined as the characteristic number, see [2]. Also we used the following notations

$$\varphi(x, t_\ell) = \varphi_\ell(x),$$

$$F(t_\ell, t_j) = F_{\ell, j},$$

$$f(x, t_\ell) = f_\ell(x). \quad (2.2)$$

The boundary condition (1.3), becomes

$$\int_{-a}^{a} \phi_k(x)\, dx = P_k \quad (P_k \text{ are constants}), \quad (2.3)$$

Let, in (1.2), $m = 1$ and $\lambda \to \infty$, such that the term $(\mathbf{x} - \mathbf{y})$ is very small, then using the relation **[7]**

$$\int_0^\infty \frac{\cos v z}{v} dv = -\ln z + d \quad (d \text{ is a constant}), \quad (2.4)$$

the conditions (2.3), take the form

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \frac{[-\ln(x - y) + d]}{\sqrt{a^2 - y^2}} T_{n_j}\left(\tfrac{y}{a}\right) dy = \begin{cases} \pi P_\ell \left[\ln\left(\tfrac{2}{a}\right) + d \right] & n_\ell = 0 \\ \pi T_{n_\ell}\left(\tfrac{x}{a}\right), \ \ell = 1, 2, \ldots, N & n_\ell \ge 1 \end{cases} \quad (3.1)$$

where $T_{\mathbf{n_j}}(\mathbf{x})$, j = 1, 2, …, ℓ are the **CPs** of the first find and order n .

Proof: The proof of (3.1) depends on the following lemmas

Lemma 1: For all positive integers n_j, $\mathbf{a} = 1$, we have

$$I_{n_j}(u) = 2\left[\frac{1}{2} P_{n_j}^{(-1,0)}\left(2u^2 - 1\right) + n_j u^2 \ln\left(\tfrac{2}{u} + d\right).P_{n_j - 1}^{(0,1)}\left(2u^2 - 1\right) \right] \quad (3.2)$$

where $P_{n_j}^{(\alpha,\beta)}(x)$ are Jacobi polynomials (**JPs**)

Proof: For proving (3.2) let $\mathbf{g}_\ell^+(\mathbf{y}) = T_{2\mathbf{n}_\ell}(\mathbf{y})$, where $T_{2\mathbf{n}_\ell}(\mathbf{y})$ are the **CPs** of the first kind, then (2.8) can be written in the form

$$I_{n_j}(u) = \frac{2}{\pi}\left[D_{n_\ell}(u) + u \ln\left(\frac{2}{u} + d\right) \frac{d}{du} D_{n_\ell}(u) \right] \quad (3.3)$$

where

$$D_{n_\ell}(u) = \int_0^u \frac{T_{2n_\ell}(s)\, ds}{\sqrt{u^2 - s^2}}. \quad (3.4)$$

Using the substation $\mathbf{s} = \mathbf{ut}$ and the relation $T_{2\mathbf{n_j}}(\mathbf{x}) = T_{\mathbf{n_j}}\left(2\mathbf{x}^2 - 1\right)$, the formula (3.4) takes the form

$$D_{n_\ell} = \int_0^1 \left(1 - t^2\right)^{\frac{-1}{2}} T_{n_\ell}\left(2t^2 u^2 - 1\right) dt. \quad (3.5)$$

Using the famous relation between **CPs**, **LPs** and **JPs**, see **[10]**

$$\int_{-1}^{1} \left(1 - t^2\right)^{-1/2} T_{n_j}\left(1 - t^2 y\right) dt = \frac{\pi}{2}\left[P_{n_j}(1 - y) + P_{n_j - 1}(1 - y) \right],$$

$$2 P_{n_j}^{(-1,0)}(x) = P_{n_j}(x) - P_{n_j - 1}(x), \quad (3.6)$$

where $P_{\mathbf{n_j}}(\mathbf{x})$ are Legendre polynomials (**LPs**), the formula (3.5), yields

$$D_{n_\ell}(u) = \frac{\pi}{2} P_{n_\ell}^{(-1,0)}\left(2u^2 - 1\right) \quad (3.7)$$

