# The New Control Method of the Research Robust Stability for Linear System

Mamyrbek Aukebaevich Beisenbi, Janar Jarasovna Yermekbayeva and Arman Mamyrbekuly Beisenbin

L.N. Gumilyov Eurasian National University, Mirzoyan Street, 2, 010008, Astana, Kazakhstan

**Abstract:** The stability of control systems where the controlled object possesses dynamics is highlighted in present. For a linear system affected by parametric uncertainty, this paper focuses on robust stability analysis of system with Lyapunov functions. We construct the Lyapunov function for linear system, and then apply geometric interpretation to discover the region of stability. This paper presents some theoretical fundamental results assisting in analyzing of the behavior of control systems, depending on parameter uncertainty. The obtained results are robust stability type since the robust stability is guaranteed under certain deviations from the current state.

[Beisenbi M.A., Yermekbayeva J.J., Beisenbin A.M. **The New Control Method of the Research Robust Stability for Linear System**. *Life Sci J* 2013;10(12s):142-148] (ISSN:1097-8135). http://www.lifesciencesite.com. 26

**Keywords:** Stability, Linear systems, Robust control, Control theory, Geometrical interpretation, Lyapunov function.

## 1. Introduction

Today Robust analysis for linear systems is one of the active fields of research. Models with parametric uncertainty perform important function in both the theory and practical applications of robust control. They are described by the mathematical model containing parameters that are not precisely known, but the values are within given intervals. Such type of uncertainty can occur in the control of real processes, for example, as a result of modeling effort, inaccurate measuring (worn parts, weight change of the aircraft, temperature, fuel quality) or the influence of certain external conditions. The high research interest of robust stability analysis techniques was developed before. Nevertheless, many of them specialized for concrete systems of uncertainty structure. This paper provides a method aimed at usage of a universal approach in robust stability analysis for linear systems. An important task is to solve the problem of analysis of control systems and synthesis of control laws. All this ensures the best protection from high uncertainty of object properties. The considered problem is robust controllability of linear systems with parametric or non-parametric uncertainties [1,2]. Assuming that the linear system is controllable, a sufficient condition is proposed to preserve the properties of object (parameters of control systems) when system uncertainties are introduced. The most important idea in the study of robust stability is to specify constraints for changes in control system parameters that preserve stability.

For the purpose of studying the system dynamics and their control, we considered models of observing input and output signals of the object and the representing its behavior in the state space as most suitable.

The content of this paper is organized in next way: in section #2 we made analyses of literature review and robust stability discussion by the our considering problem. We introduce the basic equations of the model and their expanded form in section #3. Section #4 is devoted to explain the control method. We received the Lyapunov function, geometric interpretation, gradient vector components and superstability condition of system. The main points ( the radius of the robustness) of this work are presented in section #5. In section #6, we move on to the proof method of the proposed method. We show how the proposed approach is applied to the sample and construct block diagonal matrix. In the part # 7 we given a case study with practical example. Finally, the main conclusions of this work are presented in Section 8.

**2. Literature Review In Robust Stability Discussion.** The theory of robust control began in the late 1970s and early 1980s and soon developed a number of techniques for dealing with bounded system uncertainty. The Robust stability is closely related to the pioneering studies [1-4] and today we see many works in this field [3-13]. However, many of them address either linear systems or nonlinear systems with specific constraints. Successful results were reported when Lyapunov theory [4-5], [7-8] was employed to achieve robust stability of control systems with uncertain parameters. When the method of Lyapunov functions is applied, it is possible to demonstrate that asymptotic stability of zero solutions can be achieved in time-delay systems as well. The results reported in [14-20] are of particular interest, where increased robustness based on catastrophe theory lead to structurally stable systems. In particular, [7, 10] presents both analysis and synthesis steps of the process. However, linear systems are still emphasized and most of the non-linear systems with uncertainties are considered only as special cases. In addition, the examples discussed in these papers demonstrate stability. This limits the range of robustness [13]. Therefore this paper presents the approach of the construction of Lyapunov functions based on the geometric interpretation of the Lyapunov's direct method (also called the second method of Lyapunov) [13,19,20] and on gradient of dynamical systems in the state space of systems.

**3. Model Formulation**. The control system is given by the linear equation

$$\dot{x} = Ax + Bu, x \in R^n, u \in R^m, y = Cx, y \in R^l \quad (1).$$

The controller is described by the equation $u = -Kx$ or

$$u_i = -k_{i1}x_1 - k_{i2}x_2 - \ldots - k_{in}x_n, i = 1,2,\ldots,m \quad (2).$$

Where

$$A = \begin{Vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{Vmatrix}.$$

Description of parameters $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{l\times n}$, $K \in R^{m\times n}$ Matrices of the object, control, output and coefficients of control system, $x(t) \in R^n$ - state vector, $u(t) \in R^m$ - vector control, $y(t) \in R^l$ - vector output of the system.

We can provide equation (1) in expanded form

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \ldots + b_{1m}u_m \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \ldots + b_{2m}u_m \\ \ldots & \qquad \ldots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \ldots + b_{nm}u_m \end{aligned} \quad (3)$$

Let us denote $G = A - BK$ matrix of the closed system and the system (3) in matrix-vector form, we can write $\dot{x} = G\,x, x(t) \in R^n$, $g_{ij} = a_{ij} - \sum_{k=1}^{m} b_{ik}k_{kj}$.

