On dual elastic lines in dual Lorentzian space $\mathcal{D}_1^3$

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Abstract: In this paper, we derive intrinsic formulation for dual elastic line on the non-null dual unit sphere.

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I. INTRODUCTION

Nickerson and Manning studied intrinsic equations for a relaxed elastic line on an oriented surface (Nickerson and Manning, 1988).
In this paper we study intrinsic equations for non-null dual elastic line on the dual unit sphere.

In this section, definitions were taken from (Clifford, 1973) and (Uğurlu and Çalışkan , 1996), (Köse, 1988).
The set $\mathcal{D}$ of dual numbers is a commutative ring with the operations (+) and (.) (Clifford, 1973).

$$\mathcal{D}^3=\mathcal{D}\times\mathcal{D}\times\mathcal{D}=\{\widetilde{X}=x+\varepsilon\widetilde{x}_0,\quad x,x_0\in R^3\}$$

The elements of $\mathcal{D}$ are called the dual numbers.
Let $\widetilde{X}=x+\varepsilon\widetilde{x}_0$, $\widetilde{Y}=y+\varepsilon\widetilde{y}_0$ be dual unit vectors in $\mathcal{D}^3$. The Lorentzian inner product of two dual vectors $\widetilde{X}$ and $\widetilde{Y}$ is defined by

$$\left\langle\widetilde{X},\widetilde{Y}\right\rangle=\left\langle x,y\right\rangle+\varepsilon(\left\langle x,\widetilde{y}_0\right\rangle+\left\langle\widetilde{x}_0,y\right\rangle)$$

where ε is dual unit with $\varepsilon^2=0$ and $x,x_0,y,y_0\in R_1^3$. The dual space $\mathcal{D}^3$ with Lorentzian inner product is called dual Lorentzian space $\mathcal{D}_1^3$ (Köse, 1988), (Uğurlu and Çalışkan, 1996).
$\widetilde{X}$ is timelike if the vector x is timelike and $\widetilde{X}$ is spacelike if the vector x is spacelike.

$$\widetilde{S}_1^2=\{\widetilde{X}=x+\varepsilon\widetilde{x}_0,\quad \left\|\widetilde{X}\right\|=(1,0)\mid x,\cdot\in R_1^3,\left\langle x,\widetilde{x}_0\right\rangle=0,\quad x\text{ is spacelike}\}$$

is called the dual Lorentzian unit sphere in $\mathcal{D}_1^3$.

$$\widetilde{H}_0^2=\{\widetilde{X}=x+\varepsilon\widetilde{x}_0,\quad \left\|\widetilde{X}\right\|=(1,0)\mid x,\widetilde{x}_0\in R_1^3,\left\langle x,\widetilde{x}_0\right\rangle=0,\quad x\text{ is timelike}\}$$

is called the dual hyperbolic unit sphere in $\mathcal{D}_1^3$ (Köse, 1988).

Dual arc length of non-null curve β is given by

$$L=\int_{t_0}^{t}\sqrt{\left|\left\langle\frac{d\beta}{ds},\frac{d\beta}{ds}\right\rangle\right|}dt=s+\varepsilon\widetilde{s}_0.$$

$\widetilde{X}(s)=x+\varepsilon\widetilde{x}_0(s)$ dual unit vector draws a curve on a dual unit sphere, this curve corresponds to non-null ruled surface $\widetilde{X}(s)=x+\varepsilon\widetilde{x}_0(s)$ in Minkowski 3-space R_1^3.

2. INTRINSIC METHOD

Let β be non-null dual curve on dual unit sphere in $\mathcal{D}_1^3$ parametrized by arc length s.

The dual total square curvature K of β in $\mathcal{D}_1^3$ is defined by

$$K = \int_0^l \kappa^2 ds \tag{2}$$

where κ is the dual curvature function of non-null curve β.

The non-null dual arc β is called elastic line if it is an extremal for the variational problem of minimizing the value of K within the family of all arcs of length l on the dual unit sphere having the same initial point and initial direction as β.