Also, the first derivative of (3.7) takes the form

$$\frac{dD_{n_\ell}(u)}{du} = n_\ell \pi u \mathrm{P}_{n_\ell - 1}^{(0,1)}\left(2u^2 - 1\right)$$

, $\mathbf{n}_\ell$ **= 1, 2,…;** ℓ **. (** ℓ **= 1, 2, …, N)** **(3.8)**

($\mathrm{P}_{n_\ell}^{(\alpha,\beta)}(x) = 0$ **for negative integer)**

.

Finally, introducing (3.7), (3.8) in (3.3) , we obtain the required result.■

Corollary1: Put $\mathbf{u} = 1$ in (3.2), we have

$$D_{n_\ell}(1) = 2\left[\frac{1}{2}\mathrm{P}_{n_\ell}^{(-1,0)}(1) + n_\ell \ln(2+d).\mathrm{P}_{n_\ell}^{(0,1)}(1)\right] \tag{3.9}$$

Using the famous relation $\mathrm{P}_{\mathbf{n}}^{(\alpha,\beta)}(1) = \dfrac{\Gamma(\mathbf{n}+\alpha+1)}{\mathbf{n}!\,\Gamma(1+\alpha)}$, the formula (3.9) becomes

$$D_{n_\ell}(1) = 2n_\ell \ln(2+d) \tag{3.10}$$

where $\Gamma(\mathbf{x})$ is the Gamma function .

Corollary2: The value of the second derivatives $\dfrac{d}{du}\left(u\dfrac{dD_{n_\ell}}{du}\right)$ is given by

$$\frac{d}{du}\left(u\frac{dD_{n_\ell}}{du}\right) =$$

$$D_{n_\ell}^{(2)}(u) = 2n_\ell \pi u\left[\mathrm{P}_{n_\ell-1}^{(0,1)}\left(2u^2-1\right) + (n_\ell+1)u^2\mathrm{P}_{n_\ell-2}^{(1,2)}\left(2u^2-1\right)\right]$$

. (3.11)

Lemma 2: The value of the following integral

$$\mathrm{A}_{\mathbf{n}_\ell}(\mathbf{x}) = \int_{\mathbf{x}}^{1}\frac{\mathbf{du}}{\sqrt{\mathbf{u}^2-\mathbf{x}^2}}\frac{\mathbf{d}}{\mathbf{du}}\left[\mathbf{u}\frac{\mathbf{d}}{\mathbf{du}}\int_0^{\mathbf{u}}\frac{\mathrm{T}_{2\mathbf{n}_\ell}(\mathbf{s})\mathbf{ds}}{\sqrt{\mathbf{u}^2-\mathbf{s}^2}}\right], \tag{3.12}$$

takes the form

$$\mathrm{A}_{n_\ell}(x) = \frac{\pi^{\frac{3}{2}} n_\ell!}{\sqrt{2}\,\Gamma\left(n_\ell - \frac{1}{2}\right)}\cdot\frac{1}{\sqrt{1-y}}\left[\frac{1-y}{2n_\ell - 1}\mathrm{P}_{n_\ell-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(y) - (1+y)\mathrm{P}_{n_\ell-1}^{\left(\lambda-\frac{1}{2},\frac{3}{2}\right)}(y)\right]$$

$$+\frac{\sqrt{2}n_\ell\pi}{\sqrt{1-y}}$$

$$, \quad y = 2x^2 - 1$$

$$, n_\ell = 1,2,\ldots;\ \ell = 1,2,\ldots,\mathrm{N} \tag{3.13}$$

Proof: For proving the lemma, we introduce (3.12) in (3.13) to have

$$\mathrm{A}_{n_\ell}(x) = 2n_\ell\pi\left[\int_x^1\frac{u\mathrm{P}_{n_\ell-1}^{(0,1)}\left(2u^2-1\right)du}{\sqrt{u^2-x^2}} + (n_\ell+1)\int_x^1\frac{u^3\mathrm{P}_{n_\ell-2}^{(1,2)}\left(2u^2-1\right)du}{\sqrt{u^2-x^2}}\right]. \tag{3.14}$$

