A risk-based approach to robust economic-statistical design of control charts under duncan's economic model

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Abstract: This paper deals with a robust economic-statistical design of control chart, in which the input parameters are expressed as intervals. The design procedure relies on finding the best design parameter set based on the minimax criterion for risks. Genetic algorithm (GA) has been used as a search tool to find the best design (input) parameters with which the control chart has to be designed. The proposed method minimizes the risk of not knowing the true parameters to be used in the design, and is robust to the true parameter values. A numerical example is used to illustrate the performance of the proposed economic-statistical design of the $\bar{X}$ control chart.

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1. Introduction

Control charts are widely used to establish and maintain statistical control of a process. The purpose of a control chart is to detect the assignable causes, if any, in the process as early as possible and help to remove them. The design of control chart involves the optimum selection of the control chart parameters such as the sample size, the sampling interval and the control limit coefficient. The number of design parameters of the control chart depends on the type of chart under consideration. Shewart's variable control chart for averages, popularly known as $\bar{X}$ chart involves finding three design parameters namely the sample size (n), the sampling interval (h) and the control limit coefficient (k). The control chart designs can be broadly considered in three categories as (i) heuristic designs, (ii) statistical designs and (iii) economic designs. Heuristic designs are preferred in industrial practice because of their simplicity. These designs depend on rules of thumb in the selection of parameters. The sample size is generally taken equal to 4 or 5 and the control limit coefficient is taken equal to 3. The sampling interval, in general, is left to the discretion of the practitioner. The statistical design of control charts is based on their statistical performance, over a specified in-control and out-of-control regions of parameter values. Economic designs are based on either minimizing the cost or maximizing the profit, per unit time or per unit produced, without or with some constraints.

Duncan [1] introduced the design of $\bar{X}$ control chart on the basis of economic criteria. He developed a model to find the control chart parameters for a continuous process, which goes out-of-control due to a single assignable cause. Panagos et al. [2] describe a continuous process as a process in which the manufacturing activities continue during the search for an assignable cause. Goel et al. [3] introduced an algorithm to find the exact solution to the Duncan's single assignable cause model. Some manufacturing situations may not permit stopping of the process after the control chart signals an out-of-control situation and in some cases it may be advantageous to stop the process and take a remedial action. The quality characteristic observed can be a variable or an attribute. Lorenzen and Vance [4] developed a unified model for the economic design of control charts by incorporating different possible production models into it. They showed a significant monetary benefit of using an economic design over a heuristic model. But, the economic designs are criticized for their poor statistical properties. Woodall [5] pointed out the weakness of the economic design. Saniga [6] introduced the economic-statistical designs for control charts by combining the economic and statistical objectives. These designs are semi-economic designs and are costlier compared to pure economic designs. Surtihadi and Raghavachari [7] have shown that for control charts with fixed sampling intervals, the exponential process failure mechanism provides a good approximation even though the real process follows any non-exponential process failure mechanism. Control charts can be designed to have constant parameters; time varying parameters and adaptive parameters. Benerjee and Rahim [8] showed the superiority of time varying sampling intervals over fixed sampling interval in case of Weibull process failure mechanism with an increasing failure rate. Prabhu et al. [9] designed $\overline{X}$ control charts with adaptive parameters and showed that they are superior to conventional control chart designs. Engin [10] extended Duncan's economic control chart design methodology as an alternative way for estimating and optimizing machine efficiency in the case of multi-machine assignments.

In general, the complexity of the models has grown from single assignable cause models to multiple assignable cause models and exponential failures to Weibull and gamma distributions. The quality characteristics considered in models have grown from monitoring a single quality characteristic (univariate) to multiple quality characteristics (multivariate). Control charts that can use present and past information effectively have been introduced. Use of adaptive parameters has been studied. In spite of many theoretical developments in the area of control chart designs, it is observed that very little has been implemented in practice. Researchers have attributed many reasons for this situation. Keats et al. [11] analyzed the causes that act as barriers to the implementation of economical designs. They found that the robustness of the control charts to the imprecision in the input parameters as one of the reasons for lack of confidence in the economically designed control charts. This shows the necessity of robust design procedures for control charts such that the cost of operating a control chart is minimized considering the impreciseness in estimating the parameters of the process. The designs in this line will lead to the robust design procedures for the control charts.

Pignatiello and Tsai [12] have introduced the robust economic design of control charts when the cost and process parameters are precisely not known. This type of design induces confidence in the user since the design procedure is based on a range of values for every parameter instead of point estimates. Even though the process parameters are not known accurately, the losses in operating the control chart can be controlled by robust designs. Linderman and Choo [13] proposed robust economic design procedures for a single process assuming multiple scenarios. They employed three discrete robustness measures in the economic design of control charts and found the best control chart design, which works well for all scenarios considered.

In this paper, a robust economic-statistical design procedure for $\overline{X}$ control charts has been proposed under Duncan's economic model. The design procedure relies on finding the best design parameter set based on the minimax criterion for risks. Initially, a parameter space has to be formed by expressing each cost and process parameter in a range. For each parameter set chosen from the parameter space, maximum possible risk has to be calculated. The parameter set with minimum of such maximum risks has to be considered in the design of control chart. Calculation of maximum risks corresponding to a given design parameter set has been simplified by reducing the solution space. GA-based search has been employed to find the best design parameter set from the parameter space. This procedure minimizes the risk of not knowing the true parameters to be used in the design, and is robust to the true parameter values.

