Making a Mathematical Programming in Fuzzy Systems with Genetic Algorithm

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Abstract: In this article in order to assessment the efficiency of systems with have imprecise data (fuzzy), a fuzzy mathematical programming approach based on the comparison of α-cuts, using data envelopment analysis (DEA) models, is expressed. The ultimate goal of this article is determining the optimal possibility level (α) and also specifying the efficiency scores in these levels for the DMU_s, using genetic algorithm. To this goal we assess efficiency of a banking system including 16 banks and financial institutes in a two year period using suggested model, then using genetic algorithm for each DMU_s, the optimal possibility levels are derived, also the efficiency scores in these levels are calculated and then the results are compared**.**

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1. Introduction

Efficiency assessment of the manufacturing, educating and services has been a major concern among engineers and economists. Nowadays, regarding intricacy of problems, influx of information, international rivalry etc. It is absolutely vital to apply a scientific approach to elicit the desired goal. Efficiency assessment is an important issue in this regard. Since the history of applying efficiency assessment ways back in years ago, different approaches have been presented. DEA is one of the techniques applying to efficiency assessment. DEA is a technique in which using linear programming, the degree of efficiency of a DMU in comparison to decision making units which product similar outputs by similar inputs, is calculated. Actually DEA specifies the efficient and non efficient units as well as determining the efficiency. Efficiency assessment backing by DEA, has attracted researchers' attention in the last two decades. DEA was first developed by Charnes, Cooper and Rhodes a CCR model in 1987. Then it was rapidly developed both theoretically and practically.

In most real systems, data are inaccurate. In efficiency assessment this inaccuracy must be taking into account. Inaccuracy has been investigated differently by researchers. Sometimes it is presented as fuzzy. The word "fuzzy" means vague, inaccurate, disorder. In expression and dealing with human behavior mathematical models are confronted with inaccurate and indefinite situation, consequently many human judgments and real problems in the universe cannot be expressed using classical mathematics and scientists believe that economical, social and human systems can be expressed and modeled using a new branch of mathematics called fuzzy mathematics.

One of the most important applicability of fuzzy theory is fuzzy optimization and fuzzy linear programming. As an example we can refer to researches done by Cooper, Jahanshahloo and Khodabakhshi[1, 2, 3, 4]. In [5] some fuzzy DEA models are presented. In [6] a fuzzy mathematical programming approach based on the comparison of α-cuts is explained. The ultimate goal of this article is determining the optimal possibility level (α) and also specifying the efficiency scores of these levels for the DMU_s, using genetic algorithm.

The paper unfolds as follows: Section 2 contains some fuzzy concept [6]. In section 3 the mentioned approach for the assessment of efficiency is presented [6]. In section 4 another model with the assumption that data are symmetric triangle fuzzy numbers is presented [6]. In section 5 we assess efficiency of a banking system including 16 banks and financial institutes in a two year period using suggested model[6], then using genetic algorithm for each DMU_s, the optimal possibility levels are derived, also the efficiency scores in these levels are calculated and then the results are compared. Section 6 contains final conclusions.

2. Some fuzzy definitions and concept [6]

Definition1. α-cut set is a set that its elements belong to X such that their degrees of
than or equal to α. Namely:

$$\left[\overline{M}\right]_{\alpha} = \overline{M}_{\alpha} = \left\{x \in X \,\middle|\, \mu_{\overline{M}}(x) \geq \alpha\right\} \tag{1}$$

Definition2. Fuzzy number $\overline{M}$ is a LR number which its membership function defined by:

$$\mu_{\overline{M}}(r) = \begin{cases} L\left(\dfrac{m^L - r}{\alpha^L}\right) & r \leq m^L \\ 1 & m^L \leq r \leq m^R \\ R\left(\dfrac{r - m^R}{\alpha^R}\right) & r \geq m^R \end{cases} \tag{2}$$

Where $L, R : [\circ, +\infty) \to [\circ, 1]$ are upper semi-continuous and strictly decreasing functions in $SUPP(\overline{M}) = \left\{r \middle| \mu_{\overline{M}}(r) > \circ\right\}$ such that $L(\circ) = R(\circ) = 1$.

And $\overline{M} = (m^L, m^R, \alpha^L, \alpha^R)_{L,R}$.

