Computing of Julian Calendar by Congruence Relation

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Abstract: In this paper we discuss the application of linear congruence. We give an interesting application of linear congruences. Through this we find what day will fall on some required specific date from a given date both backward and onward and no matter how far it is. By using our method which is both naval and easier we can count that how many times a specific day of a week will occur in a given month of a given year.
[Umar Farooq, Muhammad Sarwar and Muhammad Waqar. **Computing of Calendar by Congruence Relation.** *Life Sci J* 2013; 10(10s): 265-269]. (ISSN: 1097-8135). http://www.lifesciencesite.com. 43

Keywords: Julian Calendar, days, linear congruence, Conjecture.

Introduction and preliminaries:

Number theory, also known as higher arithmetic, is a branch of mathematics concerned with the properties of integers, rational numbers, irrational numbers, and real numbers. Sometimes the discipline is considered to include the imaginary and complex numbers as well.

Number theory is the study of the properties of the integers, particularly concerning prime numbers. It is often one of the first courses after calculus that math majors take. Some parts of number theory have important uses in computing science (see [4]).

It might be better to call it "integer theory", but the name "number theory" has been around for more than a century so we think we are stuck with it.

Linear congruence is one of the important topic in number theory, there are many applications (for vmore examples see [1],[2] and [3]). Thomos Koshy has discussed one application in his book "Elementary Number Theory with Applications" CHAPTER 5 Congruence Applications page 282 topic "5.6 The Perpetual Calendar (optional)" [1]. This application is briefly discussbelow.

The Perpetual Calendar (optional)

Thomos Koshy developed an interesting formula [1] to determine the day of the week for any date in any year. Since the same day occurs every seventh day, he employed congruence modulo 7 to accomplish this goal. But first a few words about the historical background of this theory and its derivation.

Around 738 B.C, Romulus, the legendary founder of Rome is said to have introduced a calendar consisting of 10 months, comprising a year of 304 days. His successor, Nauma, is credited with adding two months to the calendar. This new calendar was followed until Julius Caesar introduced the Julian Calendar in 46 B.C, to minimize the distortions between the solar calendar and the Roman year. The Julian Calendar consisted of 12 months of 30 and 31days, except for February, which had 29 days, and every fourth year 30 days. The first Julian year began on January 1, 45B.C. It contained 365.25 days, was 11 minutes and 14 seconds longer than the solar year, and made every fourth year a leap year of 366 days. By 1580, the Julian calendar, although the primary calendar in use, was 10 days off. It was, however, widely used until 1582.

In October 1582, astronomers Fr. Christopher Clavius and Aloysius Giglio introduced the Gregorian Calendar at the request of Pope Gregory XIII, to rectify the errors of the Julian calendar. The accumulated error of 10 days was compensated by dropping 10 days in October, 1582. (October 5 became October 15). The Gregorian Calendar designates those century years divisible by 400 as leap years; all noncentury years divisible by 4 are also leap years. For example, 1776 and 2000 were leap years, but 1900 and 1974 were not. T he Gregorian Calendar, now used throughout the world, is so accurate that it differs from the solar year only by about 24.5376 seconds. This discrepancy exists because a Gregorian year contains about 365.2425 days, whereas a solar year contains about 365.242216 days. The result is an error of 3 days every 10,000 years.

With this in mind, we can now return to our goal; determine the day d of the week for the r^{th} day in a given month m of any given year y in the Gregorian Calendar. The first century leap year occurred in 1600 (18 years after the introduction of the Gregorian calendar); so Koshy developed the formula which would hold for years beyond 1600. Also, since a leap year adds a day to February, he would count the New Year beginning with March 1. For example, January 3000 is considered the eleventh month of 2999, whereas April 3000 is the second month of year 3000; also February 29 of 1976 is the last day of the 12th month of 1975. So he assigned the numbers 1 through 12 for March through February, and 0 through 6 for Sunday through Saturday; so, $1 \leq m \leq 12, 1 \leq r \leq 31$, and $0 \leq d \leq 6$. For example, $m = 3$ denotes May and $d = 5$. indicates Friday. The derivation is lengthy and complicated, so Koshy developed the formula in small steps. Let d_y denote the day of the week of March 1 (the first day of the year) in year y, where $y \geq 1600$.

