Simulation of Butterfly Option Using Two Dimensional Black-Scholes Equation.

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Abstract: Abstract: An innovative space adaptive finite difference technique is proposed to simulate butterfly spread call option using two assets Black-Scholes equation. In this technique the grid is refined near exercise prices of the butterfly option and a coarse grid is generated otherwise. The non uniform finite difference discretization is used in this computation. The numerical experiments show that the adaptive finite difference method is much more efficient than the method with uniform spacing. The proposed grid adaptative technique reduces the points drastically for two dimensional Black-Scholes equation which in turn decreases the computational cost and makes the algorithm efficient.

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1. Introduction

The finite difference scheme that was developed in (Ashraf et al. 2013, Mir et al. 2013) is extended to work with two assets butterfly spread option using Black-Scholes equation.There are many types of financial instruments (Duffy, 2006) which go by the name of Options. Options are traded on all of the world's major exchanges. Butterfly options (Khaliq, et al, 2007) are not only very popular in the over-the-counter markets but also important tools for designing more complex financial derivatives (Wilmott and Howison, 1996). In butterfly option, the payoff has a discontinuity at strike prices. In this work, we will focus on butterfly spread call options for two assets.

Fisher-Black and Myron-Schole (Black and Scholes, 1973) derived a celebrated partial differential equation. The Black-Scholes model is the convenient way to calculate the price of an option (Cox et al, 1979). In this article, numerical methods (Smith, 1985) will be used to solve the finite difference equation (Courtadon, 1982) of Black-Scholes. The solution to the Black-Scholes equation is smooth but the final condition has discontinuity which produces oscillations in the numerical solution. Numerical methods have been studied (Dura and Mosneagu, 2010. Zhu et al, 1988) in many application areas in order to cure these oscillations from the initial discontinuities. Finite difference methods (Khaliq et al, 2008. Wade et al, 2007) with variable space-steps are proposed in order to valuate butterfly options.

The purpose of this paper is to develop an efficient and accurate numerical technique to price options (Zhongdi and Anbo, 2009) with payoff containing discontinuities. For Butterfly options discontinuity lies only in the initial condition at strike prices, therefore we need to use small space-steps at strike prices and then use bigger space-steps otherwise. In proposed study, we focused on adaptivity (Hongjoong, 2011) for space-steps in order to see effects of variable space-steps. In this study, several numerical tests show that the adaptive finite difference methods approximate the solution more efficiently than uniform finite difference methods.

2. Discretization of Two-Asset Black-Scholes Equation

Let $S_1(t)$ *and* $S_2(t)$. be the prices of the two underlying assets S_1 *and* S_2 at time t $(0 \le t \le T)$. The interval $[0,T]$ is divided into K equally sized subintervals of length Δt. Let $V(S_1,S_2,t)$ be the option value. The corresponding Black-Scholes equation for two assets (Johnson 1987; Jeong et al. 2009) is given by

$$\frac{\partial V}{\partial t}(S_1,S_2,t)-\frac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2 V}{\partial S_1^2}(S_1,S_2,t)-\frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 V}{\partial S_2^2}(S_1,S_2,t)$$
$$-\rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2 V}{\partial S_1\partial S_2}(S_1,S_2,t)-rS_1\frac{\partial V}{\partial S_1}(S_1,S_2,t)$$
$$-rS_2\frac{\partial V}{\partial S_2}(S_1,S_2,t)+r(S_1,S_2,t)=0.......(2.1)$$

where σ_1 is the volatility of asset S_1, σ_2 is the volatility of asset S_2 and ρ is the correlation coefficient between S_1 and S_2, r is the risk free rate, E is the strike price of the rainbow option and

T is the time to expiry of the rainbow option. The solution domain is

$\{S_1 \in [0,\infty), \quad S_2 \in [0,\infty), \quad t \in [0,T]\}$ and the final condition is

$V(S_1,S_2,T) = payoff\ (S_1,S_2).$

The final condition becomes the initial condition due to the change of variable as

$V(S_1,S_2,0) = payoff\ (S_1,S_2)$.

The boundary conditions are as suggested in ((Dura &Mosneagu 2010)

- If $S_1 = 0$ and $S_2 = 0,$ the option value is $V(0,0,t) = 0.$
- If $S_1 = 0$ and $S_2 \neq 0,$ the option value V depends only on S_2 and t.
- If $S_1 \neq 0$ and $S_2 = 0,$ the option value V depends only on S_1 and t.
- If $S_1 \to \infty$ and $S_2 \to \infty,$ the option value is approximately equal to zero.
- If $S_1 \to \infty$ and S_2 is finite, the option value is approximately equal to zero.
- If S_1 is finite and $S_2 \to \infty,$ the option value is approximately equal to zero.