Therefore equation (3) can be written as

$$\begin{cases} \dot{x}_1 = \left(a_{11} - \sum_{k=1}^{m} b_{1k}k_{k1}\right)x_1 + \left(a_{12} - \sum_{k=1}^{m} b_{1k}k_{k2}\right)x_2 + \ldots + \left(a_{1n} - \sum_{k=1}^{m} b_{1k}k_{kn}\right)x_n \\ \dot{x}_2 = \left(a_{21} - \sum_{k=1}^{m} b_{2k}k_{k1}\right)x_1 + \left(a_{22} - \sum_{k=1}^{m} b_{2k}k_{k2}\right)x_2 + \ldots + \left(a_{2n} - \sum_{k=1}^{m} b_{2k}k_{kn}\right)x_n \\ \qquad \ldots \\ \dot{x}_n = \left(a_{n1} - \sum_{k=1}^{m} b_{nk}k_{k1}\right)x_1 + \left(a_{n2} - \sum_{k=1}^{m} b_{nk}k_{k2}\right)x_2 + \ldots + \left(a_{nn} - \sum_{k=1}^{m} b_{2k}k_{kn}\right)x_n \end{cases} \quad (4)$$

**4. The Geometric Interpretation Of The Lyapunov Function.** Stability is a fundamental notion in the qualitative theory of differential equations and is essential for many applications. In turn, Lyapunov functions are basic instrument for studying stability; however, there is no universal method for constructing Lyapunov functions. Nevertheless, in some special cases, a function can be constructed by applying special techniques. We construct the Lyapunov function for system and then use geometric interpretation to find the region of stability. Thus, from the geometric interpretation point of view the second method of Lyapunov, the study of stability is reduced to the construction of a family of closed surfaces surrounding the origin. As the integral curves have property to intersect each of these surfaces, then stability of the unperturbed motion will be set [2]. . If $dV/dt$ is a function with negative definite $(dV/dt < 0)$, then every integral curve starting from a sufficiently small neighborhood of the origin, will be sure to cross each of the surfaces $V(x_1(t), x_2(t), .., x_n(t)) = C, C = const$ of the outside to the inside, as the $V(x_1(t), x_2(t), .., x_n(t)) = C$ function is continuously decreasing. The gradient vector of the Lyapunov function is always directed from the origin toward the highest growth of Lyapunov functions. Also note that, in the study of stability [1] the origin corresponds to the stationary states of the system or the set of the system. The state equation (1) or (4) shall be made in respect to deviations from the steady state $X_s\left(x = \Delta x = X(t) - X_s(t)\right)$.

Therefore the left side of (1) or (4), $dx/dt$ expresses the velocity vector changes and deviations. We can assume that the velocity vector of deviations submitted to the stability of a system to the origin.

Thus, we can assume that the velocity vector changes deviations directed towards the origin. Components of the gradient vector Lyapunov functions in the opposite direction, but they are equal in absolute value. Then, if the Lyapunov function $V(x)$ is specified as a vector of functions $V(V_1(x), V_2(x), \ldots, V_n(x))$, then gradient vector Lyapunov function can be written as $\partial V/\partial x = -dx/dt = -(A - BK)x.$ Vector components of the gradient of a potential function $V(x_1, \ldots, x_n)$ are given in the form of vector Lyapunov functions with components $\left(V_1(x_1, x_2, \ldots, x_n), V_2(x_1, x_2, \ldots, x_n), \ldots, V_n(x_1, x_2, \ldots, x_n)\right)$ we write in the form:

$$\begin{cases} -\dfrac{dx_1}{dt} = \dfrac{\partial V_1(x)}{\partial x_1} + \dfrac{\partial V_1(x)}{\partial x_2} + \ldots + \dfrac{\partial V_1(x)}{\partial x_n} \\ -\dfrac{dx_2}{dt} = \dfrac{\partial V_2(x)}{\partial x_1} + \dfrac{\partial V_2(x)}{\partial x_2} + \ldots + \dfrac{\partial V_2(x)}{\partial x_n} \\ \ldots \\ -\dfrac{dx_n}{dt} = \dfrac{\partial V_n(x)}{\partial x_1} + \dfrac{\partial V_n(x)}{\partial x_2} + \ldots + \dfrac{\partial V_n(x)}{\partial x_n} \end{cases} \quad (5)$$

In this system by substituting values of the components of the velocity vector we get:

$$\begin{cases} \dfrac{\partial V_1(x)}{\partial x_1} + \dfrac{\partial V_1(x)}{\partial x_2} +,\ldots,+ \dfrac{\partial V_1(x)}{\partial x_n} = -\left(a_{11} - \sum_{k=1}^m b_{1k}k_{k1}\right)x_1 - \left(a_{12} - \sum_{k=1}^m b_{1k}k_{k2}\right)x_2 -,\ldots,-\left(a_{1n} - \sum_{k=1}^m b_{1k}k_{kn}\right)x_n, \\ \dfrac{\partial V_2(x)}{\partial x_1} + \dfrac{\partial V_2(x)}{\partial x_2} +,\ldots,+ \dfrac{\partial V_2(x)}{\partial x_n} = -\left(a_{21} - \sum_{k=1}^m b_{2k}k_{k1}\right)x_1 - \left(a_{22} - \sum_{k=1}^m b_{2k}k_{k2}\right)x_2 -,\ldots,-\left(a_{2n} - \sum_{k=1}^m b_{2k}k_{kn}\right)x_n \\ \ldots \\ \ldots \\ \dfrac{\partial V_n(x)}{\partial x_1} + \dfrac{\partial V_n(x)}{\partial x_2} +,\ldots,+ \dfrac{\partial V_n(x)}{\partial x_n} = -\left(a_{n1} - \sum_{k=1}^m b_{nk}k_{k1}\right)x_1 - \left(a_{n2} - \sum_{k=1}^m b_{nk}k_{k2}\right)x_2 -,\ldots,-\left(a_{nn} - \sum_{k=1}^m b_{nk}k_{kn}\right)x_n \end{cases} \quad (6)$$