We end this introduction with a brief outline of the paper; the next section introduces the notation and the basic assumptions required for the model. Section 3 defines the economic-statistical design and introduces the loss-cost function of Duncan [1]. Section 4 introduces the risk concept and the approach to find the maximum risk corresponding to a given design parameter set. In Section 5, the GA-based search to find the best design parameter set from the parameter space, using minimax criterion has been discussed. A control chart design problem has been considered for analysis at different precision scenarios and robust design solutions are provided. Finally, a summary of the work is given in Section 6.

2. Notation and assumptions

ATS average time to signal out-of-control situation

b fixed cost per sample

c cost per unit sampled

d time to search and fix the process

D design vector of control chart parameters obtained from imprecise input parameters

D^* design vector of control chart parameters obtained from true input parameters

g time to test and interpret the result per sample unit

h sampling interval

k control limit coefficient

M hourly cost penalty of running the process in out-of-control

MR_{ϖ} maximum risk corresponding to a design parameter set ϖ

n sample size

p power of the chart (1- β)

sus stochastic universal sampling

V_0 profit per hour while the process is running in-control

V_1 profit per hour while the process is running out-of-control

W cost of finding and fixing an assignable cause

xosp single point cross over

Y cost of false alarm

α probability of type I error
β probability of type II error
δ ratio of magnitude of shift in process mean to the standard deviation of the process, in short, shift parameter
Ω parameter space
ψ a true parameter set
ϖ a design (input) parameter set
$(.)_D$ denotes a design parameter
$(.)_T$ denotes a true parameter

The features to be studied in this paper are as follows (Duncan [1]):

1. The process is either in-control or out-of-control state only and is in-control state at the beginning.
2. The process follow a normal distribution with a mean μ and standard deviation σ.
3. There is one assignable cause in the production process.
4. The process will have a shift in the process mean of $\delta\sigma$, if assignable cause occurs.
5. The standard deviation is assumed to remain invariant when process shifts.
6. The failure rate of a assignable cause follows an exponential whit parameter λ.
7. Production is continuous during the search and repair.
8. The detection probability when assignable cause occurs is greater than p_L.
9. The type I error of the control chart is less than α_H.

3. The method of economic-statistical design of control charts

An economic-statistical design of a control chart can be defined as that the loss-cost function is minimized subject to the constrained maximum value of probabilities of type I and type II errors. Apart from constraining the type I and type II errors of the control chart, Saniga [6] introduced one more constraint to take care of the average time to signal the out-of-control ($ATS_{.}$). This acts as a constraint on the sampling interval of the control chart.

Let ϖ be the set of design parameters and L be the expected hourly loss-cost function of an $\overline{X}$ control chart economic model. Then, the economic-statistical model of the $\overline{X}$ control chart can be formulated as:

minimize $L(\varpi)$

$$\text{subject to} \begin{cases} \alpha \le \alpha_H \\ p \ge p_L \\ ATS_0 \le ATS_H. \end{cases} \tag{1}$$

ATS_H and α_H provide upper bounds on the average time to signal an assignable cause and on type I error probability, respectively. The power constraint p_L has a lower bound on power such that the control chart is assured always with certain minimum power.

The solution of this model is an improvement to the economic design because both the statistical properties and minimization of loss cost have been considered. A solution without the constraints is the optimum economic design of the control charts.

3.1. Duncan's loss-cost function

Duncan's economic control chart design model is based on the assumption that the cycle time in a production process consists of four time intervals (Fig.1), namely, (1) the interval during which the process is in control; (2) the time to signal; (3) the time required to sample, inspect, evaluate and plot a sample mean; (4) the time required to search and repair for the assignable cause. Duncan derived an expected loss-cost function, minimizing of which gives the optimal control chart parameters. The expected loss-cost function is as follows:

$$L = \frac{b+cn}{h} + \frac{\lambda MB + \alpha Y/h + \lambda W}{1+\lambda B} \tag{2}$$

where,

$$M = V_{.} - V$$

$$B = ah + gn + d$$

$$a = \frac{1}{p} - 0.5 - \lambda h/12$$

$$\alpha = 2\Phi(-k)$$

$$p = \Phi(-\delta\sqrt{n} - k) + \Phi(\delta\sqrt{n} - k)$$

In economic-statistical designs of control chart, the control chart parameters are obtained by minimizing the expected loss-cost function (2) under the statistical constraint in equation (1). The effectiveness this type design of control chart relies on the accuracy of estimation of input parameters used in the model. Conventional control chart designs consider point estimate for the input parameters. The point estimates used in the design may not represent the true parameters and sometimes may be far from true values. This situation may lead to severe cost penalties for not knowing the true values of the parameters.

In next section, we introduce a procedure to robust the design to not knowing the true values of the parameters.

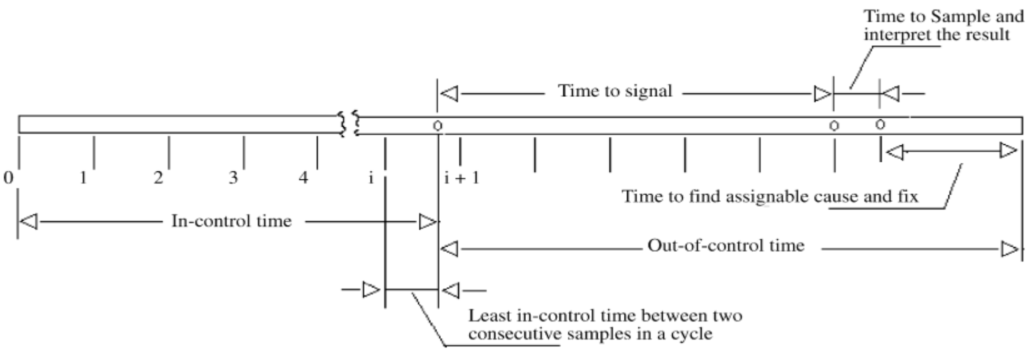


Fig. 1. A quality control cycle for a continuous process.