We choose, $S_{1\max} = \Delta S_0 + \Delta S_1 + \Delta S_2 + \cdots \Delta S_{M-1} + \Delta S_M = \sum_{m=0}^{M} \Delta S_m = \alpha_M$,

$S_{2\max} = \Delta S_0 + \Delta S_1 + \Delta S_2 + \cdots \Delta S_{N-1} + \Delta S_N = \sum_{n=0}^{N} \Delta S_n = \beta_N$.

For arbitrary interval m and n, the prices of the assets 1 and 2 will be $m\Delta S_m$ and $n\Delta S_n$ for uniform space-stepping. If the space is not uniform, the asset price at mth and nth interval is $\sum_{i=0}^{m}\Delta S_i$ and $\sum_{i=0}^{n}\Delta S_i$ respectively, where ΔS_i are the non-uniform space-steps.and assume, $\alpha_m = \sum_{i=0}^{m}\Delta S_i$, $\beta_n = \sum_{i=0}^{n}\Delta S_i$. We used same point placement in both directions S_1 and S_2.

In this method, we use forward difference for time derivative, central difference for first space derivative, we first use forward difference and then backward difference in case of second space derivative, i,e.

$$\frac{\partial V}{\partial t}(\alpha_m,\beta_n,k\,\Delta t) = \frac{V_{m,n}^{k+1} - V_{m,n}^{k}}{\Delta t} + O(\Delta t),........(2.2)$$

$$\frac{\partial V}{\partial S_1}(\alpha_m,\beta_n,k\,\Delta t) \approx \frac{V_{m+1,n}^{k} - V_{m-1,n}^{k}}{\Delta S_m + \Delta S_{m-1}},..........(2.3)$$

$$\frac{\partial V}{\partial S_2}(\alpha_m,\beta_n,k\,\Delta t) \approx \frac{V_{m,n+1}^{k} - V_{m,n-1}^{k}}{\Delta S_n + \Delta S_{n-1}},.......(2.4)$$

$$\frac{\partial^2 V}{\partial S_1 \partial S_2}(\alpha_m,\beta_n,k\,\Delta t) \approx \frac{V_{m+1,n+1}^{k} - V_{m+1,n-1}^{k} - V_{m-1,n+1}^{k} + V_{m-1,n-1}^{k}}{(\Delta S_m + \Delta S_{m-1}) \times (\Delta S_n + \Delta S_{n-1})},...(2.5)$$

$$\frac{\partial^2 V}{\partial S_1^2}(\alpha_m,\beta_n,k\,\Delta t) \approx \frac{\Delta S_{m-1}(V_{m+1,n}^{k} - V_{m,n}^{k}) - \Delta S_m (V_{m,n}^{k} - V_{m-1,n}^{k})}{(\Delta S_m)^2 \times \Delta S_{m-1}},....(2.6)$$

$$\frac{\partial^2 V}{\partial S_2^2}(\alpha_m,\beta_n,k\,\Delta t) \approx \frac{\Delta S_{n-1}(V_{m,n+1}^{k} - V_{m,n}^{k}) - \Delta S_n (V_{m,n}^{k} - V_{m,n-1}^{k})}{(\Delta S_n)^2 \times \Delta S_{n-1}}..(2.7)$$

Using the above substitutions in equation (2.1), we get:

$$\frac{V_{m,n}^{k+1} - V_{m,n}^{k}}{\Delta t} - \frac{1}{2}\sigma_1^2\left(\sum_{i=0}^{m}\Delta S_i\right)^2 \left\{\frac{\Delta S_{m-1}(V_{m+1,n}^{k} - V_{m,n}^{k}) - \Delta S_m (V_{m,n}^{k} - V_{m-1,n}^{k})}{(\Delta S_m)^2 \times \Delta S_{m-1}}\right\}$$
$$-\frac{1}{2}\sigma_2^2\left(\sum_{i=0}^{n}\Delta S_i\right)^2 \left\{\frac{\Delta S_{n-1}(V_{m,n+1}^{k} - V_{m,n}^{k}) - \Delta S_n (V_{m,n}^{k} - V_{m,n-1}^{k})}{(\Delta S_n)^2 \times \Delta S_{n-1}}\right\}$$
$$-\rho\sigma_1\sigma_2\left(\sum_{i=0}^{m}\Delta S_i\right)\left(\sum_{i=0}^{n}\Delta S_i\right)\left\{\frac{V_{m+1,n+1}^{k} - V_{m+1,n-1}^{k} - V_{m-1,n+1}^{k} + V_{m-1,n-1}^{k}}{(\Delta S_m + \Delta S_{m-1}) \times (\Delta S_n + \Delta S_{n-1})}\right\}$$
$$-r\sum_{i=0}^{m}\Delta S_i\left(\frac{V_{m+1,n}^{k} - V_{m-1,n}^{k}}{\Delta S_m + \Delta S_{m-1}}\right) - r\sum_{i=0}^{n}\Delta S_i\left(\frac{V_{m,n+1}^{k} - V_{m,n-1}^{k}}{\Delta S_n + \Delta S_{n-1}}\right) + rV_{m,n}^{k}$$
$$= 0.$$