Assume in (3.14) the substitution $2u^2 - 1 = y,\ 2x^2 - 1 = z$, to have

$$A_{n_\ell}(x) = A_{n_\ell}(z) = \frac{\pi n_\ell}{\sqrt{2}}\int_z^1\frac{\mathrm{P}_{n_\ell-1}^{(0,1)}(y)dy}{\sqrt{y-z}} + \frac{n_\ell(n_\ell+1)\pi}{2\sqrt{2}}\int_z^1\frac{y\mathrm{P}_{n_\ell-2}(y)dy}{\sqrt{y-z}}$$

$$+\frac{n_\ell(n_\ell+1)\pi}{2\sqrt{2}}\int_z^1\frac{\mathrm{P}_{n_\ell-2}^{(1,2)}(y)dy}{\sqrt{y-z}}. \tag{3.15}$$

If we put $y = 1-(1-z)v$, then (3.15) yields

$$A_{n_\ell}(z) = \frac{\pi n_\ell}{\sqrt{2}}\sqrt{1-z}\int_0^1(1-v)^{\frac{-1}{2}}\mathrm{P}_{n_\ell-1}^{(0,1)}[1-(1-z)v]dv$$

$$+\frac{\pi n_\ell(n_\ell+1)}{2\sqrt{2}}\sqrt{1-z}(1+z)\int_0^1(1-v)^{\frac{-1}{2}}\mathrm{P}_{n_\ell-2}^{(1,2)}[1-(1-z)v]dv$$

$$+\frac{n_\ell(n_\ell+1)\pi}{2\sqrt{2}}(1-z)^{\frac{3}{2}}\int_0^1(1-v)^{\frac{1}{2}}\mathrm{P}_{n_\ell-2}^{(1,2)}[1-(1-z)v]dv$$

. (3.16)

If we use the famous formulas **[10]**

$$\int_0^1 z^{\lambda-1}(1-z)^{r-1}\mathrm{P}_n^{(\alpha,\beta)}(1-\gamma z)dz =$$

$$\frac{\Gamma(\alpha+n+1)\Gamma(\lambda)\Gamma(r)}{n!\,\Gamma(1+\alpha)\Gamma(\lambda+r)}\,{}_3F_2\left(-n,n+\alpha+\beta+1;\lambda,\alpha+1,\lambda+r;\frac{\gamma}{2}\right)$$

$$(R_e\lambda > 0, R_e r > 0) \tag{3.17}$$

and

$$\mathrm{P}_n^{(\alpha,\beta)}(v) = \binom{n+\alpha}{n}F\left(-n,n+\alpha+\beta+1;\frac{1-v}{2}\right) \tag{3.18}$$

where ${}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;z)$ is the generalized hypergeometric series and $F(\alpha,\beta;\gamma;z)$ is the hypergeometric Gauss function , the first integral term of (3.16) becomes

$$\int_0^1(1-v)^{\frac{-1}{2}}\mathrm{P}_{n_\ell-1}^{(0,1)}[1-(1-z)v]dv = \frac{\sqrt{\pi}(n_\ell-1)\,!}{\Gamma\left(n_\ell+\frac{1}{2}\right)}\mathrm{P}_{n_\ell-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(z) \tag{3.19}$$