Definition3 [5]. Let $\overline{M}$ and $\overline{N}$ be two fuzzy numbers. Then,

$$\overline{M} \underset{\approx}{>} \overline{N} \Leftrightarrow \overline{M} \vee \overline{N} = \overline{M} \tag{3}$$

Lemma1 [5, 7]. Let $\overline{M}$ and $\overline{N}$ be two fuzzy numbers. Then, $\overline{M} \vee \overline{N} = \overline{M}$ if, and only if, $\forall h \in [\circ, 1]$ the two statements below hold:

$$\begin{aligned} &\inf\left\{s : \mu_{\overline{M}}(s) \geq h\right\} \geq \inf\left\{t : \mu_{\overline{N}}(t) \geq h\right\} \\ &\sup\left\{s : \mu_{\overline{M}}(s) \geq h\right\} \geq \sup\left\{t : \mu_{\overline{N}}(t) \geq h\right\}. \end{aligned} \tag{4}$$

In particular, for two LR-fuzzy numbers, $\overline{M} = (m^L, m^R, \alpha^L, \alpha^R)_{L,R}$, $\overline{N} = (n^L, n^R, \beta^L, \beta^R)_{L',R'}$, (4) holds if, and only if,

$$\begin{aligned} &m^L - L^*(h)\alpha^L \geq n^L - L'^*(h)\beta^L \quad \forall h \in [\circ, 1] \\ &m^R + R^*(h)\alpha^R \geq n^R + R'^*(h)\beta^R \quad \forall h \in [\circ, 1] \end{aligned} \tag{5}$$

Where

$$\begin{aligned} &L^*(h) = \sup\left\{z : L(z) \geq h\right\} \\ &L'^*(h) = \sup\left\{z : L'(z) \geq h\right\} \\ &R^*(h) = \sup\left\{z : R(z) \geq h\right\} \\ &R'^*(h) = \sup\left\{z : R'(z) \geq h\right\} \end{aligned} \tag{6}$$

Moreover, if $\overline{M} = (m^L, m^R, \alpha^L, \alpha^R)_{L,R}$ and $\overline{N} = (n^L, n^R, \beta^L, \beta^R)_{L',R'}$ have bounded support and $L = L'$ and $R = R'$ holds, then (6) becomes

$$\begin{aligned} &m^L \geq n^L \\ &m^R \geq n^R \\ &m^L - \alpha^L \geq n^L - \beta^L \\ &m^R + \alpha^R \geq n^R + \beta^R \end{aligned} \tag{7}$$

Definition 4[5]. Let $\overline{N}, \overline{M}$ be two fuzzy numbers and h a real number, $h \in [\circ, 1]$. Then $\overline{M} \overset{h}{\underset{\approx}{>}} \overline{N}$ if, and only if, $\forall k \in [h, 1]$ the following two statements hold:

$$\begin{aligned} &\inf\left\{s : \mu_{\overline{M}}(s) \geq k\right\} \geq \inf\left\{t : \mu_{\overline{N}}(t) \geq k\right\} \\ &\sup\left\{s : \mu_{\overline{M}}(s) \geq k\right\} \geq \sup\left\{t : \mu_{\overline{N}}(t) \geq k\right\} \end{aligned} \tag{8}$$

For LR-fuzzy numbers with bounded support, and using this ranking method, for a given h, (8) expression becomes

$$\begin{aligned} &m^L - L^*(k)\alpha^L \geq n^L - L'^*(k)\beta^L \quad \forall k \in [h, 1] \\ &m^R + R^*(k)\alpha^R \geq n^R + R'^*(k)\beta^R \quad \forall k \in [h, 1] \end{aligned} \tag{9}$$

And for LR-fuzzy numbers $\overline{M} = (m^L, m^R, \alpha^L, \alpha^R)_{L,R}$ and $\overline{N} = (n^L, n^R, \beta^L, \beta^R)_{L',R'}$ with bounded support that $R = R'$, $L = L'$ hold, expression (9) becomes

$$\begin{aligned} &m^L \geq n^L \\ &m^R \geq n^R \\ &m^L - L^*(h)\alpha^L \geq n^L - L^*(h)\beta^L \\ &m^R + R^*(h)\alpha^R \geq n^R + R^*(h)\beta^R \end{aligned} \tag{10}$$

3. A model in fuzzy data envelopment analysis (FDEA) [6]

For assessing efficiency with DEA models here we use the input-oriented BCC model. We are willing to evaluate the efficiency of n DMU_s which use m inputs to produce s outputs. We assume that data are imprecise and they can be defined as LR-fuzzy numbers with bounded support satisfying [5] :

$$\begin{aligned} &L_{i1} = L_{i2} = \dots\dots = L_{in} := L_i, && i = 1,\dots, m \\ &L'_{r1} = L'_{r2} = \dots\dots = L'_{rn} := L'_r, && r = 1,\dots, s \\ &R_{i1} = R_{i2} = \dots\dots = R_{in} := R_i, && i = 1,\dots, m \\ &R'_{r1} = R'_{r2} = \dots\dots = R'_{rn} := R'_r, && r = 1,\dots, s \end{aligned} \tag{11}$$