4. Robust economic-statistical design of control charts

The economic-statistical design of $\overline{X}$ control chart involves the selection of chart parameters by minimizing the loss-cost function. This requires nine input parameters (*b, c, W, Y, M, λ, δ, g, d*) for a continuous process. When the input parameters are not known precisely, a range can be specified to each of the parameters depending on the quantum of data used in the estimation and also based on the knowledge of similar processes. The true values of the parameters are assumed to lie in the range specified for each parameter. Specifying a range for each parameter gives rise to a possibility of theoretically infinite true values for each parameter. That is, the true value of a parameter can be any value among the infinite values in the specified range for it. By suitably dividing the parameter ranges into appropriate number of parts, possible true values of each parameter can be made finite. In other words, if the range of a parameter is divided into *x* parts, then the true parameter can be assumed to take any one of these *x* discrete values only. Considering nine input parameters for the design of control charts, and dividing the range of each parameter into *x* parts, the true parameter set the process can experience is one true parameter set out of $x^{٩}$ combinations. But this true parameter set realized by the process is unknown in the process of control chart design. In order to find the optimum control chart parameters, one has to find the best design parameter set, to be considered in the design. The design parameter set can be chosen from the parameter space formed by $x^{٩}$ possible combinations of design parameters. Arbitrarily choosing one parameter set from the available parameter space for the design of control chart gives rise to two design scenarios.

(i) The design parameters and true parameters are same.

(ii) The design parameters and true parameters are different.

The first possibility gives optimum design of control chart but is a very rare incident. The second possibility leads to a situation where there is a cost penalty for not knowing the real scenario.

Denoting the input (design) parameters used in the control chart design with subscript D, the loss-cost equation with an arbitrarily chosen design parameter set $\varpi = (b_D, c_D, W_D, Y_D, M_D, \lambda_D, \delta_D, g_D, d_D)$ can be written as:

$$L = \frac{b_D + c_D n}{h} + \frac{\lambda_D M_D B_D + \alpha_D Y_D / h + \lambda_D W_D}{١ + \lambda_D B_D} \qquad (3)$$

where $\alpha_D = 2\Phi(-k)$

$B_D = a_D h + g_D n + d_D$

and

$$a = \left(\frac{1}{p_D} - \frac{1}{2} + \frac{\lambda_D h}{12}\right)$$

$$p_D = \Phi(-\delta_D\sqrt{n} - k) + \Phi(\delta_D\sqrt{n} - k)$$

By minimizing the loss-cost function (3), the optimum parameters (*n, h, k*) corresponding to the design parameter set can be obtained. When the design parameters chosen are different from the true parameters, and the control chart is used in the actual scenario, the loss-cost is not optimum but will be more than the optimum loss-cost. The actual loss-cost is given by

$$L(D,\psi) = \frac{b_T + c_T n}{h} + \frac{\lambda_T M_T B_T + \alpha_T Y_T / h + \lambda_T W_T}{١ + \lambda_T B_T} \qquad (4)$$

where $\alpha_T = 2\Phi(-k)$

$B_T = a_T h + g_T n + d_T$

$$a_T = \left(\frac{1}{p_T} - \frac{1}{2} + \frac{\lambda_T h}{12}\right)$$

$$p_T = \Phi(-\delta_T\sqrt{n} - k) + \Phi(\delta_T\sqrt{n} - k)$$

In the above Eq. (4), the loss-cost is represented as function of *D* and ψ, where *D* represents the vector of parameters of the control chart (*n, h, k*) obtained

with design parameter set and ψ represents the true parameter set $(b_T, c_T, W_T, Y_T, M_T, \lambda_T, \delta_T, g_T, d_T)$. In other words, the actual loss-cost is a function of (*n, h, k*) values obtained by minimizing the loss-cost function by considering a design parameter set, and the process true parameter set.
If the design parameter set and the true parameter set of the process are same, it leads to optimal loss-cost. Denoting the control chart parameters in this case as (n^*, h^*, k^*), the optimal loss-cost can be written as:

$$L(D^*,\psi)=\frac{b_T+c_T n^*}{h}+\frac{\lambda_T M_T B^*+\alpha^* Y_T/h^*+\lambda_T W_T}{1+\lambda_T B^*} \tag{5}$$

where $\alpha^* = 2\Phi(-k^*)$

$B^* = a^*h^* + g_T n^* + d_T$

$a^* = \left(\frac{1}{p^*} - \frac{1}{2} + \frac{\lambda_T h^*}{12}\right)$

$p^* = \Phi(-\delta_T\sqrt{n^*} - k^*) + \Phi(\delta_T\sqrt{n^*} - k^*)$

In the optimal loss-cost equation $L(D^*,\psi)$، D^*, represents the vector of optimal parameters of the control chart (n^*, h^*, k^*) obtained by optimizing the loss-cost function with true parameter set ψ.
From Eqs. (4) and (5), the cost penalty for not knowing the actual scenario can be written as:

$$\text{Cost penalty} = L(D,\psi) - L(D^*,\psi) \tag{6}$$

Defining risk as the cost penalty, expressed as a percentage, for not knowing the actual scenario and designing the chart for a different scenario, the risk can be written as under:

$$\text{Risk(\%)} = 100 \times \frac{L(D,\psi)-L(D^*,\psi)}{L(D^*,\psi)} \tag{7}$$

For a chosen design parameter set, *D* represents the vector of control chart parameters (*n, h, k*), which is unique for that design parameter set. However, the true parameters can have any value within their parameter limits. This gives rise to different risks corresponding to different values of ψ. That is, the risk will be zero if true parameters and the design parameters are same and will be greater than zero for all other cases. Depending on the true parameters realized by the process, the risks would vary. Out of all such risks, there exists a maximum value of risk corresponding to the given design parameter set. This is the maximum risk corresponding to the given design parameter set.

