A Three-Step Family Of Iterative Methods With Optimal Eighth Order Of Convergence

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Abstract: The aim of this paper is to construct iterative methods to solve non-linear equation with high efficiency index. We proposed a family of iterative methods of optimal order of convergence eight for solving nonlinear equation $f(x) = 0$, where $f(x)$ is a real valued nonlinear function. It is based on Noor M.A. et al. algorithm [6] and weight function approach. To support the new findings, analysis of convergence and efficiency index are studied. Per iteration these families of methods require three functional evaluations and one evaluation of its derivative, so the method is optimal according to Kung-Traub conjecture [3] which states that an optimal iterative method based on $k+1$ evaluations could achieve a maximum convergence order of 2^k. The efficiency index of the new eighth order method is 1.682. Several numerical tests show that the proposed methods are more efficient and perform better than classical Newton's method, Noor M.A. et al. method [6] and some other existing method [5],[7].

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1. Introduction

A wide class of problems which arise in various disciplines of pure and applied sciences can be studied and formulated as nonlinear equation of the form $f(x) = 0$. The main objective of this paper is to solve the nonlinear equation of the form $f(x) = 0$ for simple root α, that is $f(\alpha)=0$ but $f'(\alpha) \neq 0$. recently, a large number of methods which are based on the Newton's method are proposed to solve nonlinear equations. All these modified methods are in the direction of improving the efficiency index and order of convergence [1],[2]. The efficiency index gives a measure of efficiency of the method which is calculated by the formula $E.I = p^{1/\lambda}$, where p the order of convergence of the method and λ is is the number of function evaluations per step. To improve the order of convergence, many modified methods are proposed in literature [1] to [10]and reference therein.

Bi et al. [1], proposed a family of eighth order convergent method which is based on King's method with $\beta = \frac{-1}{2}$ given by

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= x_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)}\frac{f(y_n)}{f'(x_n)} \\ z_n &= x_n - H(\mu_n)\frac{f(z_n)}{f[z_n,y_n] + f[z_n,x_n,x_n](z_n - y_n)} \end{aligned}\right\} \quad (1)$$

Where $\mu_n = \frac{f(y_n)}{f(x_n)}$ and $H(t)$ represents a real valued function with $H(0) = 1, H'(0) = 2$ and $|H''(0)| < \infty$

In [2] Bi et al. presented a new family of eighth order
iterative methods by a scheme

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - h(\mu_n)\frac{f(y_n)}{f'(x_n)} \\ z_n &= x_n - \frac{f(x_n)+(\gamma+2)f(z_n)}{f(x_n)+\gamma f(z_n)}\frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)} \end{aligned}\right\} (2)$$

Where $\gamma \epsilon R$ and $\mu_n = \frac{f(y_n)}{f(x_n)}$ and $H(t)$ is a real valued function with $H(0) = 1,\ H'(0) = 2,\ H''(0) = 10$ and $|H'''(0)| < \infty$

Another eighth-order method developed by Noor M.A et al. [7], is given by

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)} \end{aligned}\right\} \quad (3)$$

Thukral and Petkovic obtained eighth order method
in [10], which is based on King's method with $\beta = 0$

$$\left.\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)} \\ z_i &= w_i - \frac{f(x_i)}{f(x_i)-2f(w_i)}\frac{f(w_i)}{f'(x_i)} \\ x_{i+1} &= z_i - \\ &\left[\frac{f(x_i)^2}{f(x_i)^2-2f(x_i)f(w_i)-f(w_i)^2}+\frac{f(z_i)}{f(w_i)-\alpha f(z_i)}+\frac{4f(z_i)}{f(x_i)}\right]\frac{f(z_i)}{f'(x_i)} \end{aligned}\right\} \quad (4)$$

By getting motivation of the research in this direction we develop a new optimal eighth order method in section 2.