From here we can find the components of the gradient vector for the component vector functions $\left(V_1(x_1,x_2,\ldots,x_n),V_2(x_1,x_2,\ldots,x_n),\ldots,V_n(x_1,x_2,\ldots,x_n)\right)$ :

$$\begin{cases} \dfrac{\partial V_1(x)}{\partial x_1} = -\left(a_{11} - \sum_{k=1}^m b_{1k}k_{k1}\right)x_1, \dfrac{\partial V_1(x)}{\partial x_2} = -\left(a_{12} - \sum_{k=1}^m b_{1k}k_{k2}\right)x_2, \ldots, \dfrac{\partial V_1(x)}{\partial x_n} = -\left(a_{1n} - \sum_{k=1}^m b_{1k}k_{kn}\right)x_n \\ \dfrac{\partial V_2(x)}{\partial x_1} = -\left(a_{21} - \sum_{k=1}^m b_{2k}k_{k1}\right)x_1, \dfrac{\partial V_2(x)}{\partial x_2} = -\left(a_{22} - \sum_{k=1}^m b_{2k}k_{k2}\right)x_2, \ldots, \dfrac{\partial V_2(x)}{\partial x_n} = -\left(a_{2n} - \sum_{k=1}^m b_{2k}k_{kn}\right)x_n \\ \ldots \\ \dfrac{\partial V_n(x)}{\partial x_1} = -\left(a_{n1} - \sum_{k=1}^m b_{nk}k_{k1}\right)x_1, \dfrac{\partial V_n(x)}{\partial x_2} = -\left(a_{n2} - \sum_{k=1}^m b_{nk}k_{k2}\right)x_2, \ldots, \dfrac{\partial V_n(x)}{\partial x_n} = -\left(a_{nn} - \sum_{k=1}^m b_{nk}k_{kn}\right)x_n \end{cases} \quad (7)$$

Total time derivative of the components of the vector Lyapunov function $V_i(x)$ given by the equation of motion (1) and (4) is determined by

$$\frac{dV_i(x)}{dt} = -\left[\left(a_{i1} - \sum_{k=1}^m b_{ik}k_{k1}\right)x_1 + \left(a_{i2} - \sum_{k=1}^m b_{ik}k_{k2}\right)x_2 +,,,+\left(a_{in} - \sum_{k=1}^m b_{ik}k_{kn}\right)x_n\right]^2, i = 1,2,\ldots,n \quad (8)$$

From the expressions (8) that the total time derivative of the vector-Lyapunov $V_i(x)$ functions in the performance of the initial assumptions resulting from the geometric interpretation of a theorem A.M. Lyapunov will be negative sign function. This means that the conditions for asymptotic stability of the system will always be performed (4).

Now, using components of the gradient vector we will restore components of the vector Lyapunov functions:

$$V_i(x_1,x_2,\ldots,x_n) = -\left(a_{i1} - \sum_{k=1}^m b_{ik}k_{k1}\right)x_1^2 - \left(a_{i2} - \sum_{k=1}^m b_{ik}k_{k2}\right)x_2^2 -,\ldots,-\left(a_{in} - \sum_{k=1}^m b_{ik}k_{kn}\right)x_n^2, i = 1,2,\ldots,n$$

The positive definiteness of all components of the vector Lyapunov function will be expressed by $-\left(a_{ij} - \sum_{k=1}^m b_{ik}k_{kj}\right) > 0, i = 1,2,\ldots,n, j = 1,2,..,n$ (9). This condition characterized superstability of transposed matrix of a closed system [4].

**5.The Radius Of The Robustness.** Let us investigate the robust stability of the vector-Lyapunov functions. Then let us transform the condition of robust stability of the components of the vector Lyapunov function. For this, we can turn to a parametric family of coefficients the vector-Lyapunov functions, such as the interval family, defined as [4]: $d_{ij} = d_{ij}^0 + \Delta_{ij}, \left|\Delta_{ij}\right| \le \gamma\, m_{ij}, i,j = 1,2,\ldots,n$ , where the nominal rate $d_{ij}^0 = -\left(a_{ij}^0 - \sum_{k=1}^m b_{ik}^0 k_{kj}^0\right)$ corresponds to a positive-definite Lyapunov functions, i.e.

$$\sigma(D_0) = \min_i \min_j -\left(a_{ij}^0 - \sum_{k=1}^m b_{ik}^0 k_{kj}^0\right) > 0$$

Now, we require that the positivity condition coefficients stored for all functions of the family: $-\left(a_{ij}^0 - \sum_{k=1}^m b_{ik}^0 k_{kj}^0\right) + \Delta_{ij} > 0, i = 1,2,\ldots,n; j = 1,2,\ldots,n$ . Clearly, this inequality holds for all admissible $\Delta_{ij}$ if and only if $-\left(a_{ij}^0 - \sum_{k=1}^m b_{ik}^0 k_{kj}^0\right) + \gamma\, m_{ij} > 0, i = 1,2,\ldots,n; j = 1,2,\ldots,n$ i.e.

$$\gamma < \gamma^* = \min_i \min_j \frac{-\left(a_{ij}^0 - \sum_{k=1}^m b_{ik}^0 k_{kj}^0\right)}{m_{ij}}$$

when (10). In particular, if $m_{ij} = 1$ (scale factors of a member of Lyapunov functions are the same), then $\gamma^* = \sigma(D_0)$ (11)

Thus, the stability radius of interval family of positive definite functions is the smallest value of the coefficients of the vector Lyapunov functions.