Mathematically, the maximum risk MR_ϖ corresponding to a particular design parameter set ϖ, can be written as:

$$MR_\varpi(\%) = 100 \times \max\left\{\frac{L(D,\psi)-L(D^*,\psi)}{L(D^*,\psi)}\right\}$$

$$\forall \psi = (b_T, c_T, W_T, Y_T, M_T, \lambda_T, \delta_T, g_T, d_T)$$

Such that

$(b_L \le b_T \le b_H)$ $(c_L \le c_T \le c_H)$
$(W_L \le W_T \le W_H)$
$(Y_L \le Y_T \le Y_H)$ $(M_L \le M_T \le M_H)$
$(\lambda_L \le \lambda_T \le \lambda_H)$
$(\delta_L \le \delta_T \le \delta_H)$ $(g_L \le g_T \le g_H)$
$(d_L \le d_T \le d_H)$ (8)

From Eq. (8), it can be observed that calculating the maximum risk corresponding to a given design parameter set involves finding the risks corresponding to all possible combinations of true parameters from their respective ranges. For a given design parameter set, one should find the true parameter set that produces maximum risk. Likewise, for all possible different design parameter sets, one has to find the corresponding maximum risks. Finally, the design parameter set which corresponds to the minimum of all maximum risks is to be considered as the best design parameter set with which the control chart has to be designed. This design parameter set assures a safe design in a situation when nothing can be known about the distribution of true parameters within their corresponding ranges.

4.1. Maximum risk corresponding to a given design parameter set

Designing the control chart with a design (input) parameter set chosen arbitrarily from the approximate parameter space will lead to risk, and the maximum risk that may be possible with that particular design parameter set depends on the true parameter values the process realizes. In other words, the design parameter set that has the parameters very close to true parameters of the process will lead to low risk. Similarly, the design parameters that are far off from the true parameters will lead to higher risks. Since the true parameters of the process are unknown, one has to consider each possible true parameter set form the parameter space to find the risks.
In order to find the possible true parameter set, which causes maximum risk for a given design parameter set, studies have been carried out using genetic algorithm (GA) as an efficient tool for search. The procedure adopted is as follows:

- Step 1: choose a design parameter set arbitrarily from the parameter space. Find the optimum parameters (*n*, *h*, *k*) of the control chart by minimizing the loss-cost function, *L*, a function of the chosen design parameter set.
- Step 2: choose arbitrarily some possible true parameter sets from the parameter space. This creates a population of true parameter sets in the genetic algorithmic sense.
- Step 3: find the optimum costs corresponding to each possible true parameter set of step 2.
- Step 4: using the control chart parameters (*n*, *h*, *k*) obtained in the step 1 and the true parameters find the actual costs. Find the risks for each true parameter set using the actual costs and the optimum costs obtained in the step 3.
- Step 5: find the fitness value of each true parameter set based on the risk it causes. Choose the true parameter sets based on the risks obtained. The true parameter sets with higher risks are to be preferred in the selection process.
- Step 6: with the selected true parameter sets, form new parameter sets to represent the true parameter sets for the next generation. To form the new parameter sets, the genetic operations like crossover and mutation with appropriate probabilities have to be used.
- Step 7: repeat the steps from 3 to 6 obtain the new true parameter sets. Continue the process till there is no improvement in the maximum value of the risk obtained from the true parameter sets.
- Step 8: stop the search process. Find the true parameter set that has the highest risk from population corresponding to the last generation. This gives the true parameter set corresponding to maximum risk.

The above procedure has been employed to find the maximum risks corresponding to different design parameter sets of the same control chart design problem and also with different control chart design problems. In all the cases, it has been observed that for any design parameter set, the maximum risks occur with a true parameter set formed by the extreme values of the parameters. Based on these results, a proposition regarding the maximum risks corresponding to a given design parameter set has been stated and proved graphically using GA.

Proposition. For any given design (input) parameter set, the maximum risk will occur with the true parameter set formed by the extreme values of the parameter ranges.

- Graphical proof:

The above proposition can be proved graphically by considering any control chart design problem, completely specified by all parameters and the precision with which the parameters are estimated. It can be shown that for the maximum risk to occur, the true parameters will tend towards the extreme values to either low or high values.

- Numerical example:

A control chart design problem with the following cost and process parameters in Table 1 has been considered from Duncan [2] to prove the above proposition.

Table 1
Control chart input parameters: Duncan's example no. 1

b	0.5
c	0.1
W	25
Y	50
M	100
λ	0.01
δ	2
g	0.05
d	2

Precision in the estimates of each input parameter is assumed to be ±50%. Table 2 shows the range for each parameter, covering their true parameters.

Table 2
Range of cost and process parameters at ±50% precision

S. no.	Cost/process parameter	Range of each parameter
1	Fixed cost of sampling (b)	$0.25 \leq b_T \leq 0.75$
2	Variable cost of sampling (c)	$0.05 \leq c_T \leq 0.15$
3	Cost of finding assignable cause and fixing (W)	$12.5 \leq W_T \leq 37.5$
4	Cost of false alarms (Y)	$25 \leq Y_T \leq 75$
5	Cost of running out-of-control (M)	$50 \leq M_T \leq 150$
6	Process failure rate (λ)	$0.005 \leq \lambda_T \leq 0.015$
7	Average shift in the process mean (δ)	$1.0 \leq \delta_T \leq 3.0$
8	Time to sample and interpret per unit (g)	$0.025 \leq g_T \leq 0.075$
9	Time to find assignable cause and repair it (d)	$1.0 \leq d_T \leq 3.0$

Considering the mean values of the parameter ranges for the design parameter set, the numerical values for all parameters are given in Table 3.