2. Development of New Methods

Muhammad Aslam Noor et al obtained the following algorithm (2.9) in [6]

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= x_n - \left[\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right]\frac{f(x_n)}{f'(x_n)} \end{aligned}\right\} \quad (5),$$

The order of convergence of (5) is four. In order to increase its order of convergence and to make it optimal. We introduce Newton's method in the third step of (5) and usedthe divided difference to approximate $f'(z_n)$ (see[8]) in third step by,

$$f'(z_n) = \frac{f[x_n,z_n]f[y_n,z_n]}{f[x_n,y_n]} \quad (6)$$

Where $f[x_n,y_n] = \frac{f(y_n)-f(x_n)}{y_n-x_n}$,

$f[y_n,z_n] = \frac{f(y_n)-f(z_n)}{y_n-z_n}$ and

$f[x_n,z_n] = \frac{f(z_n)-f(x_n)}{z_n-x_n}$ are first order divided difference. So we obtained the following algorithm

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= x_n - \left[\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right]\frac{f(x_n)}{f'(x_n)} \\ w_n &= z_n - \frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]} \end{aligned}\right\} \quad (7)$$

It involves three functional evaluations and one evaluation of first derivative. The order of convergence of (7) is seven and the efficiency index is $7^{1/4} = 1.63$ which is same as the seventh order method obtained in [11].To Increase the order of convergence of the method. We introduce a weight function $H(\lambda_n)$ in the third step of above method (7), we obtained the following scheme

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= x_n - \left[\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right]\frac{f(x_n)}{f'(x_n)} \\ w_n &= z_n - H(\lambda_n)\frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]} \end{aligned}\right\} \quad (8)$$

Where $\lambda_n = \frac{f(z_n)}{f(x_n)}$ and $H(t)$ is weight function.

The method in (8) is eighth order optimal method as it uses three functional evaluation and one evaluation of first derivative at each step and having efficiency index $8^{1/4} = 1.682$, which is same as given in [8],[9].

3. Analysis of Convergence

The following theorem shows that method (8) has eighth order of convergence.

Theorem 1 Let $f : I \subseteq R \to R$ is a sufficiently differentiable function where I is an open interval. Let $\alpha \in I$ be a simple zero of f and $f'(\alpha) \neq 0$. If the initial point x_0 is sufficiently close to α then the new method described by (8) has the eighth-order of convergence, provided the weight function $H(t)$ satisfies the $H(0) = 1,\ H'(0) = 1, |H''(0)| < \infty$ and error equation is given by

$$\tilde{e}_{n+1} = [3c_2^7 + c_4c_2^4 - 7c_3c_2^5 + 4c_3^2c_2^3 - c_4c_3c_2^2]e_n^8 + O(e_n^9)$$

Where $\tilde{e}_n = w_n - \alpha$, $c_k = \frac{f^{(k)}(\alpha)}{f'(\alpha)\,k\,!}$

where $k = 2,3,4,\ldots$

Proof

Let $e_n = x_n - \alpha$ be the error in the nth iterate. By Taylor's series expansion, we obtain

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + O(e_n^9)] \quad (9)$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7 + 9c_9e_n^8 + O(e_n^9)] \quad (10)$$

Where $c_k = \frac{f^{(k)}(\alpha)}{f'(\alpha)\,k\,!}$ for $k \epsilon N$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + (-8c_2^4 + 20c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5)e_n^5 + (-52c_3c_2^3 + 28c_4c_2^2 + 33c_2c_3^2 + 16c_2^5 + 5c_6 - 17c_4c_3)e_n^6 + (-32c_2^6 + c_2c_4c_3 + 36c_5c_2^2 - 72c_4c_2^3 - 126c_3^2c_2^2 - 16c_2c_6 + 18c_3^3 - 12c_4^2 - 22c_5c_3 + 128c_3c_2^4 + 6c_7)e_n^7 + (-348c_4c_3c_2^2 + c_2^7 + c_2c_4^2 - 92c_5c_2^3 - 135c_2c_3^3 - 19c_2c_7 + 75c_4c_3^2 + 408c_3^2c_2^3 - 304c_3c_2^5 - 31c_4c_5 - 27c_3c_6 + 7c_8 + 176c_4c_2^4 + 44c_6c_2^2 + 118c_2c_3c_5)e_n^8 + O(e_n^9) \quad (11)$$