As an example, we consider the system described in state space.

Let n=2,m=1 i.e.,

$$\dot{x} = Ax + Bu,\ A = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix},\quad B = b = \begin{Vmatrix} b_1 \\ b_2 \end{Vmatrix}, u = -Kx,\quad K = k = \begin{Vmatrix} k_1 & k_2 \end{Vmatrix}$$

$$\dot{x}_1 = (a_{11} - b_1k_1)x_1 - (a_{12} - b_1k_2)x_2$$

$$\dot{x}_2 = (a_{21} - b_2k_1)x_1 - (a_{22} - b_2k_2)x_2$$

. Then

$$G = A + BK = \begin{Vmatrix} a_{11} - b_1k_1 - (a_{12} - b_1k_2) \\ a_{21} - b_2k_1 - (a_{22} - b_2k_2) \end{Vmatrix}$$

With inequality [4] characteristic equation has roots with negative real parts.

$$\begin{cases} a_{22} - b_2k_2 - a_{11} + b_1k_1 > 0 \\ (a_{12} - b_1k_2)(a_{21} - b_2k_1) - (a_{11} - b_1k_1)(a_{22} - b_2k_2) > 0 \end{cases}$$

We investigate the stability of the system using the idea of Lyapunov functions. Let us investigate the components of the gradient vector components vector functions $V_1(x_1, x_2)$ and $V_2(x_1, x_2)$:

$$\frac{\partial V_1(x_1,x_2)}{\partial x_1} = -(a_{11} - b_1k_1)x_1, \quad \frac{\partial V_1(x_1,x_2)}{\partial x_2} = +(a_{12} - b_1k_2)x_2$$

$$\frac{\partial V_2(x_1,x_2)}{\partial x_1} = -(a_{21} - b_2k_1)x_1, \quad \frac{\partial V_2(x_1,x_2)}{\partial x_2} = +(a_{22} - b_2k_2)x_2$$

We discover the total time derivative of the Lyapunov function by the formula (8):

$$\frac{dV(x_1,x_2)}{dt} = -[(a_{11} - b_1k_1)x_1 + (a_{12} - b_1k_2)x_2]^2 - [(a_{21} - b_2k_2)x_1 + (a_{22} - b_2k_2)x_2]^2 < 0$$

The next step - discovering vector Lyapunov functions

$$V_1(x_1,x_2) = -\frac{1}{2}(a_{11} - b_1k_1)x_1^2 + \frac{1}{2}(a_{12} - b_1k_2)_2^2 \quad V_2(x_1,x_2) = -\frac{1}{2}(a_{21} - b_2k_1)x_1^2 + \frac{1}{2}(a_{22} - b_2k_2)x_2^2$$

Conditions for the stability of the system obtained in the form:

$$-(a_{11} - b_1k_1) > 0,\ (a_{12} - b_1k_2) > 0,\ -(a_{21} - b_2k_1) > 0,\ (a_{22} - b_2k_2) > 0$$

and

$$-(a_{11} - b_1k_1) > (a_{21} - b_2k_1), (a_{22} - b_2k_2) > -(a_{12} - b_1k_2)$$

. From this we can get a system of inequalities $a_{22} - b_2k_2 - a_{11} + b_1k_1 > 0$

$$(a_{12} - b_1k_2)(a_{21} - b_2k_1) - (a_{11} - b_1k_1)(a_{22} - b_2k_2) > 0$$

Thus, from (9) and (10) we can determine the radius of robust stability of a second order system, if system parameters are uncertain: $\gamma^* = \min\{-(a_{11} - b_1k_1), (a_{12} - b_1k_2), -(a_{21} - b_2k_1), (a_{22} - b_2k_2)\}$

**6. The Proof Of The Proposed Method.** We will show how the proposed approach is applied to the sample and construct block diagonal matrix from matrix $A$.

$\tilde{A} = P^{-1}AP = diag\{\Lambda, J_1, ..., J_m, J_1', ..., J_k'\}$, (12). A block diagonal matrix is a block matrix which is a square matrix, and having main diagonal blocks square matrices, such that the off-diagonal blocks are zero matrices. A block diagonal matrix $A$ has the form $\Lambda = diag\{s_1, ..., s_l\}$; (13)

$$J_i = \begin{Vmatrix} s_i & 1 & \dots & 0 & 0 \\ 0 & s_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & s_i & 1 \\ 0 & 0 & \dots & 0 & s_i \end{Vmatrix} \quad N_i \times N_i, i = 1, ..., m$$

(14);

$$J_j' = \begin{Vmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{Vmatrix}, \quad j = 1, ..., k \qquad (15)$$

where $s_1, ..., s_l$ - Simple Real, $s_i$ -real, $N_i$ - multiple, $s_j = \alpha_j \pm \beta_j$ - complex conjugates eigenvalues of the matrix $A$, and, naturally, $l + N_1 + ... + N_m + 2k = n$.