Note that if data are trapezoid or triangle fuzzy numbers then (11) holds. The extended BCC model can be defined as the following fuzzy linear program:

$$\begin{aligned} \min \quad & \theta_\circ \\ s.t \quad & \sum_{j=1}^{n} \lambda_j \bar{x}_{ij} \underset{\approx}{<} \theta_\circ \bar{x}_{io} && i = 1,\dots,m \\ & \sum_{j=1}^{n} \lambda_j \bar{y}_{rj} \underset{\approx}{>} \bar{y}_{ro} && r = 1,\dots,s \qquad \text{model (1)} \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \ge \circ \ , && j = 1,\dots,n \end{aligned}$$

Here we consider LR fuzzy number for all DMU_s, especially for DMU_o:

$$\begin{aligned} \bar{x}_{io} &= (x_{io}^L, x_{io}^R, \alpha_{io}^L, \alpha_{io}^R)_{L_i,R_i} && i = 1,\dots,m \\ \bar{y}_{ro} &= (y_{ro}^L, y_{ro}^R, \beta_{ro}^L, \beta_{ro}^R)_{L'_r,R'_r} && r = 1,\dots,s \\ \bar{x}_{ij} &= (x_{ij}^L, x_{ij}^R, \alpha_{ij}^L, \alpha_{ij}^R)_{L_i,R_i} && {\scriptstyle i=1,\dots,m \atop j=1,\dots,n} \\ \bar{y}_{rj} &= (y_{rj}^L, y_{rj}^R, \beta_{rj}^L, \beta_{rj}^R)_{L'_r,R'_r} && {\scriptstyle r=1,\dots,s \atop j=1,\dots,n} \end{aligned} \tag{12}$$

Then, according to lemma 1:

$$\begin{cases} x_{ij}^L \le x_{io}^L \\ x_{ij}^R \le x_{io}^R \\ x_{ij}^L - \alpha_{ij}^L \le x_{io}^L - \alpha_{io}^L \\ x_{ij}^R + \alpha_{ij}^R \le x_{io}^R + \alpha_{io}^R \end{cases} \Leftrightarrow \quad \bar{x}_{ij} \underset{\approx}{<} \bar{x}_{io} \tag{13}$$

And

$$\begin{cases} y_{rj}^L \ge y_{ro}^L \\ y_{rj}^R \ge y_{ro}^R \\ y_{rj}^R + \beta_{ij}^R \ge y_{ro}^R + \beta_{ro}^R \\ y_{rj}^L - \beta_{rj}^L \ge y_{ro}^L - \beta_{ro}^L \end{cases} \Leftrightarrow \quad \bar{y}_{rj} \underset{\approx}{>} \bar{y}_{ro} \tag{14}$$

Therefore, model (1) can be rewritten as

$$\begin{aligned} \min \quad & \theta_\circ \\ s.t \quad & \sum_{j=1}^{n} \lambda_j (x_{ij}^L, x_{ij}^R, \alpha_{ij}^L, \alpha_{ij}^R)_{L,R} \underset{\approx}{<} \theta_\circ (x_{io}^L, x_{io}^R, \alpha_{io}^L, \alpha_{io}^R)_{L,R} \\ & \qquad i = 1,\dots,m \\ & \sum_{j=1}^{n} \lambda_j (y_{rj}^L, y_{rj}^R, \beta_{rj}^L, \beta_{rj}^R)_{L',R'} \underset{\approx}{>} (y_{ro}^L, y_{ro}^R, \beta_{ro}^L, \beta_{ro}^R)_{L',R'} \\ & \qquad r = 1,\dots,s \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \ge \circ \ , \quad j = 1,\dots,n \end{aligned} \tag{15}$$