To compute d from d_{1600}:

Because $365 \equiv 1(mod\ 7)$, d_y is advanced from d_{y-1} by 1 if y is not a leap year and by 2 if y is a leap year:

$$d_y = \begin{cases} d_{y-1} + 1 & \text{if } y \text{ is not leap year} \\ d_{y-1} + 2 & \text{otherwise} \end{cases}$$

To compute d_y from d_{1600}, we need to know the number of leap years L since 1600.

By Example 2.5, (see [1])

$$L = \left\lfloor\frac{y}{4}\right\rfloor - \left\lfloor\frac{y}{100}\right\rfloor + \left\lfloor\frac{y}{400}\right\rfloor - 388 \qquad (1.1)$$

By the division algorithm m, $y = 100C + D$, where $0 \leq D < 100$, so C denotes the number of centuries in y and D the left over:

$C = \frac{y}{100}$ and $D = y\ (mod\ 100)$

(For example, if $y = 2345$, then $C = 23$ and $D = 45$).

Then

$$L = \left\lfloor\frac{100C + D}{4}\right\rfloor - \left\lfloor\frac{100C + D}{100}\right\rfloor + \left\lfloor\frac{100C + D}{400}\right\rfloor - 388$$

$$= 25C + \frac{D}{4} - C + \frac{D}{100} + \frac{C}{4} + \frac{D}{400} - 388$$

$$= 25C + \frac{D}{4} - C + \frac{C}{4} - 388, \text{since} < 100$$

$$= 24C + \frac{D}{4} + \frac{C}{4} - 388$$

$$\equiv 3C + \frac{C}{4} + \frac{D}{4} - 3(mod\ 7) \qquad (1.2)$$

Therefore,

$$d_y \equiv d_{1600} + \left(\begin{matrix}\text{one day of the year}\\ \text{year since } 1600\end{matrix}\right) + (mod\ 7)$$

$$\equiv d_{1600} + (y - 1600) = L(mod\ 7)$$

Substituting for y and L

$d_y \equiv d_{1600} + (100C + D - 1600) + 3C + \frac{C}{4} + \frac{D}{4} - 3(mod\ 7)$

$$\equiv d_{1600} + (2C + D - 4 + 3C - 3) + \frac{C}{4} + \frac{D}{4}(mod\ 7)$$

$$\equiv d_{1600} + 5C + D + \frac{C}{4} + \frac{D}{4}(mod\ 7)$$

$$\equiv d_{1600} - 2C + D + \frac{C}{4} + \frac{D}{4}(mod\ 7) \qquad (1.3)$$

We can use this formula to identify d_y, the day of March 1 in year y, provided we know d_{1600}. In fact, we can also use it to find d_{1600} from some known value of d_y

To determine d_{1600}:

Because March 1, 1994, fell on a Tuesday, $d_{1994}=2$. For $y = 1994, C = 19$, and $D = 94$, so, by formula (1.3),

$$d_{1600} \equiv 2 + 2 \cdot 19 - 94 - \frac{19}{4} - \frac{94}{4}(mod\ 7)$$

$$\equiv 2 + 3 - 3 - 4 - 2(mod\ 7)$$

$$\equiv -4 \equiv 3(mod\ 7)$$

Thus, d_{1600} was a Wednesday.

Substituting for d_{1600} in formula (1.3),

$d_y \equiv 3 - 2C + C/4 + D/4\ (mod\ 7)$

This formula enables us to determine the day on which March 1 of any year falls.

We can extend this formula for an arbitrary day of a given month of the year

We can extend this formula for an arbitrary day of a given month of the year.

To extend formula (1.4) to the r^{th} day of month m in year y:

To generalize formula (1.4), we need to know the number of days. The first of the month is moved up from that of the previous month modulo 7. For this, notice that $30 \equiv 2(mod\ 7)$ and $31 \equiv 3(mod\ 7)$. So the day of the first of the month following a month with 30 days is advanced by 2 days, whereas that following a month with 31 days is advanced by 3 days.