$$z_n = (-c_2c_3 + c_2^3)e_n^4 + (-2c_2c_4 - 2c_3^2 - 4c_2^4 + 8c_3c_2^2)e_n^5 + (10c_2^5 - 3c_2c_5 - 7c_4c_3 + 12c_4c_2^2 - 30c_3c_2^3 + 18c_2c_3^2)e_n^6 + (-20c_2^6 + 52c_2c_4c_3 + 12c_3^3 + 80c_3c_2^4 + 16c_5c_2^2 - 6c_4^2 - 4c_2c_6 - 10c_5c_3 - 80c_3^2c_2^2 - 40c_4c_2^3)e_n^7 + (252c_3^2c_2^3 - 5c_2c_7 - 13c_3c_6 + 20c_6c_2^2 + 101c_4c_2^4 - 51c_5c_2^3 + 37c_2c_4^2 + 68c_2c_3c_5 - 17c_4c_5 - 178c_3c_2^5 + 36c_2^7 - 91c_2c_3^3 - 209c_4c_3c_2^2 + 50c_4c_3^2)e_n^8 + O(e_n^9) \quad (12)$$

$$\frac{f[x_n,z_n]f[y_n,z_n]}{f[x_n,y_n]} = 1 + (c_2c_3 - c_2^3)e_n^3 + (c_2c_4 + 2c_3^2 + c_2^4 - 4c_3c_2^2)e_n^4 + (-7c_2c_3^2 + c_2c_5 + 3c_2^5 + 5c_4c_3 - 2c_4c_2^2)e_n^5 + (-3c_4^2 - c_5c_2^2 - 9c_4c_2^3 + c_2c_6 + 33c_3c_2^4 - 14c_2^6 - 6c_3^3 - 5c_3^2c_2^2 + 6c_5c_3 - 8c_2c_4c_3)e_n^6 + (-131c_3c_2^5 + c_2c_7 - 6c_2c_3c_5 + 126c_3^2c_2^3 + 7c_3c_6 - 17c_4c_3^2 - 55c_4c_3c_2^2 + 36c_2^7 -$$

$20c_3^2c_2^4 + 58c_4c_2^4 + 7c_4c_5 + 4c_2c_3^3 - 13c_5c_2^3 + 3c_2c_4^2)e_n^7 + \Big(-\frac{261c_4c_3^3}{c_2} - \frac{37c_8c_3}{c_2} + \frac{117c_6c_3^2}{c_2} - 60\,c_3^2c_2^5 - 25c_9 + 473c_3^4 - \frac{49c_5c_6}{c_2} - 5192c_4c_3c_2^3 + \frac{266c_5c_4c_3}{c_2} - 3147c_3^3c_2^2 + 5213c_3^2c_2^4 + 238c_4c_6 - \frac{45c_7c_4}{c_2} - 949c_5c_2^4 - 903c_4^2c_3 + c_2^2 + 1890c_4c_2^5 - 819c_5c_3^2 + 202c_7c_3 + 75c_2c_8 + 125c_5^2 + \frac{48c_4^3}{c_2} + \frac{10c_{10}}{c_2} + 446c_6c_2^3 - 3066c_3c_2^6 - 193c_7c_2^2 + 591c_2^8 - 717c_2c_6c_3 + 3326c_2c_4c_3^2\Big)e_n^8 + O(e_n^9)$ (13)