Or equivalently

$$\begin{aligned} \min \quad & \theta_\circ \\ s.t \quad & \Big(\sum_{j=1}^{n} \lambda_j x_{ij}^L, \sum_{j=1}^{n} \lambda_j x_{ij}^R, \sum_{j=1}^{n} \lambda_j \alpha_{ij}^L, \sum_{j=1}^{n} \lambda_j \alpha_{ij}^R\Big)_{L,R} \underset{\approx}{<} \\ & (\theta_\circ x_{io}^L, \theta_\circ x_{io}^R, \theta_\circ \alpha_{io}^L, \theta_\circ \alpha_{io}^R)_{L,R} \qquad i=1,\dots,m \\ & \Big(\sum_{j=1}^{n} \lambda_j y_{rj}^L, \sum_{j=1}^{n} \lambda_j y_{rj}^R, \sum_{j=1}^{n} \lambda_j \beta_{rj}^L, \sum_{j=1}^{n} \lambda_j \beta_{rj}^R\Big)_{L',R'} \underset{\approx}{>} \\ & (y_{ro}^L, y_{ro}^R, \beta_{ro}^L, \beta_{ro}^R)_{L',R'} \qquad r=1,\dots,s \\ & \sum_{j=1}^{n} \lambda_j = 1 \qquad \lambda_j \ge \circ \ , \quad j=1,\dots,n \end{aligned} \tag{16}$$

Now considering (13) and (14), (16) can be transformed to:

$$\begin{aligned}
&\min \quad \theta_{\circ} \\
&s.t \quad \sum_{j=1}^{n} \lambda_j x_{ij}^{L} \le \theta_{\circ} x_{io}^{L} \qquad i=1,\dots,m \\
&\sum_{j=1}^{n} \lambda_j x_{ij}^{R} \le \theta_{\circ} x_{io}^{R} \qquad i=1,\dots,m \\
&\sum_{j=1}^{n} \lambda_j x_{ij}^{L} - \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{L} \le \theta_{\circ} x_{io}^{L} - \theta_{\circ} \alpha_{io}^{L} \qquad i=1,\dots,n \\
&\sum_{j=1}^{n} \lambda_j x_{ij}^{R} + \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{R} \le \theta_{\circ} x_{io}^{R} + \theta_{\circ} \alpha_{io}^{R} \qquad i=1,\dots,r \\
&\sum_{j=1}^{n} \lambda_j y_{rj}^{L} \ge y_{ro}^{L} \qquad r=1,\dots,s \qquad \text{Model(2)} \\
&\sum_{j=1}^{n} \lambda_j y_{rj}^{R} \ge y_{ro}^{R} \qquad r=1,\dots,s \\
&\sum_{j=1}^{n} \lambda_j y_{rj}^{L} - \sum_{j=1}^{n} \lambda_j \beta_{rj}^{L} \ge y_{ro}^{L} - \beta_{ro}^{L} \qquad r=1,\dots,s \\
&\sum_{j=1}^{n} \lambda_j y_{rj}^{R} + \sum_{j=1}^{n} \lambda_j \beta_{rj}^{R} \ge y_{ro}^{R} + \beta_{ro}^{R} \qquad r=1,\dots,s \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \ge \circ \;\; \& \quad j=1,\dots,n
\end{aligned}$$

It is possible that we want to know the efficiency scores taking account into h-possibility level. In this case considering definition 4, another model can be formulated.

For LR-fuzzy data, according to definition 4:

$$\begin{cases} x_{ij}^{L} \le x_{io}^{L} \\ x_{ij}^{R} \le x_{io}^{R} \\ x_{ij}^{L} - L_i^{*}(h)\alpha_{ij}^{L} \le x_{io}^{L} - L_i^{*}(h)\alpha_{io}^{L} \\ x_{ij}^{R} + R_i^{*}(h)\alpha_{ij}^{R} \le x_{io}^{R} + R_i^{*}(h)\alpha_{io}^{R} \end{cases} \Leftrightarrow \bar{x}_{ij} \overset{h}{\underset{\approx}{<}} \bar{x}_{io} \tag{17}$$

And

$$\begin{cases} y_{rj}^{L} \ge y_{ro}^{L} \\ y_{rj}^{R} \ge y_{ro}^{R} \\ y_{rj}^{L} - L'^{*}_{r}(h)\beta_{rj}^{L} \ge y_{ro}^{L} - L'^{*}_{i}(h)\beta_{ro}^{L} \\ y_{rj}^{R} + R'^{*}_{r}(h)\beta_{rj}^{R} \ge y_{ro}^{R} + R'^{*}_{r}(h)\beta_{ro}^{R} \end{cases} \Leftrightarrow \bar{y}_{rj} \overset{h}{\underset{\approx}{>}} \bar{y}_{ro} \tag{18}$$