For example, December 1, 1992, was a Tuesday. So January 1, 1993, fell on day $(2 + 3) = day\ 5$, a Friday.

Thus, we have the following eleven monthly increments:

March 1 to April 1: 3 days
April 1 to May 1: 2 days
May 1 to June 1: 3 days
June 1 to July 1: 2 days
July 1 to August 1: 3 days
August 1 to September 1: 3 days
September 1 to October 1: 2 days
October 1 to November 1: 3 days
November 1 to December 1: 2 days
December 1 to January 1: 3 days
January 1 to February 1: 3 days

Next, we look for a function that yields these incremental values

To find a function f that produces these increments:

First, notice that the sum of the increments=29 days. So, the average number of increments $= \frac{29}{11} \approx 2.6$ days, so it was observed by Christian Zeller that the function $f(m) = 2.6m - 0.2 - 2$ can be employed to yield the above increments as m varies from 2 to 12. For example,

$$f(3) - f(2) = (7.8 - 0.2 - 2) - (5.2 - 0.2 - 2)$$

$$= (7 - 2) - (5 - 2) = 2$$

So there is an increment of 2 days from month 2 (April 1) to month 3 (May 1).

Therefore, by formula (1.4), the first day d of month m is given by

$d_y + 2.6m - 0.2 - 2(mod\ 7)$; that is,

$$d' \equiv 3 - 2C + D + \frac{C}{4} + \frac{D}{4} + 2.6m - 0.2 - 2 (mod\ 7)$$

$$\equiv 1 + 2.6m - 0.2 - 2C + D + \frac{C}{4} + \frac{D}{4} (mod\ 7)$$

To find the formula for the r[th] day of month m:
The day d of the week for the r[th] day of

$$d \equiv r + 2.6m - 0.2 - 2C + D + C/4 + D/4 (mod\ 7) \qquad (1.5)$$

According to the formula (1.5) we discuss an example. Example: Determine the day of the week on which January 13, 2020, falls.

Solution: Notice that January 2020 is the eleventh month of year 2019, so here $y = 2019$,
C$= 20, D = 19, m = 11, and\ r = 13$
Therefore, by formula (1.5),

$$d \equiv 13 + 2.6 \times 11 - 0.2 - 2 \times 20 + 19 + \frac{20}{4} + \frac{19}{4} (mod\ 7)$$

$$\equiv 13 + 28 - 40 + 19 + 5 + 4 (mod\ 7)$$

$$\equiv 1 (mod\ 7)$$

Thus, January 13, 2020, falls on a Monday.

Main work:

Now we discuss the application of congruence relation to find the followings in calendar.

I. What is "Today" after some days and what was "Today" some days ago?
II. What is "Today" after some year?
III. What was "Today" some years ago?
IV. How many Sundays in this month after some years?
V. How many Mondays in any month after some years?
VI. A specific year (after/before) start from which day?

In *part 6* we discuss the problem "*The Perpetual Calendar (optional)*"of *Thomos Koshy* book (see [1]). We solve it with a very short method.

We will use some notations in our work:
Number of days= N_d
Today = T
Number of years = N_y *(Years means that targeted year - current year)*
Current year = C_y

If "Today" is Wednesday and 13[th] March 2013

(I) What is "Today" after some days? And what was "Today" some days ago?

To find the required day we use the following Equation

$$N_d \equiv a\ (mod\ 7) \dots\dots\dots\dots (2.1)$$

$$T + a = Required\ day \dots\dots\dots\dots (2.2)$$

Example 1:

What is "Today" after 500 days?
From equation (2.1) we have
$500 \equiv 3\ (mod\ 7)$
So from (2.2) we get
"T" + *'a'* = Wednesday + 3 = Saturday
Thus after 500 days today will be Saturday.