$w_n = \alpha + (-c_2c_3 + c_2^3 + c_2c_3H(0) - c_2^3H(0))e_n^4 + (-2c_2c_4 - 2c_3^2 - 4c_2^4 + 8c_3c_2^2 + 2c_2c_4H(0) + 2c_3^2H(0) + 4c_2^4H(0) - 8c_3c_2^2H(0))e_n^5 + (10c_2^5 - 3c_2c_5 - 7c_4c_3 + 12c_4c_2^2 - 30c_3c_2^3 + 18c_2c_3^2 - 10c_2^5H(0) + 3c_2c_5H(0) + 7c_4c_3H(0) - 12c_4c_2^2H(0) + 30c_3c_2^3H(0) - 18c_2c_3^2H(0))e_n^6 + \big(-20c_2^6 + 52c_2c_4c_3 + 12c_3^3 + 80c_3c_2^4 + 16c_5c_2^2 - 6c_4^2 - 4c_2c_6 - 10c_5c_3 - 80c_3^2c_2^2 - 40c_4c_2^3 + 21c_2^6H(0) - 52c_2c_4c_3H(0) - 12c_3^3H(0) + 82c_3c_2^4H(0) - 16c_5c_2^2H(0) + 6c_4^2H(0) + 4c_2c_6H(0) - 10c_5c_3 + 81c_3^2c_2^2H(0) + 40c_4c_2^3H(0) - c_3^2c_2^2H'(0) - 2c_3c_2^4H'(0) - c_2^6H'(0)\big)e_n^7 + \big(252c_3^2c_2^3 - 5c_2c_7 - 13c_3c_6 + 20c_6c_2^2 + 101c_4c_2^4 - 51c_5c_2^3 + c_2c_4^2 + 68c_2c_3c_5 - 17c_4c_5 - 178c_3c_2^5 + 36c_2^7 - 91c_2c_3^3 - 209c_4c_3c_2^2 + 50c_4c_3^2 + 5c_2c_7H(0) + c_3c_6H(0) - 20c_6c_2^2H(0) - 104c_4c_2^4H(0) + 51c_5c_2^3H(0) - 37c_2c_4^2H(0) - 68c_2c_3c_5H(0) + 17c_4c_5H(0) + 197c_3c_2^5H(0) - 42c_2^7H(0) + 95c_2c_3^3H(0) - 26c_3c_2^5H'(0) + 212c_4c_3c_2^2H(0) + 50c_4c_3^2 + 9c_2^7H'(0) - 269c_3^2c_2^3H(0) + 4c_4c_2^4H'(0) + 21c_3^2c_2^3H'(0) - 4c_2c_3^3H'(0) - 4c_3c_2^2H'(0)\big)e_n^8 + O(e_n^9)$ (14)

Using condition on weights given in theorem 1 we have

$$w_n = \alpha + (3c_2^7 + c_4c_2^4 - 7c_3c_2^5 + 4c_3^2c_2^3 - c_4c_3c_2^2)e_n^8 + O(e_n^9)$$

So,

$\tilde{e}_{n+1} = (3c_2^7 + c_4c_2^4 - 7c_3c_2^5 + 4c_3^2c_2^3 -$ *c4c3c22en8+Oen9* (15)

From Eq. (15) it is clear that method given in (8) has eighth-order of convergence.

4. Family of Methods

We use the following weight functions which satisfy the condition given in theorem (1)

(i) $H_1(t) = 1 + t + \beta t^2,\ \beta \epsilon R$

(ii) $H_2(t) = e^t$

With these weight functions we obtained two methods.

4.1 Method (M1)

If we use weight function $H_1(t) = 1 + t + \beta t^2$, $\beta\ \epsilon\ R$ in method (11), we have the following scheme.