Then optimizing the loss-cost Eq. (2) with the above design parameter set and with statistical constraints as:

α_H =0.005, p_L=0.95, ATS_H=5
The control chart parameters are obtained as under:
n=5, h=1.37, k=3.08

Table 3
Design parameter set to find the maximum risk

b_D	0.50
c_D	0.10
W_D	25
Y_D	50
M_D	100
λ_D	0.01
δ_D	2
g_D	0.05
d_D	2

To find the true parameter set, causing the maximum risk corresponding to the above control chart parameters, GA-based search has been carried out. Table 4 provides the details of the parameters used in the GA search.

Table 4
GA parameters taken in the study

S. no.	GA parmeters	Magnitude/method
1	Population size	30
2	Selection method	Stochastic universal sampling
3	Type of cross over	Single point
4	Probability of cross over	0.7
5	Probability of mutation	0.0175
6	Strategy	Elitist
7	Generation gap	0.9
8	Maximum generations	200

Each string in the population represents a possible true parameter set for the calculation of risk. In the first generation, '30' parameter sets are taken for study. For each parameter set the risk is calculated and is taken as the objective value for each string to find its fitness. Depending on fitness values, selection is done by stochastic universal sampling. A single point crossover is performed among the best strings and the offspring are formed. Mutation is carried out with a low probability (p_c/l where p_c is the probability of cross over and l is the length of chromosome). The process is continued till the specified number of generations is over. Apart from the convergence of the objective function, i.e. the maximum risk; each parameter is observed for its convergence. It can be observed from the graphs (Fig. 2(a)–(j)), that all the true parameters are converging to either minimum or maximum, in the maximum risk condition. This shows that the maximum risk will occur when the true parameters assume the extreme values of the parameter ranges. The true parameters where the maximum risk occurs are given in Table 5. This completes the graphical proof.

Table 5
True parameter set corresponding to maximum risk

b_T	b_L (0.25)
c_T	c_L (0.05)
W_T	W_L (12.5)
Y_T	Y_L (25)
M_T	M_H (150)
λ_T	λ_H (0.015)
δ_T	δ_L (1.0)
g_T	g_L (0.25)
d_T	d_L (1.0)

For this example problem, the maximum risk obtained by keeping the design parameters at their average values = 171.81%.
Since it is observed that the maximum risks occur when the true parameters take values at the extremes of the parameter ranges, the solution space to find maximum risk becomes 2^9alternatives for a chosen design vector of (n, h, k). That is, to find the maximum risk corresponding to a particular design parameter set, one has to find the actual costs and optimum costs for all combinations of parameter sets formed by the extreme values of the parameter ranges.

5. Robust economic-statistical design of control chart with multiple parameter variation

The problem of robust control chart design can be considered as two categories:
1. Robust design with single parameter variation.
2. Robust design with multiple parameter variation.

In the case of single parameter variation, the uncertainty in the estimation of one of the important process parameters, like mean shift of the process will be considered and the corresponding parameter will be expressed in a range. The control chart has to be designed