Assume β lies in a coordinate patch $(\iota, j) \rightarrow r(\iota, j)$ of dual unit sphere in $\mathcal{D}_1^3$

$$r_\iota = \frac{\partial r}{\partial \iota}, \quad r_j = \frac{\partial r}{\partial j},$$

$$\widetilde{T}(s) = \beta'(s) = \frac{\partial r}{\partial \iota}\frac{d\iota}{ds} + \frac{\partial r}{\partial j}\frac{dj}{ds}$$

$$\widetilde{G}(s) = \rho(s) r_\iota + \chi^*(s) r_j$$

In order to obtain variational arcs of length l, it is generally necessary to extend β to an arc β^* defined for $0 \le s \le l^*$, with $l^* > l$, but sufficiently close to l so that β^* lies in the coordinate patch. Let $\mu(s)$, $0 \le s \le l^*$,be a scalar sunction of class C^2, not vanishing identically. Define

$$\eta(s) = \mu(s)\rho^*(s), \quad \xi(s) = \mu(s)\chi^*(s).$$

Along β

$$\eta(s) r_\iota + \xi(s) r_j = \mu(s)\widetilde{G}(s). \tag{3}$$

Assume also that

$$\mu(0) = 0, \quad \mu'(0) = 0. \tag{4}$$

Define

$$\Psi(\sigma; t) = r(\iota(\sigma) + t\eta(\sigma), j(\sigma) + t\xi(\sigma)), \tag{5}$$

for $0 \le \sigma \le l^*$. $\Psi(\sigma;t)$ lies in the dual coordinate patch. For constant t , $\Psi(\sigma;t)$ give an non-null dual arc with the same initial point and initial direction as β . For t =0 , $\Psi(\sigma,0) = \beta(s)$ in dual Lorentzian space.. Also, we get

$$\int_0^{\lambda(t)} \sqrt{\left| \left\langle \frac{\partial \psi}{\partial \sigma}, \frac{\partial \psi}{\partial \sigma} \right\rangle \right|} d\sigma = l. \tag{7}$$

Theorem 2.1. The analogue of the Frenet-Serret derivative formulas in the dual Lorentzian space $\mathcal{D}_1^3$ is

$$\frac{d}{ds}\begin{bmatrix} \widetilde{T} \\ \widetilde{G} \\ \widetilde{N} \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 \widetilde{\omega}_g & \varepsilon_3 \\ -\varepsilon_1 \widetilde{\omega}_g & 0 & 0 \\ -\varepsilon_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{T} \\ \widetilde{G} \\ \widetilde{N} \end{bmatrix}$$

where $\widetilde{T}=\beta'$ is the non-null unit dual tangent vector to β, $\widetilde{N}(s)$ is the non-null unit normal to the unit sphere, $\widetilde{\omega}_g$ is the dual geodesic curvature and $\langle \widetilde{T},\widetilde{T}\rangle=\varepsilon_1$, $\langle \widetilde{G},\widetilde{G}\rangle=\varepsilon_2$, $\langle \widetilde{N},\widetilde{N}\rangle=\varepsilon_3$.
For spacelike surfaces are given by

$$\widetilde{T}\times\widetilde{G}=\widetilde{N},\quad \widetilde{G}\times\widetilde{N}=-\widetilde{T},\quad \widetilde{N}\times\widetilde{T}=-\widetilde{G}.$$

where $\times$ is the Lorentzian vector product.
From Theorem 2.1 (4) and (5), we obtain

$$\left.\frac{\partial\Psi}{\partial\sigma}\right|_{t=0}=\widetilde{T},\quad 0\le\sigma\le l \tag{8}$$

which gives

$$\left.\frac{\partial^2\Psi}{\partial\sigma^2}\right|_{t=0}=\widetilde{T}'=\varepsilon_2\widetilde{\omega}_g\widetilde{G}+\varepsilon_3\widetilde{N}. \tag{9}$$

Also, it follows from (3) that

$$\left.\frac{\partial\Psi}{\partial t}\right|_{t=0}=\mu\widetilde{G}. \tag{10}$$