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= x_n - \Big[\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}\Big]\frac{f(x_n)}{f'(x_n)} \\ w_n &= z_n - \Big[1 + \frac{f(z_n)}{f(x_n)} + \beta\frac{f^2(z_n)}{f^2(x_n)}\Big]\frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]} \end{aligned}\right\} \quad (16)$$

4.2 Method (M2)

If we use weight function $H_2(t) = e^t$ in (11) we obtained M2

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= x_n - \Big[\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\Big]\frac{f(x_n)}{f'(x_n)} \\ w_n &= z_n - Exp\,\Big[\frac{f(z_n)}{f(x_n)}\Big]\frac{f[x_n,y_n]\,f(z_n)}{f[x_n,z_n]\,f[y_n,z_n]} \end{aligned}\right\} \quad (17)$$

5. Numerical Results

In this section we compare our methods denoted by M1 (16) and M2 (17) with Newton's method (NM),fourth order method by Noor M.A. et al.[6] denoted by AN4, eighth order method by Noor M.A.et al. [7] denoted by AN8, eighth order method by Thukral and petkovic [15] denoted by T8 , eighth order method by Janak Raj Sharma and Rajni Sharma [8] denoted by RJ and eighth order method by Abdullah M [5] denoted by MA.

All of the computations have been done using variable precision arithmetic, with 500 significant digits in Maple 13 and stopping criterion used is $(i)|x_{n+1} - \alpha| < \epsilon\ (ii)|f(x_{n+1})| < \epsilon$

Where $\epsilon = 10^{-15}$, where α is real root. In Table 1 the test functions and their roots are given. Table 2 shows the number of iterations and number of functional evaluations (NFE). Table 3 shows the absolute value of the difference between the approximated root x_n and the exact root, the absolute values of function $|f(x_n)|$ and the computational order of convergence. Where computational order of convergence (COC) is defined as let α be the zero of the function g and suppose that y_{n-1}, y_n and y_{n+1} are three successive iterations closer to zero. Then the computational order of convergence ρ can be calculated by formula

$$\rho \approx \frac{ln|(y_{n+1} - \alpha)/(y_n - \alpha)|}{ln|(y_n - \alpha)/(y_{n-1} - \alpha)|}$$

We have used the test functions used in paper [12]

Table 1. Test functions and their roots

Test Functions and their roots

$f_1(x) = x^5 + x^4 + 4x^2 - 15$ $\alpha = 1.3474280989683050$
$f_2(x) = sinx - \frac{x}{3}$ $\alpha = 2.2788626600758283$
$f_3(x) = 10xe^{-x^2} - 1$ $\alpha = 1.6796306104284499$
$f_4(x) = \cos\ x - x$ $\alpha = 0.7390851332151606$
$f_5(x) = e^{-x^2+x+2} - 1$ $\alpha = -1.0000000000000000$
$f_6(x) = e^{-x} + cosx$ $\alpha = 1.7461395304080124$
$f_7(x) = \ln(x^2 + x + 2) - x + 1$ $\alpha = 4.1525907367571583$
$f_8(x) = \arcsin(x^2 - 1) - \frac{x}{2} + 1$ $\alpha = 0.5948109683983692$
$f_9(x) = xe^{x^2} - sin^2(x) + 3\cos(x) + 5$ $\alpha = -1.2076478271309189$

Table 2: Number of functional Evaluations

$f(x)$	x_0	NM	AN4	AN8	T8	RJ	MA	M1	M2
$f_1(x)$	1.6								
Iterations		5	3	2	2	2	2	2	2
NFE		10	9	12	8	8	12	8	8
$f_2(x)$	2								
Iterations		5	3	2	2	2	2	2	2
NFE		10	9	12	8	8	12	8	8
$f_3(x)$	1.8								
Iterations		5	2	2	2	2	2	2	2
NFE		10	6	12	8	8	12	8	8
$f_4(x)$	1								
Iterations		4	2	2	2	2	2	2	2
NFE		8	6	12	8	8	12	8	8
$f_5(x)$	- 0.5								
Iterations		6	3	2	2	2	2	2	2
NFE		12	9	12	8	8	12	8	8
$f_6(x)$	2								
Iterations		4	2	2	2	2	2	2	2
NFE		8	6	12	8	8	12	8	8
$f_7(x)$	3.2								
Iterations		4	2	2	2	2	2	2	2
NFE		8	6	12	8	8	12	8	8
$f_8(x)$	1								
Iterations		5	3	2	2	2	2	2	2
NFE		10	9	12	8	8	12	8	8
$f_9(x)$	- 1								
Iterations		5	3	2	2	2	2	2	2
NFE		10	9	12	8	8	12	8	8