Then, model (1) can be rewritten as:

$$\begin{aligned}
&(P^{h}) \quad \min \ \theta_{\circ} \\
&s.t \ \sum_{j=1}^{n} \lambda_j (x_{ij}^{L}, x_{ij}^{R}, \alpha_{ij}^{L}, \alpha_{ij}^{R})_{L,R} \overset{h}{\underset{\approx}{<}} \theta_{\circ}(x_{io}^{L}, x_{io}^{R}, \alpha_{io}^{L}, \alpha_{io}^{R})_{L,R} \\
&\qquad i=1,\dots,m \\
&\sum_{j=1}^{n} \lambda_j (y_{rj}^{L}, y_{rj}^{R}, \beta_{rj}^{L}, \beta_{rj}^{R})_{L',R'} \overset{h}{\underset{\approx}{>}} (y_{ro}^{L}, y_{ro}^{R}, \beta_{ro}^{L}, \beta_{ro}^{R})_{L',R'} \qquad (19) \\
&\qquad r=1,\dots,s \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \ge \circ, \quad j=1,\dots,n
\end{aligned}$$

Or equivalently:

$$\begin{aligned}
&(P^{h}) \quad \min \quad \theta_{\circ} \\
&s.t \ \left(\sum_{j=1}^{n} \lambda_j x_{ij}^{L}, \sum_{j=1}^{n} \lambda_j x_{ij}^{R}, \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{L}, \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{R}\right)_{L,R} \overset{h}{\underset{\approx}{<}} \\
&\quad (\theta_o x_{io}^{L}, \theta_o x_{io}^{R}, \theta_o \alpha_{io}^{L}, \theta_o \alpha_{io}^{R})_{L,R} \quad i=1,\dots,m \\
&\left(\sum_{j=1}^{n} \lambda_j y_{rj}^{L}, \sum_{j=1}^{n} \lambda_j y_{rj}^{R}, \sum_{j=1}^{n} \lambda_j \beta_{rj}^{L}, \sum_{j=1}^{n} \lambda_j \beta_{rj}^{R}\right)_{L',R'} \overset{h}{\underset{\approx}{>}} \qquad (20) \\
&(y_{ro}^{L}, y_{ro}^{R}, \beta_{ro}^{L}, \beta_{ro}^{R})_{L',R'} \quad r=1,\dots,s \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \ge \circ, \quad j=1,\dots,n
\end{aligned}$$

Considering (17) and (18), (20) can be reformulated as:

$$\begin{aligned}
&\min \ \theta_{\circ} \\
&s.t \ \sum_{j=1}^{n} \lambda_j x_{ij}^{L} \le \theta_{\circ} x_{io}^{L} \qquad i=1,\dots,m \\
&\sum_{j=1}^{n} \lambda_j x_{ij}^{R} \le \theta_{\circ} x_{io}^{R} \qquad i=1,\dots,m \\
&\sum_{j=1}^{n} x_{ij}^{L} - L_i^{*}(h) \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{L} \le \theta_{\circ} x_{io}^{L} - L_i^{*}(h)\theta_{\circ}\alpha_{io}^{L} \quad i=1,\dots,m \\
&\sum_{j=1}^{n} \lambda_j x_{ij}^{R} + R_i^{*}(h) \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{R} \le \theta_{\circ} x_{io}^{R} + R_i^{*}(h)\theta_{\circ}\alpha_{io}^{R} \quad i=1,\dots,m \\
&\sum_{j=1}^{n} \lambda_j y_{rj}^{L} \ge y_{ro}^{L} \qquad r=1,\dots,s \qquad \text{Model}(3) \\
&\sum_{j=1}^{n} \lambda_j y_{rj}^{R} \ge y_{ro}^{R} \qquad r=1,\dots,s \\
&\sum_{j=1}^{n} \lambda_j y_{rj}^{L} - L'^{*}_{r}(h) \sum_{j=1}^{n} \lambda_j \beta_{rj}^{L} \ge y_{ro}^{L} - L'^{*}_{r}(h)\beta_{ro}^{L} \quad r=1,\dots,s \\
&\sum_{j=1}^{n} \lambda_j y_{rj}^{R} + R'^{*}_{r}(h) \sum_{j=1}^{n} \lambda_j \beta_{rj}^{R} \ge y_{ro}^{R} + R'^{*}_{r}(h)\beta_{ro}^{R} \quad r=1,\dots,s \\
&\sum_{j=1}^{n} \lambda_j = 1, \lambda_j \ge \circ \;\; \& \quad j=1,\dots,n
\end{aligned}$$

The optimal value of model (3), specifies the efficiency score of DMU at the h-possibility level, and also solving this model for different values of *h*, we can see how the efficiency scores of the DMU change when *h* varies.