Using (3), the second differentiation of (10) gives

$$\left.\frac{\partial^2\Psi}{\partial t\partial\sigma}\right|_{t=0}=-\varepsilon_1\mu\widetilde{\omega}_g\widetilde{T}+\mu'\widetilde{G} \tag{11}$$

$$\left.\frac{\partial^3\Psi}{\partial t\partial\sigma^2}\right|_{t=0}=(-\varepsilon_1\mu'\widetilde{\omega}_g-\varepsilon_1\mu\widetilde{\omega}_g')\widetilde{T}+(\mu''-\varepsilon_1\varepsilon_2\mu\widetilde{\omega}_g^2)\widetilde{G}-\varepsilon_1\varepsilon_3\mu\widetilde{\omega}_g)\widetilde{N}. \tag{12}$$

$$\left.\frac{d\lambda}{dt}\right|_{t=0}\sqrt{\left|\left\langle \left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0},\left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0}\right\rangle\right|}+\int_0^l\left\langle \left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0},\left.\frac{\partial^2\psi}{\partial\sigma\partial t}\right|_{t=0}\right\rangle\left\langle \left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0},\left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0}\right\rangle^{-1}\sqrt{\left|\left\langle \left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0},\left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0}\right\rangle\right|}ds=0 \tag{13}$$

Using (10), (11), (13) we obtain

$$\left.\frac{\partial\lambda}{\partial t}\right|_{t=0}=\varepsilon_1\int_0^l\mu\kappa\widetilde{\omega}_g ds \tag{14}$$

Let $K(t)$ denote the dual total square curvature of the arc $\Psi(\sigma;t)$. Since σ is not generally arc length for $t\neq 0$, the dual total square curvature is

$$K(t)=\int_0^{\lambda(t)}\frac{\left|\left\langle \frac{\partial\Psi}{\partial\sigma}\times\frac{\partial^2\Psi}{\partial\sigma^2},\frac{\partial\Psi}{\partial\sigma}\times\frac{\partial^2\Psi}{\partial\sigma^2}\right\rangle\right|}{\left\langle \frac{\partial\Psi}{\partial\sigma},\frac{\partial\Psi}{\partial\sigma}\right\rangle^{5/2}}d\sigma$$

A necessary condition for β being extremal is that $K'(0)=0$ for arbitrary dual μ satisfying (4). We compute,

$$K'(t)=\frac{d\lambda}{dt}\left\{\left|\left\langle\frac{\partial^2\Psi}{\partial\sigma^2},\frac{\partial^2\Psi}{\partial\sigma^2}\right\rangle\right|\left\langle\frac{\partial\Psi}{\partial\sigma},\frac{\partial\Psi}{\partial\sigma}\right\rangle^{-3/2}\right\}_{\sigma=\lambda(t)}-3\int_0^{\lambda(t)}\left\langle\frac{\partial^2\Psi}{\partial t\partial\sigma},\frac{\partial\Psi}{\partial\sigma}\right\rangle\frac{\left|\left\langle\frac{\partial\Psi}{\partial\sigma},\frac{\partial\Psi}{\partial\sigma}\right\rangle\right|}{\left\langle\frac{\partial\Psi}{\partial\sigma},\frac{\partial\Psi}{\partial\sigma}\right\rangle}\frac{\left|\left\langle\frac{\partial^2\Psi}{\partial\sigma^2},\frac{\partial^2\Psi}{\partial\sigma^2}\right\rangle\right|}{\left|\left\langle\frac{\partial\Psi}{\partial\sigma},\frac{\partial\Psi}{\partial\sigma}\right\rangle\right|^{5/2}}d\sigma$$

$$+2\int_0^{\lambda(t)}\frac{\left|\left\langle\frac{\partial^2\Psi}{\partial\sigma^2},\frac{\partial^2\Psi}{\partial\sigma^2}\right\rangle\right|}{\left\langle\frac{\partial^2\Psi}{\partial\sigma^2},\frac{\partial^2\Psi}{\partial\sigma^2}\right\rangle}\frac{\left\langle\frac{\partial^3\Psi}{\partial t\partial\sigma^2},\frac{\partial^2\Psi}{\partial\sigma^2}\right\rangle}{\left|\left\langle\frac{\partial\Psi}{\partial\sigma},\frac{\partial\Psi}{\partial\sigma}\right\rangle\right|^{3/2}}d\sigma \qquad (15)$$