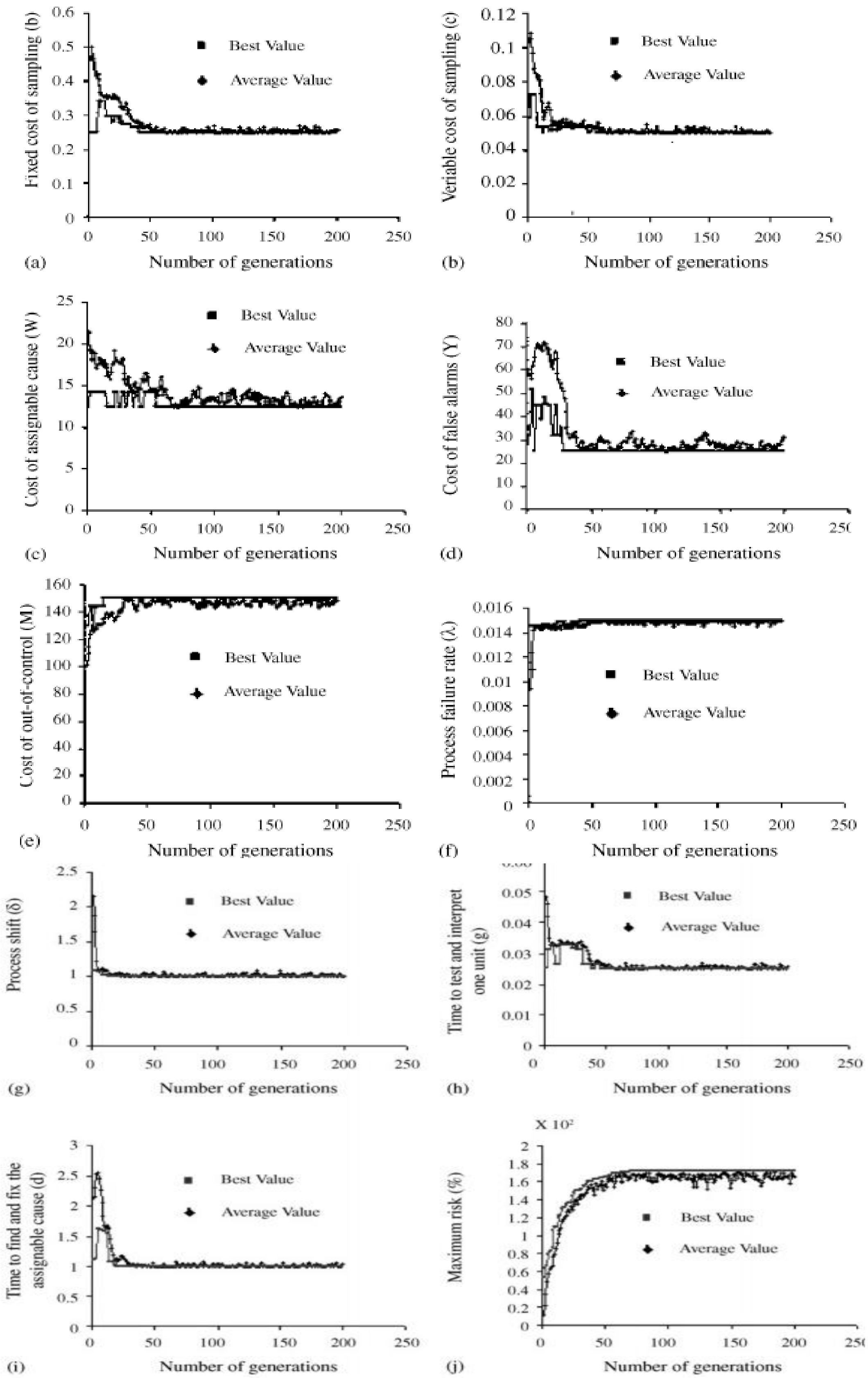


Fig. 2. (a)–(j) Convergence proof for control chart cost and process parameters.

considering the range for the single parameter under consideration. When more than one parameter is expressed in a range, the design can be categorized as robust design with multiple parameter variation. In this case, it is obvious that each cost and process parameter has its own minimum and maximum values. True value of any parameter is unknown to the designer. The cost and process parameters expressed in ranges, form a design parameter space. This design parameter space is infinite since each design parameter can take any number of values within it's own range. By suitably dividing the range of each design parameter into some appropriate number of parts, the design parameter space can be made finite. Any possible set of parameters can be taken from the finite (approximate) design parameter space for the design purpose. Similarly, the true parameters can be thought of forming a true parameter space of same dimensions as design parameter space. Hence any parameter set chosen for design would give (size of parameter space-1) situations where the actual cost is higher than optimum cost, thereby leading to cost penalties. But it is required to see that the design parameter set chosen for control chart design is superior to any other possible design parameter set. The present design procedure depends on the principle of minimizing the maximum risk that occurs as a result of not knowing the true parameters and designing the chart with other than true parameter set.

Denoting the approximate parameter space by Ω, the best design parameter set $\varpi_{\cdot}$, which corresponds to minimum of maximum risks can be written as:

$$\varpi_{\cdot} = \min_{\varpi \in \Omega} \{MR_{\varpi}\} \forall \varpi \tag{9}$$

where MR_{ϖ} represents the maximum risk corresponding to the design parameter set ϖ.

The procedure to find the best design parameter set ($\varpi_{\cdot}$) from the approximate design parameter space is given below:

1. Choose a design parameter set such that each design parameter belongs to its own range, i.e. a design parameter set is to be chosen from design parameter space.
2. Calculate the maximum risk corresponding to the design (input) parameter set chosen (as discussed in Section 4.1).
3. Based on the maximum risk of the above design parameter set, choose a new design parameter set logically such that the maximum risk may be less compared to the previous design parameter set.
4. Continue the process till a design parameter set is obtained which corresponds to a minimum of maximum risks or in other words repeat the steps 1, 2 and 3 till the improvement in the reduction of maximum risks is not possible or negligible.
5. Use the design parameter set obtained in the step 4 in the design of control chart parameters, namely, (n, h, k) values.

The above procedure clearly suggests to logically choosing a design parameter set such that the maximum risk is less compared to the previous design parameter set considered. This logical search, in the proposed method, is accomplished with the help of genetic algorithm (GA). The procedure followed to obtain the best design parameter set such that the maximum possible risk is minimized, can be explained as follows:

1. Take a pre-determined number of design parameter sets from the design parameter space. This represents the population size in GA.
2. Now to each design parameter set considered, find the maximum risk. Using the maximum risk as the basis for fitness, obtain the best design parameter sets for further selection of new design parameter sets. Here, the selection is based on the minimization of the maximum risks. That is, the design parameter sets having minimum values of maximum risks will have higher chances for selection. This represents the selection process in the genetic algorithm.
3. Using the design parameter sets obtained in the step 2, obtain new design parameter sets by using crossover principle. This leads to the creation of better offspring or the design parameter sets.
4. Mutation is allowed in the process to avoid premature convergence in the process.
5. The process is repeated till a pre-determined number of generations are over and the best design parameter set is obtained from the final population.

The control chart parameters can be found corresponding to the best design parameter set obtained from the above search. Fig. 3 gives a summary of the proposed method.