Table 3 Comparison of different methods

$f(x)$	x_0	NM	AN4	AN8	T8 $\alpha = 0$	RJ1 $\gamma = 1$	MA	M1 $\beta = 1$	M2
$f_1(x)$	1.6								
COC		2.0000	3.5747	7.9037	7.8892	8.1987	7.8064	8.1986	8.2643
$\lvert f(x_n)\rvert$		3.5e-19	1.8e-45	3.6e-39	2.8e-37	8.5e-42	5.1e-33	8.5e-42	8.8e-42
$\lvert x_n - r\rvert$		9.5e-21	2.4e-43	9.7e-41	7.7e-39	1.3e-44	1.4e-34	1.3e-44	6.6e-45
$f_2(x)$	2								
COC		2.0000	3.5724	6.7133	8.1596	8.2972	8.1951	8.2973	8.3588
$\lvert f(x_n)\rvert$		1.0e-28	9.0e-53	4.2e-57	3.8e-42	1.2e-48	4.3e-46	1.2e-48	1.3e-48
$\lvert x_n - r\rvert$		1.1e-28	1.4e-48	1.4e-48	3.9e-42	1.5e-49	4.4e-46	1.5e-49	7.8e-49
$f_3(x)$	1.8								
COC		2.0000	4.0397	7.7717	8.0622	8.2332	8.1123	8.2330	8.2883
$\lvert f(x_n)\rvert$		5.9e-29	4.9e-15	1.2e-57	5.5e-49	6.7e-56	3.0e-47	6.7e-56	6.9e-56
$\lvert x_n - r\rvert$		2.1e-29	1.8e-15	2.5e-56	2.0e-49	1.6e-57	1.1e-47	1.6e-57	8.1e-58
$f_4(x)$	1								
COC		2.0000	3.9595	6.6483	7.9434	8.1830	7.4048	8.1563	8.1050
$\lvert f(x_n)\rvert$		1.0e-20	1.8e-18	3.0e-83	2.1e-65	2.3e-71	1.1e-75	2.3e-71	2.53-71
$\lvert x_n - r\rvert$		6.3e-21	1.0e-18	1.4e-71	1.2e-65	2.1e-73	1.4e-71	3.5e-73	8.9e-73
$f_5(x)$	- 0.5								
COC		1.6541	3.4722	6.7902	7.4344	7.9157	7.2643	7.9157	8.0298
$\lvert f(x_n)\rvert$		1.3e-26	4.5e-27	1.0e-26	1.4e-20	1.7e-24	3.4e-22	1.7e-24	1.8e-24
$\lvert x_n - r\rvert$		6.6e-25	6.6e-25	6.6e-25	4.8e-21	8.2e-26	1.1e-22	8.3e-26	4.2e-26
$f_6(x)$	2								
COC		2.0000	4.0783	6.1066	8.0901	8.2850	7.1378	8.3420	8.3765
$\lvert f(x_n)\rvert$		4.5e-21	1.4e-17	9.2e-85	3.0e-61	1.2e-66	1.7e-74	1.2e-66	1.2e-66
$\lvert x_n - r\rvert$		3.9e-21	1.2e-17	1.1e-66	2.6e-60	3.4e-68	1.1e-66	1.3e-68	7.7e-69
$f_7(x)$	3.2								
COC		2.0000	4.996	6.3086	8.1186	8.3211	7.5299	8.3180	8.3692
$\lvert f(x_n)\rvert$		2.2e-18	3.8e-16	2.8e-74	1.4e-54	5.0e-60	4.5e-64	5.0e-60	5.1e-60
$\lvert x_n - r\rvert$		3.8e-18	6.3e-16	8.6e-60	2.3e-54	2.8e-61	8.6e-60	2.9e-61	1.5e-61
$f_8(x)$	1								
COC		2.0000	3.9996	7.3717	8.2419	8.2041	8.1860	8.2043	8.2442
$\lvert f(x_n)\rvert$		2.6e-27	2.8e-47	1.7e-54	5.0e-39	1.3e-49	3.6e-46	1.3e-49	9.8e-50
$\lvert x_n - r\rvert$		2.5e-27	2.7e-47	6.7e-50	4.7e-39	5.8e-50	3.4e-46	5.8e-50	2.5e-50
$f_9(x)$	-1								
COC		2.0000	3.9234	8.0178	8.0485	7.9124	8.2292	7.9124	7.9129
$\lvert f(x_n)\rvert$		1.5e-15	3.7e-55	1.7e-31	3.6e-46	4.1e-53	1.4e-19	4.1e-53	4.3e-53
$\lvert x_n - r\rvert$		7.5e-17	1.8e-56	8.6e-33	1.7e-47	2.0e-54	7.2e-21	2.0e-54	2.1e-54