Using $S_1 = \alpha_m = \sum_{i=0}^{m}\Delta S_i$, and $S_2 = \beta_n = \sum_{i=0}^{n}\Delta S_i$, the above equation becomes,

$$V_{m,n}^{k+1} = V_{m,n}^{k} + \Delta t[\frac{1}{2}\sigma_1^2\alpha_m^2\left\{\frac{\Delta S_{m-1}(V_{m+1,n}^{k} - V_{m,n}^{k}) - \Delta S_m (V_{m,n}^{k} - V_{m-1,n}^{k})}{(\Delta S_m)^2 \times \Delta S_{m-1}}\right\}$$
$$+\frac{1}{2}\sigma_2^2\beta_n^2\left\{\frac{\Delta S_{n-1}(V_{m,n+1}^{k} - V_{m,n}^{k}) - \Delta S_n (V_{m,n}^{k} - V_{m,n-1}^{k})}{(\Delta S_n)^2 \times \Delta S_{n-1}}\right\}$$
$$-\rho\sigma_1\sigma_2\alpha_m\beta_n\left\{\frac{V_{m+1,n+1}^{k} - V_{m+1,n-1}^{k} - V_{m-1,n+1}^{k} + V_{m-1,n-1}^{k}}{(\Delta S_m + \Delta S_{m-1}) \times (\Delta S_n + \Delta S_{n-1})}\right\}$$
$$+r\alpha_m\left(\frac{V_{m+1,n}^{k} - V_{m-1,n}^{k}}{\Delta S_m + \Delta S_{m-1}}\right) + r\beta_n\left(\frac{V_{m,n+1}^{k} - V_{m,n-1}^{k}}{\Delta S_n + \Delta S_{n-1}}\right) - rV_{m,n}^{k}],$$

with $m = 1,\ldots,M-1, \quad n = 1,\ldots,N-1$ and $k = 0,1,\ldots,K-1.$ The values $V_{m,n}^{0}$, $V_{0,0}^{k}$, $V_{0,n}^{k}$, $V_{m,0}^{k}, V_{M,n}^{k}, V_{m,N}^{k}, V_{M,N}^{k}$ with $n = 0,\ldots,N$ and

$m = 0, \ldots, M$ and $k = 0,1,\ldots,K$ are known from initial and boundary conditions.

3. Backward-Euler Finite Difference Scheme

In this method, we use forward difference for first time derivative, central difference for first S derivative and for second derivative, we first use forward difference and then backward difference:

$$\frac{\partial V}{\partial t}(\alpha_m, \beta_n, (k+1)\Delta t) = \frac{V_{m,n}^{k+1} - V_{m,n}^{k}}{\Delta t} + O(\Delta t),$$

$$\frac{\partial V}{\partial S_1}(\alpha_m, \beta_n, (k+1)\Delta t) \approx \frac{V_{m+1,n}^{k+1} - V_{m-1,n}^{k+1}}{\Delta S_m + \Delta S_{m-1}},$$

$$\frac{\partial V}{\partial S_2}(\alpha_m, \beta_n, (k+1)\Delta t) \approx \frac{V_{m,n+1}^{k+1} - V_{m,n-1}^{k+1}}{\Delta S_n + \Delta S_{n-1}},$$

$$\frac{\partial^2 V}{\partial S_1 \partial S_2}(\alpha_m, \beta_n, (k+1)\Delta t)$$

$$\approx \frac{V_{m+1,n+1}^{k+1} - V_{m+1,n-1}^{k+1} - V_{m-1,n+1}^{k+1} + V_{m-1,n-1}^{k+1}}{(\Delta S_m + \Delta S_{m-1}) \times (\Delta S_n + \Delta S_{n-1})},$$