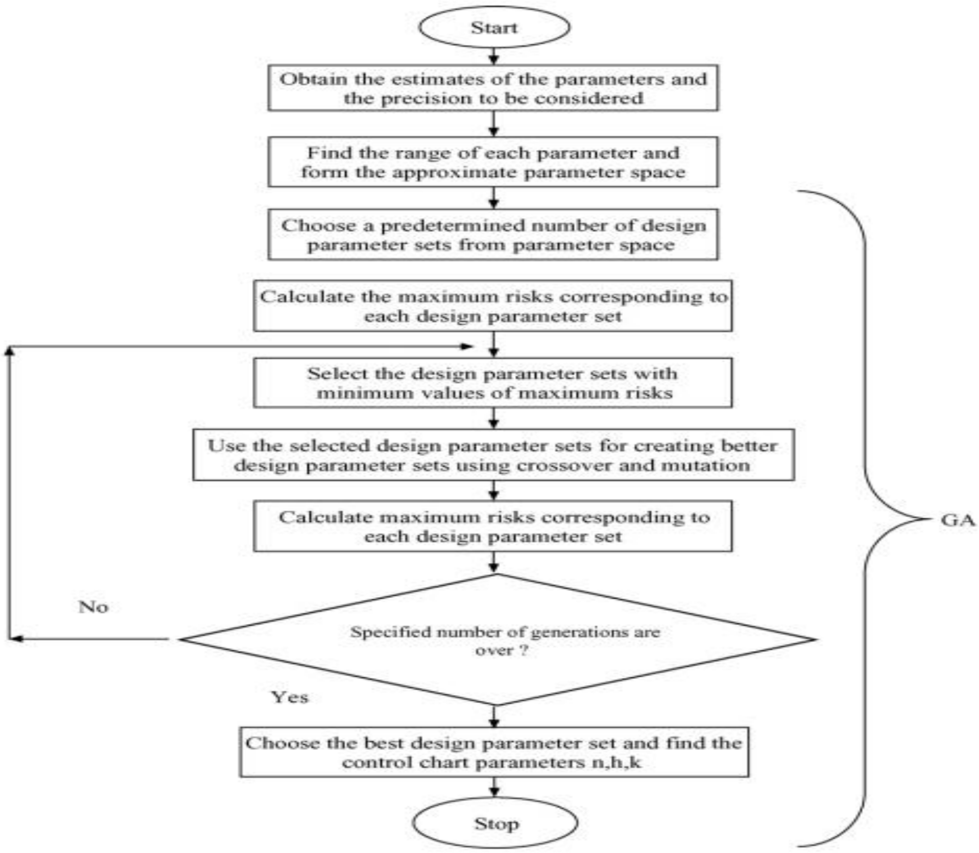


Fig. 3. Flow chart showing the procedure of risk-based robust economic design.

5.1. Application of the proposed robust economic-statistical design procedure

The proposed robust economic-statistical design procedure needs the input parameters, only to be specified in ranges. This risk-based approach is extremely useful when the designer is completely unaware of the distribution of a parameter in the considered range.

Table 6

Cost and process parameter ranges for different precision scenarios

Parameter	Estimated values	Precision scenario (±10%)		Precision scenario (±20%)		Precision scenario (±30%)		Precision scenario (±40%)		Precision scenario (±50%)	
		Low	High	Low	High	Low	High	Low	High	Low	High
b	0.5	0.450	0.55	0.400	0.600	0.350	0.65	0.300	0.700	0.250	0.750
c	0.1	0.090	0.110	0.080	0.120	0.070	0.013	0.060	0.140	0.050	0.150
W	25.0	22.50	27.50	20.00	30.00	17.50	32.50	15.00	35.00	12.50	37.50
Y	50.0	45.00	55.00	40.00	60.00	35.00	65.00	30.00	70.00	25.00	75.00
M	100.0	90.00	110.0	80.00	120.0	70.00	130.0	60.00	140.0	50.00	150.0
λ	0.01	0.009	0.011	0.008	0.012	0.007	0.013	0.006	0.014	0.005	0.015
δ	2.0	1.800	2.200	1.600	2.400	1.400	2.600	1.200	2.800	1.000	3.000
g	0.05	0.045	0.055	0.040	0.060	0.035	0.065	0.030	0.070	0.025	0.075
d	2.0	1.800	2.200	1.600	2.400	1.400	2.600	1.200	2.800	1.000	3.000

A control chart design problem (Table 1) has been considered for a process with the following cost and process parameters:

Five different precisions in the estimates at ±10, ±20, ±30, ±40 and ±50% are considered for control chart designs. The parameter ranges for different precision scenarios are shown in Table 6.

Based on the results of parametric study, the population size is fixed in the range of 100–300. The mutation probability is found to be close to (1/string size) for all scenarios. Linear ranking method has been employed in calculating the fitness of the strings. Stochastic universal sampling has been employed in the selection process. A single point cross over is found to give good results. An elitist strategy with a generation gap of 0.9 has been employed. The solutions for all precision scenarios are found to converge at less than 100 generations. Table 7 shows the combinations of GA parameters used in each scenario.

Table 7

GA parameters used with different precision scenarios

Precision scenario (%)	Population size	Cross over probability	Mutation probability	Size of sub string/ chromosome	Selection/ cross over type	Maximum generations
±10	100	0.92	0.0217	5/45	sus/xosp	100
±20	100	0.80	0.0217	5/45	sus/xosp	100
±30	150	0.84	0.0182	6/54	sus/xosp	100
±40	200	0.84	0.0156	7/63	sus/xosp	100
±50	300	0.84	0.0137	8/72	sus/xosp	100

Using the above parameters, the GA is run and the results obtained are provided in the following tables (Tables 8 and 9). In order to compare the performance of average and risk-based methods, risks surfaces have been shown at different precision scenarios. Fig. 4(a)–(e) show the risk surfaces between the process failure rate and the process mean shift, with other parameters maintained at maximum risk condition. It can be observed that the average based designs and the risk-based designs do not differ much in terms of the risk at low precision levels.