6. Conclusion

In this paper we have developed optimal family of eighth order methods. The eighth order family requires three functional evaluations and one evaluation of the first derivative of function at each step and therefore the efficiency index of the method is equal to 1.682. After comparing our developed methods with existing methods [5] and [7] we observed that our methods are better in performance and efficiency. It is noted that the new methods M1 and M2 have at least equal performance when compared with other existing optimal eighth order methods [8],[9].

Reference

1. Bi. W., Ren. H., Wu Q., Three-steps iterative methods with eighth-order convergence for solving nonlinear equations, Journal of Computational and Applied Mathematics 2009; 225,105-112.
2. Bi W., Ren. Wu Q. H., A new family of eighth-order iterative methods for solving nonlinear equations, Applied Mathematics and Computational 2009, 214,236-245.
3. Kung H.T., Traub J.F, Optimal order of one-point and multipoint iterations, Appl. Math. Comput., 1974; 21,643-651.

4. Rafiullah M., Haleem M., Three-Step Iterative Method with Sixth Order Convergence for Solving Nonlinear Equations, Int. Journal of Math. Analysis, (2010); Vol. 4, no. 50, 2459 – 2463.
5. Abdullah M., M, Phil Thesis, Some Iterative Methods for solving non-linear equations (2011).
6. Noor M.A. and Khan W.A, Fourth-Order Iterative Method Free from Second Derivative for Solving Nonlinear Equations, Applied Mathematical Sciences, 2012;Vol. 6, no. 93, 4617 – 4625.
7. Noor M.A, Noor K.I and Waseem M., Higher-order Iterative Algorithms for Solving Nonlinear Equations, World Applied Sciences journal, 2012; 16(12): 1657-1663.
8. Sharma R. and Sharma J. R., An Efficient Family of Root-Finding Methods with Optimal Eight-Order Convergence, Advance in Numerical Analysis, 2012; Volume, Article ID 3464.
9. Khattri S.K., Quadrature Based Optimal Iterative Methods with Applications in High Precision Computing, (2012); Numer Math Theo. Appl Math 5 pp. 592-601.
10. Thukral R.and Petkovic S.M , A family of three-point methods of optimal order for solving nonlinear equations, Journal of Computational and Applied Mathematics,2010; vol. 233, no. 9, pp. 2278.2284.
11. Mir N.A, Khan S.A, Naila Rafiq, Yasmin N. Some Families of Higher Order Three-Step Iterative Techniques, 2013; *Life Science Journal*;(2s).

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