Also, using the same way, the second and third integral term of (3.16), yield

$$\int_0^1 (1-v)^{\frac{-1}{2}} P_{n_\ell-2}^{(1,2)}[1-(1-z)v]dv =$$
$$\frac{1}{1-z}\left\{\frac{-2\sqrt{\pi}\Gamma(n_\ell)}{(n_\ell+1)\Gamma(n_\ell-\frac{1}{2})}P_{n_\ell-1}^{\left(\frac{-1}{2},\frac{3}{2}\right)}(z)+\frac{2}{n_\ell+1}\right\}$$
(3.20)

and

$$\int_0^1 (1-v)^{\frac{1}{2}} P_{n_\ell-2}^{(1,2)}[1-(1-z)v]dv =$$

$$\frac{-\sqrt{\pi}(n_\ell-1)\,!}{(n_\ell+1)(1-z).\Gamma(n_\ell+\frac{1}{2})}P_{n_\ell-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(z)+\frac{2}{(n_\ell+1)(1-z)}$$
(3.21)

Introducing the three formulas (3.19) - (3.21) in (3.13) the lemma is proved.■

Finally, to prove the theorem, we write (3.13) in the **CPs** form, for this purpose, we must consider the following famous formulas, see **[10, 11]**

(i)

$$P_n^{\left(\frac{-1}{2},\frac{-1}{2}\right)}(2x^2-1)=\frac{\Gamma(n+\frac{1}{2})\Gamma(\lambda)}{\sqrt{\pi}\Gamma(n+\lambda)}C_{2n}^{\lambda}(x)$$
(3.22)

(Relation between Jacobi and Gegenbauer polynomials)

(ii)

$$P_n^{\left(\lambda-\frac{1}{2},\frac{1}{2}\right)}(2x^2-1)=\frac{\Gamma(n+\frac{3}{2})\Gamma(\lambda)}{\sqrt{\pi}x\Gamma(n+\lambda+1)}C_{2n+1}^{\lambda}(x)$$
(3.23)

and

(iii) $$\lim_{\lambda\to 0}\Gamma(\lambda)C_n^{\lambda}(x)=\frac{2}{n}T_n(x)$$
$(n\geq 1)$ (3.24)

(Relation between Chebyshev and Gegenbauer polynomials)

Using these famous relations in (3.13), one has

$$A_{n_\ell}(x)=\frac{n_\ell\pi(1-T_{2n_\ell}(x))}{\sqrt{1-x^2}},$$
$(n\geq 1)$ (3.25)

Introducing (3.25) and (3.10) in (2.6) , the theorem is proved.■

By using the same way, we can prove this theorem

Theorem 2: The spectral relationships for the **SFIEs** with the kernel defined by (2.4) and the known function is odd is given by

$$\sum_{J=0}^{\ell}u_j F_{j,\ell}\int_{-1}^{1}\left[\ln\frac{1}{|x-s|}+d\right]T_{2n_j-1}(s)ds=\frac{\pi}{2n_\ell-1}T_{2n_\ell-1}(x)$$
$$(n_\ell\geq 1,\ell=1,2,\ldots,N)\quad (3.26)$$

The proof of theorem 2 can be obtained directly by following the same way of theorem 1.■

4. Conclusion and results: From the above results and discussion, the following may be concluded

(1) The contact problem of a rigid surface of an elastic material, when a stamp of length $2a$ is impressed into an elastic layer surface of a strip by a variable $P(t), 0\leq t\leq T<1$, whose eccentricity of application $e(t)$,represents a **V-FIE** of the first kind.

(2) The numerical method used transforms the **V-FIE** into **SFIEs**.

(3) The **SFIEs** depends on the number of derivatives of $F(t,\tau)$ with respect to time $t, t\in[0,T], T<1$.

(4) The displacement problems of ant plane deformation of an infinite rigid strip with width $2a$, putting on an elastic layer of thickness h is considered as a special case of this work when $t=1$, $F(t,\tau)=1$, $f(x,t)=H$ and $\varphi(x,1)=\psi(x)$. Here H represents the displacement magnitude and $\psi(x)$ the unknown function represents the displacement stress.