The columns of the nonsingular matrix $P$ in the canonical transformation (13) are determined by the eigenvectors of the matrix $A$, the rules and algorithms of which are described, for example, in [10-12].

Let's demonstrate that stated structure (12) allows verifying the validity of suggested approach to the construction of Lyapunov function and dividing the system (1) depending on proper values of any diagonal block of $\tilde{A}$ matrix.

For this purpose let's write

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u = \begin{Vmatrix} \Lambda & & 0 \\ & J & \\ 0 & & J' \end{Vmatrix} \tilde{x} + \begin{Vmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \end{Vmatrix} u \ ; \qquad (16)$$

$$\tilde{u} = -\tilde{k}^T\tilde{x} = -\begin{Vmatrix} \tilde{k}_1^T \tilde{k}_2^T \tilde{k}_3^T \end{Vmatrix} \tilde{x} \qquad (17)$$

where $\tilde{x} = P^{-1}x$, $\tilde{A} = P^{-1}AP$, $\tilde{b} = P^{-1}b$, $\tilde{k}^T = k^TP$ (18)

and here dimensions of column matrices and row matrices $\tilde{k}_1^T$, $\tilde{k}_2^T$, $\tilde{k}_3^T$ match up with dimensions of square matrices $\Lambda$, $J$, $J'$. On the basis of (16), (17) it is easy to obtain characteristic determinant of closed system

$$\left|\lambda I - (\tilde{A} - \tilde{b}\tilde{k}^T)\right| = \left|sI_1 - (\Lambda - \tilde{b}_1\tilde{k}_1^T)\right|\left|sI_2 - (J - \tilde{b}_2\tilde{k}_2^T)\right|\left|sI_3 - (J' - \tilde{b}_3\tilde{k}_3^T)\right|$$

,which clearly shows that further problem amounts to sequential development in accordance with proposed method of accepted objects. $\dot{\tilde{x}} = \Lambda\tilde{x} + \tilde{b}_1u$ (19); $\dot{\tilde{x}} = J\tilde{x} + \tilde{b}_2u$ (20); $\dot{\tilde{x}} = J'\tilde{x} + \tilde{b}_3u$ (21).

With matrices in form of (13) – (15). Set of equations (19) is written in expanded form

$$\begin{cases} \dot{\tilde{x}}_1 = (s_1 - \tilde{b}_1\tilde{k}_1)\tilde{x}_1 \\ \dot{\tilde{x}}_2 = (s_2 - \tilde{b}_2\tilde{k}_2)\tilde{x}_2 \\ \dots \\ \dot{\tilde{x}}_l = (s_l - \tilde{b}_l\tilde{k}_l)\tilde{x}_l \end{cases}$$

For candidates of gradient vector from Lyapunov function $V(x_1, ..., x_l)$ we shall obtain

$\frac{\partial V(\tilde{x})}{\partial \tilde{x}_1} = -(s_1 - \tilde{b}_1\tilde{k}_1)\tilde{x}_1$, $\frac{\partial V(\tilde{x})}{\partial \tilde{x}_2} = -(s_2 - \tilde{b}_2\tilde{k}_2)\tilde{x}_2$ ,..., $\frac{\partial V(\tilde{x})}{\partial \tilde{x}_l} = -(s_l - \tilde{b}_l\tilde{k}_l)\tilde{x}_l$

Total derivative with time from Lyapunov function

$$\frac{dV(\tilde{x})}{dt} = \sum_{i=1}^{l} \frac{\partial V(\tilde{x})}{\partial \tilde{x}_i}\frac{d\tilde{x}_i}{dt} = \sum_{i=1}^{l}(s_i - \tilde{b}_i k_i)^2 \tilde{x}_i^2$$

will be negative function. Lyapunov function we shall obtain in form

$$V(\tilde{x}) = -(s_1 - \tilde{b}_1\tilde{k}_1)\tilde{x}_1^2 - (s_2 - \tilde{b}_2\tilde{k}_2)\tilde{x}_2^2 -,...,-(s_l - \tilde{b}_l\tilde{k}_l)\tilde{x}_l^2$$

.Positive definiteness of Lyapunov function is given by in equations $s_1 - \tilde{b}_1\tilde{k}_1 < 0$, $s_2 - \tilde{b}_2\tilde{k}_2 < 0$ ,..., $s_l - \tilde{b}_l\tilde{k}_l < 0$ . Here $s_i - \tilde{b}_i\tilde{k}_i = \mu_i, i = 1,...,l$ are eigenvalues of matrix in closed system and we shall obtain acquainted result of linear principle of stability $\mu_i = s_i - \tilde{b}_i\tilde{k}_i < 0, i = 1,...,l$ .Set of equations (20) we assume in expanded form for one Jordan block:

$$\begin{cases} \dot{\tilde{x}}_i == s_i\tilde{x}_i + \tilde{x}_{i+1} - \tilde{b}_i\tilde{k}_i\tilde{x}_i \\ \dot{\tilde{x}}_{i+1} == s_i\tilde{x}_{i+1} + \tilde{x}_{i+2} - \tilde{b}_{i+1}\tilde{k}_{i+1}\tilde{x}_{i+1} \\ ... \\ \dot{\tilde{x}}_{i+N_i} == s_i\tilde{x}_{i+N_i} - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i}\tilde{x}_{i+N_i} \qquad i = 1,...,m; \end{cases}$$