$$\frac{\partial^2 V}{\partial S_1^2}(\alpha_m, \beta_n, (k+1)\Delta t)$$

$$\approx \frac{\Delta S_{m-1}(V_{m+1,n}^{k+1} - V_{m,n}^{k+1}) - \Delta S_m (V_{m,n}^{k+1} - V_{m-1,n}^{k+1})}{(\Delta S_m)^2 \times \Delta S_{m-1}},$$

$$\frac{\partial^2 V}{\partial S_2^2}(\alpha_m, \beta_n, (k+1)\Delta t) \approx \frac{\Delta S_{n-1}(V_{m,n+1}^{k+1} - V_{m,n}^{k+1}) - \Delta S_n (V_{m,n}^{k+1} - V_{m,n-1}^{k+1})}{(\Delta S_n)^2 \times \Delta S_{n-1}}.$$

After simplifying and re-arranging, the above equation takes the form:

$$V_{m,n}^{k+1} = V_{m,n}^{k} + \Delta t$$

$$[\frac{1}{2}\sigma_1^2 \alpha_m^2 \left\{ \frac{\Delta S_{m-1}(V_{m+1,n}^{k+1} - V_{m,n}^{k+1}) - \Delta S_m (V_{m,n}^{k+1} - V_{m-1,n}^{k+1})}{(\Delta S_m)^2 \times \Delta S_{m-1}} \right\}$$

$$+\frac{1}{2}\sigma_2^2 \beta_n^2 \left\{ \frac{\Delta S_{n-1}(V_{m,n+1}^{k+1} - V_{m,n}^{k+1}) - \Delta S_n (V_{m,n}^{k+1} - V_{m,n-1}^{k+1})}{(\Delta S_n)^2 \times \Delta S_{n-1}} \right\}$$

$$-\rho \sigma_1 \sigma_2 \alpha_m \beta_n \left\{ \frac{V_{m+1,n+1}^{k+1} - V_{m+1,n-1}^{k+1} - V_{m-1,n+1}^{k+1} + V_{m-1,n-1}^{k+1}}{(\Delta S_m + \Delta S_{m-1}) \times (\Delta S_n + \Delta S_{n-1})} \right\}$$

$$+r\alpha_m \left(\frac{V_{m+1,n}^{k+1} - V_{m-1,n}^{k+1}}{\Delta S_m + \Delta S_{m-1}} \right) + r\beta_n \left(\frac{V_{m,n+1}^{k+1} - V_{m,n-1}^{k+1}}{\Delta S_n + \Delta S_{n-1}} \right) - rV_{m,n}^{k+1}],$$

This system of equations can be solved by Gauss-Seidel method. The values V_n^0, V_0^m, V_N^m with $n = 0, \ldots, N$ and $m = 0, \ldots, M$ are known from initial and boundary conditions.

4. Numerical Experiments

We demonstrate some numerical experiments for two assets butterfly spread call option. The butterfly can be created by using call or put options.The strategy is termd "Butterfly" due to the shape of the risk characteristics graph we see, the two wings and the larger body.The butterfly spread is constructed through buying 1 long In The Money(ITM) call, shorting two At The Money (ATM) calls and buying 1 long Out of the money(OTM) call. The ratio between the three options is 1:2:1 and the distance between the strike prices of long options should be equidistant from the short call strike. For example, a butterfly spread could be made of 3 call options with strikes of $E_1 = 10, E_2 = 20$ *and* $E_3 = 30$. The butterfly will result in a net debit transaction as the ITM and OTM call options will total a larger value than the two short ATM calls. Let $S_{max} = \max(S_1, S_2)$

The payoff function of butterfly spread call option for two assets is given by

$$V(S_1, S_2, T) = max(\text{Smax} - E_1, 0) - 2max(\text{Smax} - E_2, 0) + max(\text{Smax} - E_3, 0)$$

In butterfly spread call option, the payoff is acting as the initial condition and has a piecewise discontinuity at the strike prices. We use the following parameters for the computation of the butterfly spread call option for two asset.

$$M = N = 80, S^1 = S^2 = 40, \sigma_1 = \sigma_2 = 0.2, r = 0.1, \rho = 0.1, T = 0.5, K = 401, E1 = 10,$$

$E2 = 20$ *and* $E3 = 30$, different schemes are applied for option valuation. It can be observed that same option values are obtained by using less number of points in adaptive space-stepping as compared to uniform space-stepping and adaptive space-stepping converges more rapidly than uniform space-stepping.