From (8),(9),(11) and (14), we obtain

$$K'(0)=\varepsilon_1\int_0^l\mu\tilde{\omega}_g ds\{\left|\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3\right|\}_{\sigma=\lambda(0)}+2\int_0^l\tilde{\omega}_g(\mu''-\varepsilon_1\varepsilon_2\mu\tilde{\omega}_g^2)\frac{\left|\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3\right|}{\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3}ds$$

$$-2\int_0^l(\varepsilon_1\varepsilon_3\mu\tilde{\omega}_g)\frac{\left|\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3\right|}{\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3}ds-2\int_0^l(\varepsilon_1\varepsilon_3\mu\tilde{\omega}_g)\frac{\left|\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3\right|}{\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3}ds+3\varepsilon_1\int_0^l\mu\tilde{\omega}_g\left|\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3\right|ds$$

Integration by parts and (4),

$$2\int_0^l\mu''\tilde{\omega}_g ds=2\mu'(l)\tilde{\omega}_g(l)-2\mu(l)\tilde{\omega}_g(l)+2\int_0^l\mu\tilde{\omega}_g ds \qquad (16)$$

2.1 Intrinsic equations for dual elastic line on dual unit Lorentzian sphere

In this case, $\tilde{T}$ is timelike, $\tilde{G}$ ve $\tilde{N}$ are spacelike .

$$\left\langle\tilde{T},\tilde{T}\right\rangle=\varepsilon_1=-1,\quad\left\langle\tilde{G},\tilde{G}\right\rangle=\varepsilon_2=1,\left\langle\tilde{N},\tilde{N}\right\rangle=\varepsilon_3=1.$$

For $\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3>0$,

$$\left|\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3\right|=\tilde{\omega}_g^2+\varepsilon_3. \qquad (17)$$

Substituting (8), (11), (14), (16) and (17) in (15), we find

$$K'(0)=\int_0^l\mu\{2\tilde{\omega}_g''+\tilde{\omega}_g(-\tilde{\omega}_g^2(l)-2-\tilde{\omega}_g^2)\}ds+2\mu'(l)\tilde{\omega}_g(l)-2\mu(l)\tilde{\omega}_g'(l)$$

In order that $K'(0)=0$ for all choices of the dual function $\mu(s)$ satisfying (4), with arbitrary values of $\mu(l)$ and $\mu'(l)$, the given dual timelike arc β must satisfy two boundary conditions and differential equation

(BC1) $\tilde{\omega}_g(l)=0$

(BC2) $\tilde{\omega}_g'(l)=0$ (18)

(DE) $2\tilde{\omega}_g''+\tilde{\omega}_g(-\tilde{\omega}_g^2(l)-2-\tilde{\omega}_g^2)=0.$

2.2 Intrinsic equations for dual elastic line on dual unit Lorentzian sphere in dual Lorentzian space $\mathcal{D}_1^3$

In this case, $\tilde{G}$ is timelike, $\tilde{T}$ and $\tilde{N}$ are spacelike.

If $\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3>0$

$$\left|\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3\right|=-\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3. \tag{19}$$

Substituting (8), (11), (14), (16) and (19) in (15), we find $K'(0)$ is given by

$$K'(0)=\int_0^l \mu\left\{2\tilde{\omega}''_g+\tilde{\omega}_g(-\tilde{\omega}_g^2(l)+2-\tilde{\omega}_g^2)\right\}ds+2\mu'(l)\tilde{\omega}_g(l)-2\mu(l)\tilde{\omega}'_g(l)$$

In order that $K'(0)=0$ for all choices of the dual function $\mu(s)$ satisfying (4), with arbitrary values of $\mu(l)$ and $\mu'(l)$, the given dual timelike arc β must satisfy two boundary conditions and differential equation

$$\begin{aligned} &\text{(BC1)} \quad \tilde{\omega}_g(l)=0\\ &\text{(BC2)} \quad \tilde{\omega}'_g(l)=0\\ &\text{(DE)} \quad 2\tilde{\omega}''_g+\tilde{\omega}_g(-\tilde{\omega}_g^2(l)+2-\tilde{\omega}_g^2)=0. \end{aligned} \tag{20}$$

2.3 Intrinsic equations for dual elastic line on dual unit hyperbolic sphere in dual Lorentzian space $\mathcal{D}_1^3$

The case $\tilde{T}$, $\tilde{G}$ is spacelike and $\tilde{N}$ is timelike,

For $\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3>0$

$$\left|\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3\right|=\varepsilon_2\tilde{\omega}_g^2+\varepsilon_3. \tag{21}$$