Gradient vector candidates of Lyapunov vector function in accordance with suggested approach will be equal to:

$$\frac{\partial V_i(\tilde{x})}{\partial \tilde{x}_{i+1}} = -(s_i - \tilde{b}_i\tilde{k}_i)\tilde{x}_i; \quad \frac{\partial V_i(\tilde{x})}{\partial \tilde{x}_{i+1}} = -\tilde{x}_{i+1}$$

$$\frac{\partial V_{i+1}(\tilde{x})}{\partial \tilde{x}_{i+1}} = -(s_i - \tilde{b}_{i+1}\tilde{k}_{i+1})\tilde{x}_{i+1}; \quad \frac{\partial V_{i+1}(\tilde{x})}{\partial \tilde{x}_{i+2}} = -\tilde{x}_{i+2} \quad ...$$

$$\frac{\partial V_{i+N_i}(\tilde{x})}{\partial \tilde{x}_{i+N_i}} = -(s_i - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i})\tilde{x}_{i+N_i};$$

Complete derivatives with time from Lyapunov vector functions have form:

$$\frac{dV_i(\tilde{x})}{dt} = -(s_i\tilde{x}_i + \tilde{x}_{i+1} - \tilde{b}_i\tilde{k}_i\tilde{x}_i)^2;$$

$$\frac{dV_{i+1}(\tilde{x})}{dt} = -(s_i\tilde{x}_{i+1} + \tilde{x}_{i+2} - \tilde{b}_{i+1}\tilde{k}_{i+1}\tilde{x}_{i+1})^2; ...;$$

$$\frac{dV_{i+N_i}(\tilde{x})}{dt} = -(s_i\tilde{x}_{i+N_i} - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i}\tilde{x}_{i+N_i})^2.$$

Complete derivatives with time are negative functions and meet the condition of asymptotic stability.

Candidates of Lyapunov vector function will be equal to:

$$V_i(\tilde{x}) = -(s_i - \tilde{b}_i\tilde{k}_i)\tilde{x}_i^2 - \tilde{x}_{i+1}^2,$$

$$V_{i+1}(\tilde{x}) = -(s_i - \tilde{b}_{i+1}\tilde{k}_{i+1})\tilde{x}_{i+1}^2 - \tilde{x}_{i+2}^2,...,$$

$$V_{i+N_i-1}(\tilde{x}) = -(s_i - \tilde{b}_{i+N_i-1}\tilde{k}_{i+N_i-1})\tilde{x}_{i+N_i-1}^2 - \tilde{x}_{i+N_i}^2,$$

$$V_{i+N_i}(\tilde{x}) = -(s_i - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i})\tilde{x}_{i+N_i}^2.$$

Condition of positive definiteness of Lyapunov function for system (20) we shall obtain in form

$$s_i - \tilde{b}_i\tilde{k}_i < 0,$$

$$s_i + 1 - \tilde{b}_{i+1}\tilde{k}_{i+1} < 0,..., s_i + 1 - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i} < 0, \quad i = 1,...,m. \qquad (22)$$

Set of inequations (22) also evaluates the condition of negativeness of real-valued roots of secular equation in closed system. Let's observe the system (21) in expanded form for one block:

$$\begin{cases} \dot{\tilde{x}}_i = \alpha_i\tilde{x}_i + \beta_i\tilde{x}_{i+1} - \tilde{b}_i\tilde{k}_i\tilde{x}_i \\ \dot{\tilde{x}}_{i+1} = -\beta_i\tilde{x}_i + \alpha_i\tilde{x}_{i+1} - \tilde{b}_{i+1}\tilde{k}_{i+1}\tilde{x}_{i+1} \quad i = 1,..,k \end{cases}$$

If we construct Lyapunov functions in form of vector functions with candidates $V_i(\tilde{x})$ and $V_{i+1}(\tilde{x})$, we shall obtain gradient vector candidates of Lyapunov function as follows

$$\frac{\partial V_i(\tilde{x})}{\partial \tilde{x}_i} = -(\alpha_i - \tilde{b}_i\tilde{k}_i)\tilde{x}_i, \quad \frac{\partial V_i(\tilde{x})}{\partial \tilde{x}_{i+1}} = -\beta_i\tilde{x}_{i+1},$$

$$\frac{\partial V_{i+1}(\tilde{x})}{\partial \tilde{x}_i} = \beta_i\tilde{x}_i,$$

$$\frac{\partial V_{i+1}(\tilde{x})}{\partial \tilde{x}_{i+1}} = -(\alpha_i - \tilde{b}_{i+1}\tilde{k}_{i+1})\tilde{x}_{i+1}$$

.Complete derivatives with time from Lyapunov vector function candidates

$$\frac{dV_i(\tilde{x})}{dt} = -(\alpha_i\tilde{x}_i + \beta_i\tilde{x}_{i+1} - \tilde{b}_i\tilde{k}_i\tilde{x}_i)^2,$$

$$\frac{dV_{i+1}(\tilde{x})}{dt} = -(-\beta_i\tilde{x}_i + \alpha_i\tilde{x}_{i+1} - \tilde{b}_{i+1}\tilde{k}_{i+1}\tilde{x}_{i+1})^2$$

are negative function and meets the conditions of asymptotic stability. Lyapunov function in scalar form is given in form $V_i(\tilde{x}) = -2(\alpha_i - \tilde{b}_i\tilde{k}_i)\tilde{x}_i^2$, $i = 1,..,k$ . Conditions of Lyapunov function positive definiteness is written $\alpha_i - \tilde{b}_i\tilde{k}_i < 0$, $i = 1,..,k$ (23). Condition (22) evaluates negativeness of real part of performance equation roots in closed system. Thus, the correctness of suggested approach is supported by results of linear conception of stability, q.e.d.