(5) The problems of infinite rigid strip with width $2a$ impressed in a viscous liquid layer of thickness h , when the strip has a velocity resulting from the impulsive force $v=v_0e^{-iwt}, i=\sqrt{-1}$, where v_0 is the constant velocity , w is the angular velocity resulting rotating the strip about z-axis are considered as special case of this work , when $F(t,\tau)=$ constant , and $t=1$ see [4] .

(6)In the discussion (4) and (5) , when $h\to\infty$, this means the depth of the liquid (Fluid mechanics) or the thickness of elastic material (contact problem) becomes an infinite .

(7)The three kinds of the displacement problem, in the theory of elasticity and mixed contact problems, which discussed in Aleksandrov et al. [4],Muskelishvili [5], Green [6] and Popov [7].,are considered special cases of this work .

(8) Many important relationships can be derived from (3.1)

If $n_j = 2m_j$, $\frac{x}{a} = \frac{\sin\frac{\xi}{2}}{\sin\frac{\alpha}{2}}$, $\frac{y}{a} = \frac{\sin\frac{\eta}{2}}{\sin\frac{\alpha}{2}}$; and if $n_j = 2m_j + 1$, $\frac{x}{a} = \frac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}$, $\frac{y}{a} = \frac{\tan\frac{\eta}{2}}{\tan\frac{\alpha}{2}}$, we have the following **SFIEs**

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-\alpha}^{\alpha} \left[\ln \frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|} + d \right] \psi_j(\xi) d\xi = h_k(\eta)$$

(4.1)

The above system leads to the following **SRs**

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-\alpha}^{\alpha} \left[\ln \frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|} + d \right] \frac{T_{2n_j}\left(\frac{\sin\frac{\eta}{2}}{\sin\frac{\alpha}{2}}\right) \cos\left(\frac{\eta}{2}\right) d\eta}{\sqrt{2(\cos\eta - \cos\alpha)}}$$

$$= \begin{cases} \pi P_\ell \left(\ln\frac{2}{\sin\alpha} + d \right) & m_\ell = 0 \\ \frac{\pi}{2m_\ell} T_{m_\ell}\left(\frac{\sin\frac{\xi}{2}}{\sin\frac{\alpha}{2}} \right) & m_\ell \geq 1, \ell = 1, 2, \ldots, N \end{cases}$$

(4.2)

and

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-\alpha}^{\alpha} \left(\ln \frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|} \right) \frac{T_{2m_j+1}\left(\frac{\tan\frac{\eta}{2}}{\tan\frac{\alpha}{2}}\right) \cos\frac{\eta}{2} d\eta}{\sqrt{2(\cos\eta - \cos\alpha)}}$$

$$= \frac{\pi}{2m_\ell + 1} T_{2m_j+1}\left(\frac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}} \right) \qquad m_\ell \geq 0 \quad (4.3)$$

(ii) Differentiating (3.1) with respect to x, we have

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \frac{T_{n_j}\left(\frac{y}{a}\right)}{y - x} \frac{dy}{\sqrt{a^2 - y^2}} = \pi U_{n_\ell - 1}\left(\frac{x}{a}\right) \qquad n_\ell \geq 1$$

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \frac{dy}{(y - x)\sqrt{a^2 - y^2}} = 0$$

(4.4)

where $U_{n_\ell}\left(\frac{x}{a}\right)$ are the **CPs** of the second kind.