Substituting (8), (11), (14), (16) and (21) in (15), we find $K'(0)$ can be written as

$$K'(0)=\int_0^l \mu\left\{2\tilde{\omega}''_g+\tilde{\omega}_g(\tilde{\omega}_g^2(l)+\tilde{\omega}_g^2-2)\right\}ds+2\mu'(l)\tilde{\omega}_g(l)-2\mu(l)\tilde{\omega}'_g(l)$$

In order that $K'(0)=0$ for all choices of the function $\mu(s)$ satisfying (4), with arbitrary values of $\mu(l)$ and $\mu'(l)$, the given dual timelike arc β must satisfy two boundary conditions and differential equation

$$\begin{aligned} &\text{(BC1)} \quad \tilde{\omega}_g(l)=0\\ &\text{(BC2)} \quad \tilde{\omega}'_g(l)=0\\ &\text{(DE)} \quad 2\tilde{\omega}''_g+\tilde{\omega}_g(\tilde{\omega}_g^2(l)+\tilde{\omega}_g^2-2)=0. \end{aligned} \tag{22}$$

3.APPLICATIONS

Theorem 3.1. On dual hyperbolic unit sphere $\tilde{H}_0^2$, an dual arc is dual elastic line if and only if it lies on dual hyperbolic circle.

Proof. The third equation in (22) reduces to

$$2\tilde{\omega}_g+\tilde{\omega}_g^3-2\tilde{\omega}_g=0. \tag{23}$$

With integrating factor $\tilde{\omega}_g$, the first integral is

$$(\tilde{\omega}_g)^2+\frac{1}{4}\tilde{\omega}_g^4-\tilde{\omega}_g^2=const.$$

The boundary conditions in (22), which reduces to $\tilde{\omega}_g(l)=0$, require that the constant is zero. Thus, we have $\tilde{\omega}_g\equiv 0$.

If $\tilde{\omega}_g \equiv 0$, the dual curvature $\tilde{\kappa} = \sqrt{\left|\varepsilon_2 \tilde{\omega}_g^2 + \varepsilon_3\right|} = 1$, $\tau = 0$.

Conversely, any dual circle on $\tilde{H}_0^2$ satisfies (22), trivially.

Corollary 3.1. A spacelike ruled surface in 3-dimensional Minkowski space is dual elastic if and if it correspond a dual hyperbolic circle on the unit dual Hyperbolic sphere.

Theorem 3.2. On dual Lorentzian sphere $\tilde{S}_1^2(r)$, a dual arc is dual elastic line if and only if it lies on dual Lorentzian circle.

Proof. On dual Lorentzian sphere $\tilde{S}_1^2(r)$, the third equation in (18) reduces to

$$2\tilde{\omega}_g'' - \tilde{\omega}_g(\tilde{\omega}_g^2 + 2) = 0. \tag{24}$$

With integrating factor $\tilde{\omega}_g$, the first integral is

$$(\tilde{\omega}_g')^2 - \frac{1}{4}\tilde{\omega}_g^4 - \tilde{\omega}_g^2 = const.$$

The boundary conditions in (18), which reduces to $\tilde{\omega}_g'(l) = 0$, require that the constant is zero. But then, we must have $\tilde{\omega}_g \equiv 0$.

Similarly, the third equation in (20) reduces to

$$2\tilde{\omega}_g'' - \tilde{\omega}_g(-\tilde{\omega}_g^2 + 2) = 0. \tag{25}$$

With integrating factor $\tilde{\omega}_g'$, the first integral is

$$(\tilde{\omega}_g')^2 - \frac{1}{4}\tilde{\omega}_g^4 + \tilde{\omega}_g^2 = const.$$

The boundary conditions in (20), which reduces to $\tilde{\omega}_g'(l) = 0$, require that the constant is zero. We have $\tilde{\omega}_g \equiv 0$.

Conversely, any arc of a dual geodesic on dual Lorentzian sphere $\tilde{S}_1^2(r)$ satisfies (20), trivially.

Corollary 3.2. An timelike ruled surface in 3-dimensional Minkowski space is dual elastic if and if it correspond a dual Lorentzian circle on dual Lorentzian sphere.

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