Example 2: What are "Today" after " 971537" days?
From equation (2.1) we have

$$971537 \equiv 0\ (mod\ 7)$$

So after "971537" days "Today" is same day because
From equation (2.2) we have
"T" + 0 = Wednesday + 0 = Wednesday

Example 3:
What was today 800 days ago?
From equation (2.1) we have

$$800 \equiv 2 (mod\ 7)$$

So $a = 2$ where it was 800 days before so $a = -2 = 5$
From equation (2.2) we have
T – 2 =Wednesday – 2 = Monday
So 800 days ago today was Monday.

Now we introduce some (three) conjunctures for the calculation of years, because we cannot use equation (2.1).There is a problem of leap year and all years are not of equal number of days, so we define it in three conjectures which we proved with the help of some examples.

$$N_y + b \equiv a\ (mod\ 7) \dots\dots\dots (3.1)$$

Where

$$b = \frac{N_y + r}{4} \dots\dots\dots (3.2)$$

And

$$N_y \equiv r\ (mod 4) \dots\dots\dots (3.3)$$

***(II)*What will be "Today" after some years?**

Example 4:

Let's find what will be "Today" after 18 years (on 13[th] March 2031)
Then
From conjunctures (3.1), (3.2) and (3.3) we have $18 + b \equiv 4\ (mod\ 7)$
For b:

$$b = \frac{years + r}{4}$$

And for *"r"*

$2013 \equiv 1(mod4)$.
So $r = 1$

$$b = (18+1)/4 = 19/4 = 4.75$$

But we take only 4 (do not take fraction) So $b = 4$

Then conjecture (3.1) become

$$18+4 \equiv 8(mod\ 7)$$
$$18+4 \equiv 1(mod\ 7)$$

So
"Today" is Wednesday so "T" + 1 is Thursday
So on 13th march 2031 "Today" is Thursday

Example 5:

If "Today" is 13th March 2019, then what is "Today" on 13th March 2098?

From equation "*Sarwar Conjecture*" we have

$$N_y = 79$$

$$r = 2019 \equiv 3\ (mod\ 4)$$

And $b = (79+3)/4 = 20.5$ So we take $b = 20$
From conjunctures (3.1), (3.2) and (3.3) we have

$$79 + b \equiv a\ (mod\ 7)$$

$$79 + 20 \equiv 22(mod\ 7)$$
$$\equiv 1\ (mod\ 7)$$

So today +1= Wednesday+1 = Thursday
So on 13th March 2098 is Thursday.

(III) What was "Today" some years ago?

Example 6:

What was "Today" on 13thMarch 1950?
From 1950 to 2013 there are 63 years

$$N_y = 63$$
$$C_y = 2013$$
$$2013 \equiv 1\ (mod4)$$

So $r = 1$
And $b = (63+1)/4 = 16$
From conjunctures (3.1), (3.2) and (3.3) we have

$$63 + 1 \equiv 16(mod\ 7)$$
$$\equiv 2(mod\ 7)$$

Where '1950' has passed some years ago year so we will subtract it from "Today"
So
"Today" – 2 = Wednesday– 2 = Monday

So on 13th March 1950 was Monday (verified from calendar).

Now we discuss another application in the next parts

(IV)How many Sundays will be in this month after some years?

Example 7:

How many Sunday in the month of March in 2036?

For this, first of all we find what is "Today" (13thMarch) in 2033.
So,

$$N_y = 23$$
$$r = 2013 \equiv 1\ (mod\ 4) = 1$$
$$b = \frac{23+1}{4} = 6$$

From conjunctures (3.1), (3.2) and (3.3) we have

$$23 + 6 \equiv 8\ (mod\ 7)$$
$$\equiv 1\ (mod\ 7)$$

So on 13thMarch 2036 is Wednesday + 1 =Thursday
Now,

If 13th March in 2036 is Thursday then coming Sunday is on 16th march.
So,

$$16 \equiv 2\ (mod7)$$

So, every date of March 2036 which satisfies the relation

$X \equiv 2\ (mod7)$ is Sunday for example.

$$2 \equiv 2\ (mod7)$$
$$9 \equiv 2\ (mod7)$$
$$16 \equiv 2\ (mod7)$$
$$23 \equiv 2\ (mod7)$$

And

$$30 \equiv 2\ (mod7)$$

So all these dates i.e. $2, 9, 16, 23$ and 30 will be Sunday in March 2036

(V) How many Mondays will be in any month after some years?