Also (4.4) yields

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-\alpha}^{\alpha} \cot\frac{\eta-\xi}{2} \frac{T_{n_j}\left(\frac{\tan\frac{\eta}{2}}{\tan\frac{\alpha}{2}}\right)}{\sqrt{2(\cos\eta - \cos\alpha)}} \cos\left(\frac{\eta}{2}\right) d\eta$$

$$= \begin{cases} 0 & n_\ell = 0 \\ 2\csc\left(\frac{\alpha}{2}\right) U_{2m_\ell - 1}\left(\frac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}\right) & n_\ell = 2m_\ell \\ \left[2\csc\left(\frac{\alpha}{2}\right) U_{2m_j-1}\left(\frac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}\right) + (-1)^2 \frac{\sin\alpha}{1+\cos\alpha} \left[\tan\frac{\alpha}{4}\right]^{2m_\ell - 2} \right] & n_\ell = 2m_\ell - 1 \end{cases},$$

(4.5)

$$\frac{1}{2} \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \frac{\sec\frac{\eta}{2} . \cot\frac{\eta-\xi}{2}}{\sqrt{2(\cos\eta - \cos\alpha)}} T_{n_j}\left(\frac{\tan\frac{\eta}{2}}{\tan\frac{\alpha}{2}}\right) d\eta$$

$$= \begin{cases} \csc\left(\frac{\alpha}{2}\right) \sec^2\left(\frac{\xi}{2}\right) U_{n_\ell - 1}\left(\frac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}\right) & n \geq 1 \\ \sec\left(\frac{\alpha}{2}\right) . \tan\left(\frac{\xi}{2}\right) & n_\ell = 0 \end{cases} \quad (4.6)$$

(9) The mixed integral equation with Carleman kernel can be established from this work by using the following relation

$$\ln|x - y| = h(x, y)|x - y|^{-\upsilon} \qquad 0 < \upsilon < 1 \quad (4.7)$$

where

$h(x, y) = |x - y|^{\upsilon} \ln|x - y|$ is a smooth function

The importance of Carleman kernel came from the work of Arutiunion [12] who has shown that, the contact problem of nonlinear theory of plasticity, in its first approximation reduce to FIE of the first kind with Carleman kernel.

(10) The relation between the eigenvalues n and the corresponding Chebyshev polynomial T_n are obtained in the following figures

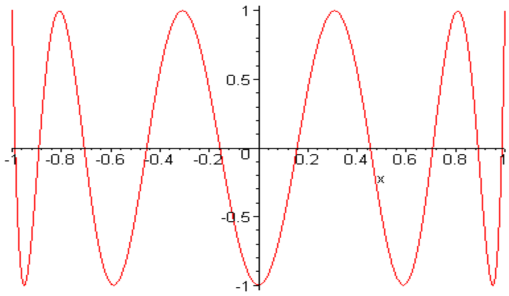

Fig. 1 n=5

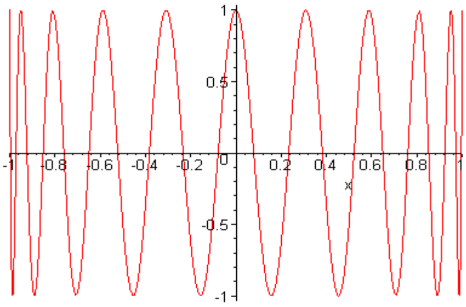

Fig.2: ***n*****=10**

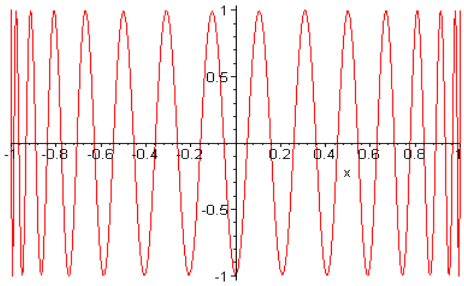

Fig.3: ***n*****=30**

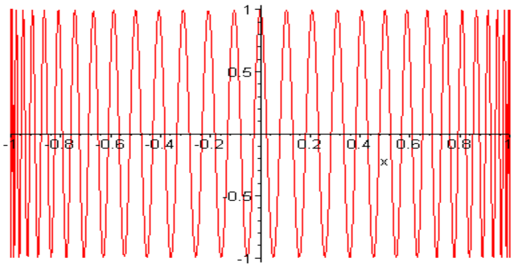

Fig.4.6: *n*=15

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