4. A simpler model for symmetric triangle fuzzy data [6]

If we consider inputs and outputs as symmetric triangle fuzzy numbers such as

$$\bar{x}_{ij} = (x_{ij}, \alpha_{ij}) \quad i = 1, \ldots, m \;\; \& \;\; j = 1, \ldots, n \qquad (21)$$
$$\bar{y}_{rj} = (y_{rj}, \beta_{rj}) \quad r = 1, \ldots, s \;\; \& \;\; j = 1, \ldots, n$$

Model (3) can be substantially simplified. Recalling that for symmetric triangle fuzzy numbers

$$L_i^{*}(h) = R_i^{*}(h) = L'^{*}_{r}(h) = R'^{*}_{r}(h) = 1 - h \quad \circ \le h \le 1 \qquad (22)$$
$$i = 1, \ldots, m \;\; \& \;\; r = 1, \ldots, s$$

and also, the two constraints with the main values reduce to only one, model (3) for these data becomes

$$\begin{aligned}
\min \quad & \theta_{\circ} \\
s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij} - (1-h)\sum_{j=1}^{n} \lambda_j \alpha_{ij} \le \theta_{\circ} x_{io} - (1-h)\theta_{\circ}\alpha_{io} \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j x_{ij} + (1-h)\sum_{j=1}^{n} \lambda_j \alpha_{ij} \le \theta_{\circ} x_{io} + (1-h)\theta_{\circ}\alpha_{io} \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - (1-h)\sum_{j=1}^{n} \lambda_j \beta_{rj} \ge y_{ro} - (1-h)\beta_{ro} \quad r = 1, \ldots, s \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + (1-h)\sum_{j=1}^{n} \lambda_j \beta_{rj} \ge y_{ro} + (1-h)\beta_{ro} \quad r = 1, \ldots, s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge \circ \;\; \& \;\; j = 1, \ldots, n
\end{aligned} \qquad \text{Model (4)}$$

By using the recommended models we can evaluate the efficiency of systems with imprecise nature.

5. Efficiency assessment of a banking system and determining the optimal possibility level (α) for each DMU$_s$ using genetic algorithm

In this section having the data on sixteen banks of America's banking system, efficiency assessment of this system is accomplished [6]. Data on DMUs in the years 2003 and 2004 is shown in tables 1 and 2.

The sixteen target institutes have three inputs called "Base Salary, Option Adjusted compensation, Non-interest expense" and two outputs called "Net Income, Revenue". Data are considered in symmetric triangle fuzzy numbers. Fuzzy data are presented in tables 3 and 4. The results of the efficiency assessment of these banks, over the two year period, based on model 4 in different *h* levels with *h*=0.25are presented in table 5[6].

Ultimately using genetic algorithm we have determined the optimal possibility level (α) and also specified the efficiency scores in these levels for each bank. Results are shown in table 6.

Table 1. Data of Sixteen financial institutes in 2003

DMUs	**Base Salary**	**Option Adjusted Compensation**	**Non-interest Expense**	**Net Income**	**Revenue**
Babette E. Heimbuch	463,680	943,809	54,809	1,157,000	5,605,000
Daniel H. Cuddy	81,200	89,450	70,108	55,572	150,112
David A. Daberko	1,000,000	9,567,286	4,048,034	573,942	2,429,850
Dunson K. Cheng	700,000	7,722,470	55,140	2,117,064	7,964,007
Ernest S. Rady	348,330	865,106	282,482	396,365	971,325
Herbert M. Sandler	900,000	2,807,143	720,515	6,202,000	28,381,000
Jerry A. Grundhofer	1,000,038	15,408,324	5,550,700	1,064,903	4,909,344
John A. Allison IV	887,186	4,520,538	2,933,888	41,077	135,508
John Adam Kanas	2,014,500	10,300,891	332,477	64,475	163,500
Joseph R. Ficalora	850,000	3,039,266	155,857	1,106,099	2,521,714
Marion O. Sandler	900,000	2,805,190	720,515	1,106,099	2,521,714
Michael R. Melvin	182,989	260,672	2,896	123,605	824,432
Richard M. Kovacevich	995,000	22,132,238	17,136,000	4,627	10,084
Robert G. Wilmers	480,768	944,245	1,387,793	323,371	668,962
Thomas A. Renyi	1,000,000	18,818,479	3,524,000	3,732,600	12,502,300
Salomon Levis	1,800,000	3,600,000	185,802	321,299	593,252