**7. Case study**. Then, as an example, we define the following initial conditions and find conditions for the stability of the system, the radius and transients. When the initial settings are follow:

$$A = \begin{Vmatrix} 6 & 5 \\ 13 & 0.03 \end{Vmatrix}, B = \begin{Vmatrix} 9.2 \\ 7 \end{Vmatrix}, K = \begin{Vmatrix} 2 & 0.001 \end{Vmatrix}$$

In this case, the radius will be equal ($\gamma^* = 0.0160$). The overall the transition process of the system shows on the Figure 1 with green color. The second case, when the initial settings are follow:

$$A = \begin{Vmatrix} 7 & 5.6 \\ 13 & 0.4 \end{Vmatrix}, B = \begin{Vmatrix} 11 \\ 8 \end{Vmatrix}, K = \begin{Vmatrix} 2 & 0.001 \end{Vmatrix}$$

. In this second case, the radius will be equal $\gamma^* = 0.3840$ and eigenvalues are real sample. The overall the transition process of the system shows on the Figure 1 with red color.

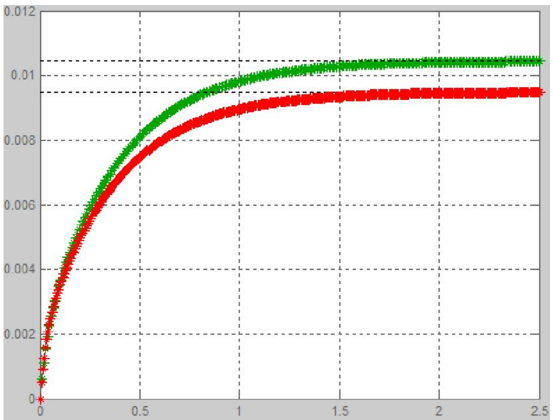

Fig. 1.The transition process, exp.1-2.

The third case, when the initial settings are follow:

$$A = \begin{Vmatrix} 11.6 & 15 \\ 11.7 & 0.45 \end{Vmatrix}, B = \begin{Vmatrix} 9.8 \\ 6.2 \end{Vmatrix}, K = \begin{Vmatrix} 2 & 0.001 \end{Vmatrix}$$

In this case, the radius will be equal $\gamma^* = 0.3876$ and eigenvalues are real simple. The overall the transition process of the system shows on the Figure 2 with blue color. For 4-d case, when the initial settings are follow:

$$A = \begin{Vmatrix} 11.6 & 15 \\ 11.7 & 0.4 \end{Vmatrix}, B = \begin{Vmatrix} 9 \\ 6 \end{Vmatrix}, K = \begin{Vmatrix} 2 & 0.001 \end{Vmatrix}$$

In this case, the radius will be equal $\gamma^* = 0.3880$ and eigenvalues are real simple. The overall the transition process of the system shows on the Figure 2 with margin color.

Complex analysis of all examples shown on Figure 3.For all the cases the stability conditions of the robust control system are executed. The experiment part of the proposed system obtained.

**Conclusion.**

In our theory robust stability perform an important function in the theory of control of dynamic objects is [13,14]. The main point of robust stability study is to specify constraints on the change control system parameters that preserve stability. These limits are determined by the region of stability in an uncertain and are selected, i.e. changing parameters [15,16,17,18].

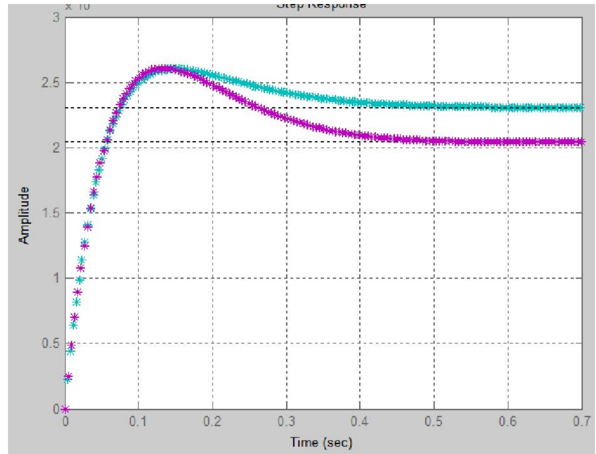


Fig. 2. The transition process, exp.3-4.

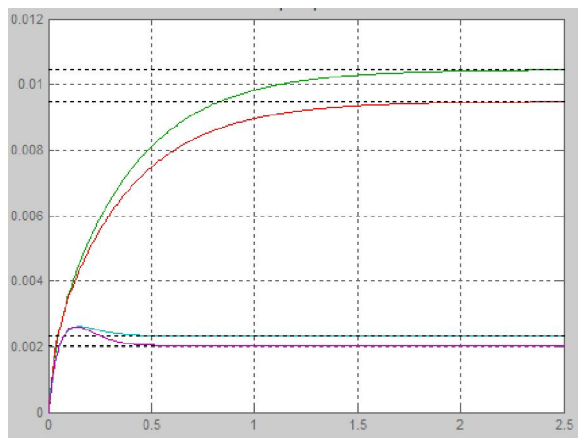

Fig.3. Compare analysis .