Example 8:

If "Today" is 13thMarch 2013 then how many Mondays in the month of may 2020?
First we find what is "Today" (13thMarch) in 2020
So,

$$N_y = 7$$

$$r = 2013 \equiv 1\ (mod4) = 1$$
$$b = (7+1)/4 = 2$$

From conjunctures (3.1), (3.2) and (3.3) we have $7 + 2 \equiv 2\ (mod7)$

It is for 13th March 2020 while we are concerned about May. So,

$$7 + 2 + 61 \equiv 2 + 5\ (mod7)$$

Where 61 represent the number of days of two months,

(March has 31 days and April has 30 days)
Thus

$$7 + 2 + 61 \equiv 7\ (mod\ 7)$$
$$\equiv 0\ (mod\ 7)$$

So on 13th May 2020 is same day (Wednesday)
And Monday is 2 days later on 11thMay 2020.
While $11 \equiv 4\ (mod\ 7)$

So, every date of May 2020 which satisfies the relation $X \equiv 4\ (mod\ 7)$ is Monday, for example,

$$4 \equiv 4\ (mod\ 7)$$
$$11 \equiv 4\ (mod\ 7)$$
$$18 \equiv 4\ (mod\ 7)$$
$$25 \equiv 4\ (mod\ 7)$$

So,
4, 11, 18 and 25 will be Monday in the month of May 2020.

***(VI)* A specific year starts on which day (after/ago)?**

In this part we calculate as to a specific year starts on which day of the week? This work has already been discussed in [1]. But we calculate this by a new method with the help of "*Sarwar Conjecture*".
To determine as to the year starts on which day, we make a table which is congruent to modulo 7.

Month	Days	(Mod 7)
January	31	3
February	280r 29(29 in case of leap year)	0 or 1 (1 in Case of leap year)
March	31	3
April	30	2
May	31	3
June	30	2
July	31	3
August	31	3
September	30	2
October	31	3
Number	30	2
December	31	3

Example 9:

The year 2031 start on which day?

First of all we find what is "Today" in 2031 (13th March 2031).

From conjunctures (3.1), (3.2) and (3.3) we have

$$18 + b \equiv a\ (mod\ 7)$$
$$b = (18 + 1)/4 = 19/4 = 4.75$$

But we take only 4 (do not take fraction). So $b = 4$

Then *Conjecture* (3.1) becomes

$$18 + 4 \equiv a(mod\ 7)$$
$$18 + 4 \equiv 8(mod\ 7)$$
$$18 + 4 \equiv 1(mod\ 7)$$

So
"Today" is Wednesday so "Today" + 1 is Thursday So on 13th March 2031 "Today" will be Thursday.
Now we calculate as to what is the day on 1st January 2031?
For "Today"

$$18 + 4 \equiv 8(mod\ 7)$$
$$22 \equiv 1(mod\ 7)$$

And for 1st January 2031

1st January has passed 12 days and 2 months before "Today",

So,
22–12–January–February ≡ 1–12–January–February $(mod\ 7)$

(–ve sign because 1st January passed before 13th March)

From table
January =3

And February =0 because 2031 is not leap year,

$$22-\ 12-\ 3 - 0 \equiv 1 - 12 - 3 - 0(mod\ 7)$$
$$22-\ 12-\ 3 \equiv -\ 14(mod\ 7)$$

$$22-\ 12-\ 3 \equiv 0(mod\ 7)$$

So

1st January 2031 will be Wednesday because "Today" is also Wednesday.

References:

[1] Elementary Number Theory with Applications by Thomos Koshy.

[2] Introduction to Modern Number Theory Fundamental Problems, Ideas and Theories 2nd Edition -Manin I., Panchishkin A.

[3] A First Course in Number Theory by K. C. Choudhary.

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9/25/2013