Table 2. Data of sixteen financial institutes in 2004

ceo	Base salary	Option adjusted compensation	Non-interest expense	Net income	Revenue
Thomas A. Renyi	1,000,000	12,693,258	4,092,000	1,440,000	6,295,000
Dunson K. Cheng	775,000	1,629,984	90,660	86,813	231,082
Robert G. Wilmers	544,808	1,060,167	1,516,018	722,521	2,677,541
David A. Daberko	1,000,000	10,158,289	4,571,158	2,779,934	8,972,370
John Adam Kanas	2,076,923	12,575,816	555,802	552,996	1,423,724
Richard M. Kovacevich	995,000	45,530,663	17,550,000	7,014,000	30,055,000
John A. Allison IV	900,000	5,443,708	2,890,345	1,558,375	5,467,494
Daniel H. Cuddy	79,085	88,085	74,854	37,176	134,290
Babette E. Heimbuch	486,840	1,144,031	66,064	65,842	182,520
Herbert M. Sandler	900,000	1,457,736	840,126	1,279,721	2,912,528
Marion O. Sandler	900,000	1,455,463	840,126	1,279,721	2,912,528
Ernest S. Rady	362,115	2,240,958	295,607	207,962	923,880
Michael R. Melvin	186,865	278,641	3,062	4,916	10,484
Joseph R. Ficalora	975,000	2,919,338	205,072	355,086	737,040
Jerry A. Grundhofer	1,100,042	22,954,061	5,784,500	4,166,800	12,630,500
Salomon Levis	2,400,000	11,109,560	209,052	489,625	716,282

Table 3. Input of sixteen financial institutes in terms of symmetric triangle fuzzy numbers.

DMU_s	Base Salary	Option Adjusted Compensation	Non-Interest Expense
1 BebetteE. Heimbucl	(475260 , 11580)	(1043920 , 100111)	(60436.5 , 5627.5)
2 Daniel H. Cuddy	(80142.5 , 1057.5)	(88767.5, 682.5)	(72481 , 2373)
3 David A. Daberko	(1000000, 0)	(9862787.5, 687926)	(4309596 , 261562)
4 Dunson K. Cheng	(737500 , 37500)	(4676227, 3046243)	(72900 , 17760)
5 Ernest S. Rady	(355222.5 , 6892.5)	(1553032 , 687926)	(289044.5 , 6562.5)
6 Herbert M. Sandler	(900000, 0)	(2132439.5 , 674703.5)	(780320.5 , 59805.5)
7 Jerry A. Grundhofer	(1050040, 50002)	(19181192.5 , 3772868.5)	(5667600 , 116900)
8 John A. Allison IV	(893593, 6407)	(4982123, 461585)	(2912116.5 , 21771.5)
9 John Adam Kanas	(2045711.5 , 312211.5)	(11438353.5 , 1137462.5)	(439139.5 , 116662.5)
10 Joseph R. Ficalora	(912500 , 62500)	(2979302 , 59964)	(180464.5 , 24607.5)
11 Marion O.Sandler	(900000, 0)	(2130326.5, 674863.5)	(780320.5 , 59805.5)
12 Marin R. Melvin	(184927 , 1938)	(269656.5 , 8948.5)	(2979 , 83)
13 Richard M. Kovacevi	(995000 , 0)	(33831450.5 , 1169921.5)	(17343000 , 207000)
14 Robrt G. Wilmers	(512788, 32020)	(1002206, 57961)	(1451905.5 , 64112.5)
15 Salomon Levis	(2100000 , 300000)	(7354780, 3754780)	(197427 , 11625)
16 Thomas A. Reni	(1000000 , 0)	(15755868.5 , 3062610.5)	(3808000 , 284000)

Table 4. Outputs of sixteen financial institutes in terms of symmetric triangle fuzzy numbers

DMU_s	Net Income	Revenue
1	(611421 , 545579)	(2893760, 2711240)
2	(46347 , 9171)	(142201, 7911)
3	(1676938, 1102996)	(5701110, 3271260)
4	(1101938.5, 1015126)	(4097544.5, 3866462.5)
5	(302163.5, 94201.5)	(947602.5, 23722,5)
6	(3740760.5, 2461139.5)	(15646764, 12734236)
7	(2615851.5, 1550948.5)	(8769922, 3860578)
8	(799726, 758649)	(2801501, 2665993)
9	(308735.5, 244260.5)	(793612, 630112)
10	(730592.5, 375506.5)	(1629377, 892337)
11	(1192910, 86811)	(2717121, 195407)
12	(64260.5, 59344.5)	(417458, 406974)
13	(3509313.5, 3504686.5)	(15032542, 15022458)
14	(522946, 199575)	(1673251, 1004290)
15	(405462, 84163)	(654767, 61515)
16	(2586300 , 1146300)	(9398650 , 3103650)

Table 5. The results of the efficiency assessment DMUs at different levels from 0 to 1 using model 4