This paper describes a novel idea - a new theoretical method of robust stability for linear system. This method is an extension of the notion of stability where the Lyapunov function is replaced by a geometric interpretation of the Lyapunov function with dependence on the uncertain parameters [17,18,19]. The radius of stability coefficients interval family of positive definite functions is equal to the smallest value of the coefficients of the vector Lyapunov functions. Theoretical results obtained in this paper are an important contribution to the theory of stability, to the theory of robust stability of linear control systems. Thus, for a wide class of systems, we believe the theory is sufficiently well developed that work can begin on developing efficient approach to aid control engineers in incorporating the parametric approach into their analysis and design toolboxes. The practical importance of these results should motivate new theoretical studies on typical application techniques, clarification area of the robust control and stability [19,20]. Finally, this is the main results that theoretical approaches represent the most promising direction. These studies are especially important for the designing more effective control systems.

**Acknowledgment**

Authors would like to thank L.N. Gumilyov Eurasian National University for funding this project (under Grant No.264).

**Corresponding Author:**
Dr. Beisenbi
L. N. Gumilyov Eurasian National University,
Mirzoyan Street, 2, 010008, Astana, Kazakhstan

**References**

1. Barbashin, E.A., 1967. Introduction in the theory of stability. Moscow: Publishing Nauka, 255 p.
2. Malkin, I.G., 1966. The theory of stability of motion. 2-d Publishing. Moscow: Nauka, 540 p.
3. Siljak, D.D., 1989. Parameter space methods for robust control design: a guided tour. IEEE Transactions on Automatic Control, V.AC-34, 7: 674-688.
4. Polyak, B. and P. Scherbakov, 2002. Robust Stability and Control. Moscow: Nauka, ISBN 5-02-002561-5, 303 p.
5. Polyak, B. and P. Scherbakov, 2002. Superstable linear control systems I, II. Avtomatika i Telemekhanika, 8: 37–53. ISSN 005-2310.
6. Polyak, B., 2010. Introduction to optimization. Optimization Software. New York: Inc. Publications Division, 438 p.
7. Gilmore R., 1981. Catastrophe Theory for Scientists and Engineers. Moscow: Publ. Mir. 300 p.
8. Voronova, A.A., V.M. Matrosova, 1987. Methods of Lyapunov vector functions in theory of stability. Moscow: Nauka, 312 p.
9. Pupkov, K.A., N.D. Egugov, 2004. Methods of classical and modern theory of automatic control. Arithmetic models, dynamic behavior and analysis of automatic control systems. Vol.1. Moscow: Publishing office of Bauman University, 656 p.
10. Gantmacher, F. R., 2002. The Theory of Matrices. AMS Chelsea Publishing: Reprinted by American Mathematical Society, 660 p.
11. Strejc, V, 1985. State Space Theory of Discrete Linear Control. Moscow: Nauka, 200 p.
12. Lancaster, P., 1978.The theory of matrices. Translated from English. Moscow: Nauka, 280 p.
13. Beisenbi, M.A., K.S. Kulniyazova, 2007. Research of robust stability in the control systems with Lyapunov A.M. direct method. In the Proceedings of 11-th Inter-University Conference on Mathematics and Mechanics. Kazakhstan. pp. 50-56.
14. Beisenbi, M.A., 2011. Methods of increasing the potential of robust stability control systems. Astana: Publishing office of L.N.Gumilyov Eur Nat University, 352 p.
15. Abitova, G.A., M.A. Beisenbi, V.V. Nikulin and A. Ainagulova, 2012. Design of Control Systems for Nonlinear Control Laws with Increased Robust Stability. In the Proceedings of the CSDM 2012 France. pp.150-155.
16. Abitova, G.A., M.A. Beisenbi, V.V. Nikulin and A. Ainagulova, 2012. Design of Control System Based on Functions of Catastrophe. In the Proceedings of the IJAS 2012, Massachusetts, pp.236-242.
17. Abitova, G.A., M.A. Beisenbi, V.V. Nikulin, V.A. Skormin and A. Ainagulova, 2012. Control System with High Robust Stability Characteristics Based on Catastrophe Function. In the Proceedings of the ICECCS 2012 (IEEE). France, Paris. pp.273-279.
18. Abitova, G.A., M.A. Beisenbi and V.V. Nikulin, 2012. Design of Complex Automation System for Effective Control of Technological Processes of Industry. In the Proceedings of the IEOM 2012. Turkey, pp.120-126.
19. Yermekbayeva, J.J., M.A. Beisenbi, A.N. Omarov and G.A. Abitova, 2012. The Control of Population Tumor Cells via Compensatory Effect, Proceedings of the International Conference on Math. Sciences & Computer Engn. (ICMSCE 2012), Malaysia. pp.46-52.
20. Beisenbi, M.A. and L.G. Abdrakhmanova, 2013. Research of dynamic properties of control systems with increased potential of robust stability in a class of two-parameter structurally stable maps by Lyapunov Function, ICCNCE 2013. Atlantis Press, pp: 201-203.

11/10/2013