Figure 1, shows the payoff function of two asset butterfly call option. It is a three dimensional plot with x-axis as S^1 asset price, y-axis as S^2 asset price and z-axis is the payoff function at zero time. We choose ε(epsilon) as 5 and we refined grid around the intervals $[E1 - \varepsilon, E1 + \varepsilon], [E2 - \varepsilon, E2 + \varepsilon]$ *and* $[E3 - \varepsilon, E3 + \varepsilon]$ to cure oscillations caused by discontinuity. In these intervals, grid intensity is higher than the other part of the domain as is obvious from the figure. The simulation results of this grid are shown in Figure 2, this figure represents four subplots, two surface plots and two plots one for constant S_1 and varying S_2 while the other is for fixed S_2 and varying S_1. The two different views for surface plots are shown. The figure shows that the more points are placed in the vicinity of E_1, E_2 and E_3 for both assets. The lower half of the figure shows two plots. The left graph is taken at $S_1 = E_1$ *and* S_2 varies from zero to its maximum value. The right graph shows option value for $S_2 = E_1$ *and* S_1 taking all its values.

Figure 3, shows the comparison plot of option value for three different grids, uniform dense grid, adaptive grid and a uniform coarse grid. The course grid underpredict the option value while the uniform dense grid and adaptive grid has same answer but the points used in adaptive grid are about the same as for the coarse unifrom grid. This shows that adaptive space-stepping is much better than uniform space-stepping. Similar results can be obtained for space-stepping by Backward-Euler scheme.

5. Conclusions

We have simulated the butterfly option using two asset Black-Scholes equation. The oscillations at discontinuities are eliminated by using adaptive space-stepping. The computational domain is discretized by embedding more points near the singularities and coarse grid otherwise. The adaptive space-stepping speeds up the solution convergence as compared to the uniform space-stepping. The adaptive finite difference scheme needs less points in its computation and hence is very efficient.

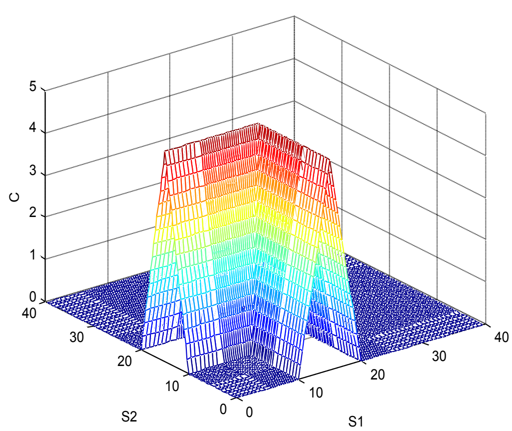

Figure 1. Adaptive grid and Payoff function for two asset butterfly spread

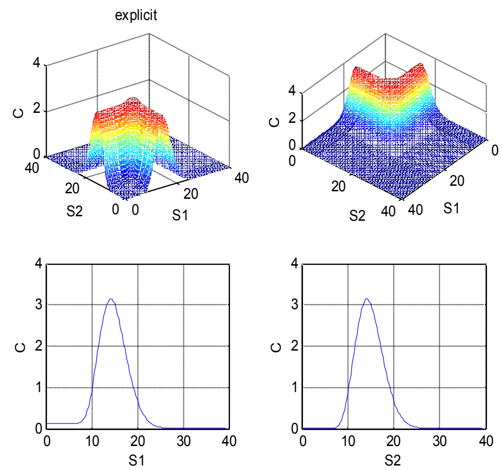

Figure 2. Butterfly option simulation using adaptive explicit scheme

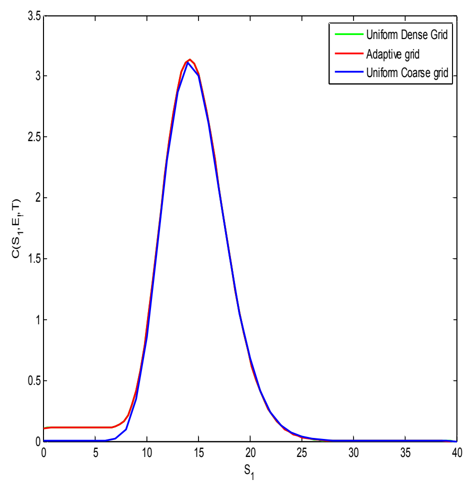

Figure 3. Comparison results for various grids at E1

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