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
h=0	1	1	0.663458	1	0.966844	1	1	0.315034	0.126230	1	0.873538	1	1	0.674814	0.887724	1
h=0.25	1	1	0.503655	1	0.569529	1	0.840939	0.308785	0.124188	0.660843	0.625269	1	1	0.523926	0.490153	1
h=0.5	1	1	0.443255	1	0.451468	1	0.664388	0.300625	0.122203	0.660843	0.499860	1	1	0.443114	0.283978	0.771579
h=0.75	1	1	0.442692	1	0.423166	1	0.637227	0.290351	0.119964	0.555094	0.424059	1	1	0.390391	0.207911	0.680348
h=1	1	1	0.441990	1	0.403900	1	0.619354	0.276778	0.117275	0.504653	0.373197	1	0.871880	0.362539	0.171411	0.643789

Table 6.The Optimal Efficiency and Possibility Levels Of 16 DMUs

DMUs	Optimal Efficiency	Optimal Possibility Level (α)
1	1	0.9999
2	1	0.6002
3	0.7816	0
4	1	0.9846
5	1	0.0010704
6	1	0.0357
7	1	0.0357
8	0.3754	$0.952 * 10^{-15}$
9	0.3404	0.7845
10	1	1
11	1	0
12	1	0.666
13	1	$0952*10^{-15}$
14	1	0
15	1	0.7781
16	1	0.0149

This study suggests that DMUs No 1, 2, 4, 6 and 12 are efficient in all levels. DMUs No 3, 5, 8, 9, 11, 14, 15 are inefficient in all levels. DMU No 7 in h=0 is efficient. However it is not efficient in other levels. There is Similarity in this regard DMUs No 7 and 10. Bank No 13 is efficient in all levels except h=1. DMU No 16 is efficient in h=0 and h=0.25 but it is inefficient at other levels. So, according to definition 6

$$\overline{Ef} = \left\{ (1,1),(2,1),(4,1),(6,1),(7,\circ),(10,\circ), (12,1),(13,0/75),(16,0/25) \right\}$$

It can be inferred from table 5 that the least amount of efficiency occurs in h=1. Efficiency assessed at this level equals traditional crisp model for assess efficiency. The most amount of efficiency occurs in h=0 and efficiency amount of all DMUs at other levels is located between 2 externals.

This study suggests that using presented fuzzy method, and by obtaining efficiency in different levels one can determine appropriate levels and sensitive DMUs. For example considering DMU 16 one can notice that h=0 and h=0.25 are appropriate levels for this DMU, because the efficiency degree at this levels is 1. But at other levels is inefficient. At some DMUs such as DMU No 13 one can observe that how the target unit at h=1 i.e. in assessing efficiency using traditional crisp model, is inefficient while at the other levels is efficient. This issue accentuates the efficiency assessment using a fuzzy method. It should be noted that efficiency assessment affects decision making of principals. Consequently, applying an accurate efficiency assessment is of vital. To this goal the method presented in this essay, which is a fuzzy data envelopment analysis (FDEA) model, can be an appropriate option [6]. Ultimately using genetic algorithm we have determined the optimal possibility level (α) and also specified the efficiency scores in these levels for each bank. Results are shown in table 6.

6. Comparison and conclusions

In this article in order to assess the efficiency of systems with imprecise data (fuzzy), a fuzzy mathematical programming approach based on the comparison of α-cut sets, using data envelopment analysis (DEA) models, is expressed. The ultimate goal of this article is determining the optimal possibility level (α) and also specifying the efficiency scores in these levels for the DMU_s, using genetic algorithm. Using this method the efficiency of a banking system in the US, including sixteen banks and financial institutes, over a period of two years i.e. 2003-2004 is assessed [6]. Each DMU's data during this two year period is in symmetric triangle fuzzy numbers and according to this ranking, which is based on comparison of α-cut sets, assessment of efficiency of DMUs in different levels is done. This study suggests that DMUs No 1, 2, 4, 6 and 12 are efficient in all levels. DMUs No 3, 5, 8, 9, 11, 14, 15 are inefficient in all levels. While other DMUs are efficient at some levels, they are inefficient at other levels. This assess shows that the least amount of efficiency occurs in h=1. Efficiency assessed at this level equals traditional crisp model for assess efficiency. This issue determines the significant of applying a fuzzy method in assessing efficiency. The results which have derived by intelligent method are the best α-cut levels, in the case that in the previous method the results were found with try and fault and it causes error increases and therefore the optimal levels doesn't derive and determined solutions don't be accurate, consequently the optimal solution for objective function doesn't derive[6], but taking account into the presented method, through specifying the optimal α, using genetic algorithm, the optimal and the most stable solution for DMU_o is derived.

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