Table 8

Design parameters at different precision levels with $\alpha \le 0.005$, $p \ge 0.95$, $ATS_o \le 5$

Precision scenario (%)	b	c	W	Y	M	λ	δ	g	d	Risk (%)
±10	0.5306	0.1087	25.726	45.323	102.903	0.0097	2.1097	0.048	1.826	0.87
±20	0.4516	0.0955	21.613	40.000	85.161	0.0101	2.0645	0.046	1.600	6.38
±30	0.6119	0.1109	18.691	43.095	121.429	0.0071	1.8762	0.046	1.400	14.98
±40	0.6370	0.0883	21.929	36.614	72.598	0.0093	1.7165	0.045	2.800	31.69
±50	0.6304	0.1496	14.265	27.353	62.941	0.0104	1.5098	0.055	1.949	58.57

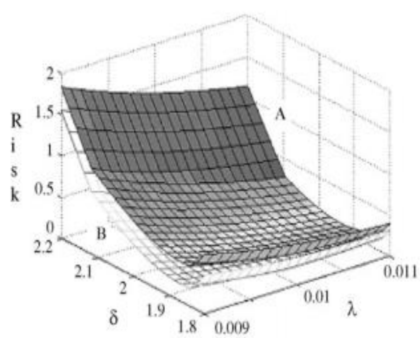
Table 8 gives the details of the best design parameter sets at each precision scenario, to find the control chart parameters. The last column of the table indicates the maximum possible risk with each scenario. For example, in the ±40% precision scenario, the maximum possible risk is 31.69%, which indicates that the risk will not be more than this value for any real scenario of the process.

Table 9 gives the results of the risk-based robust economic-statistical designs at different precision scenarios.

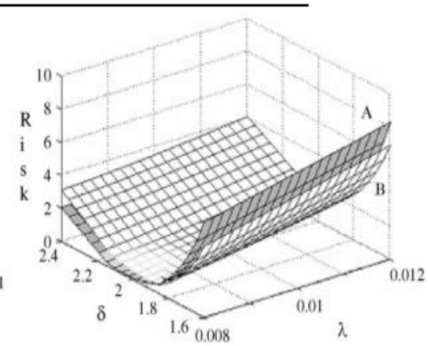
Table 9

Control chart design under statistical constraints

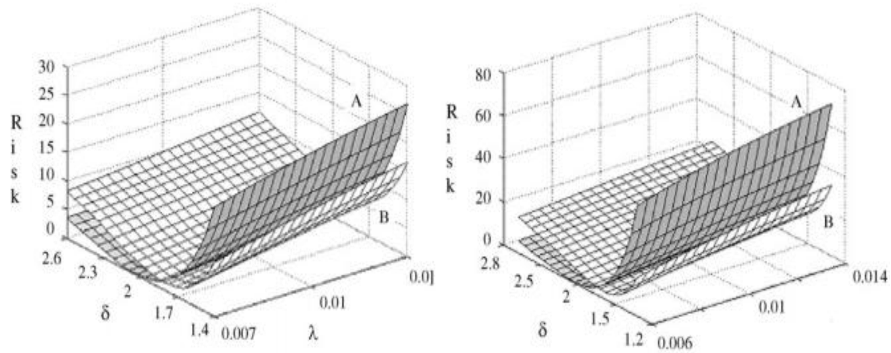
Precision scenario (%)	n	h	k	Type-I error	Power	ATS_o
±10	5	1.457	3.072	0.0021	0.9501	1.533
±20	5	1.478	2.971	0.0030	0.9501	1.555
±30	6	1.724	2.950	0.0032	0.9501	1.815
±40	7	1.929	2.896	0.0038	0.9501	2.030
±50	9	2.427	2.823	0.0048	0.9560	2.539



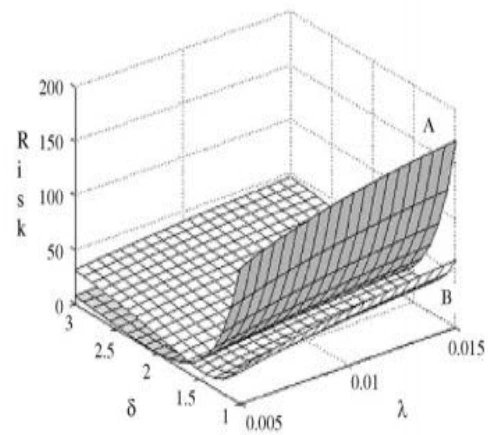
(a) Risk Variation at ± 10%.



(b) Risk Variation at ± 20%.



(c) Risk Variation at ± 30%. (d) Risk Variation at ± 40%.



(e) Risk Variation at ± 50%.

Fig. 4. Variation of risk between process failure rate and process mean shift at different precision levels (at maximum risk condition). A: Risk surface of average based design. B: Risk surface of risk-based design.

6. Summary and conclusions

A simple robust economic design methodology for an $\overline{X}$ control chart has been discussed. The design is made robust to the values of the true cost and process parameters in their specified ranges. The principle of minimization of maximum risk has been used in the design, which is simple to understand. Apart from simplicity of the approach, this method has the advantage of incorporating statistical constraints into the design, which is very much essential in the control chart designs. Genetic algorithm has been used as an efficient search tool for finding the best design parameter set. This methodology assumes primarily that no information is known about the distribution of the parameter in its own range. This will help to find a robust economic-statistical design when very little information about the parameter is available. Results of the robust designs indicate that the average based designs and the risk-based designs do not differ much in terms of the risk at low precision levels. At higher levels of uncertainty in the estimation of parameters, the advantages of using robust design are profound.

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