GENERALIZED TOPOLOGICAL VECTOR SUBSPACES

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Abstract: In this paper, we will emphasize the Generalized topology [1] endowed upon the vector space. Whereas the addition and scalar multiplication of vector space, follows the principle of Generalized continuity [3]. The Generalized topology and Generalized continuity enhances the horizon of understanding and application by easier approach.

Key words: Generalized topology, generalized continuity, direct sum, vector subspaces, quotient vector space

Introduction

Á. Cs ász ár [1] introduced the notion of generalized topology. And later on, he gave the idea of generalized continuity [3]. We will apply these fundamental results on the vector spaces and subspaces over the field K (Real or Complex). This will be done by the application of generalized continuity over addition and scalar multiplication. Further more, we will give the example of generalized topological space which fails to be a topological vector space. Generalized topology will also be studied upon the subspaces and the quotient spaces by virtue of generalized continuity over addition and scalar multiplication. The significance of generalized topology over the usual topology is realized by broadening the research horizon as lesser conditions required to be satisfied.

1. Preliminaries

Definition 2.1.1 [1]**.** Let X be any non empty set, $\mu \subset \exp(X)$ is said to form generalized topology if the following conditions are satisfied:

(1). $\varphi \in \mu$

(2). Arbitrary union of elements of μ are closed.

The pair (X, μ) is called the generalized topological space. Since generalized topological space need not to be closed under the finite intersection. Note that every topological space is generalized topological space but converse is not generally true.

Examples 2.1.2.[4]. Let R be the real line and let Λ = {[a, b] : a, b ∈ R, a ≤ b} and μ be the collection of all union of members of Λ. It is trivial that μ is a GT on R.

Examples 2.1.3 Let X= {a,b,c} μ ={ φ , { a},{b},{a,b}, {a,c}, {b,c},{a,b,c}}

(i). $\varphi \in \mu$

(ii). Arbitrary union is closed in μ.

Therefore μ forms the generalized topology in spite of the fact that {a,c}∩{b,c}={c} ∉ μ.

Definition 1.1: [5] A vector space V over the field K (Real or Complex) along with a topology τ endowed on it which can be expresses as a pair (V , τ) . Moreover, the scalar multiplication and vector addition are continuous functions

$$'m' : V \times V \to V$$
$$'M' : K \times V \to V$$

Where $V \times V$ and $K \times V$ are given the respective product topologies. Thus $x, y \in X$ and $U \in \tau$ is an open neighborhood of $x + y$ there must be open neighborhoods U_x, $U_y \in \tau$ of x and y, respectively, such that $U_x + U_y \subseteq U$. Similarly, the continuity of scalar multiplication implies that for given $\alpha \in K$, $x \in X$ and any neighborhood W of (α, x), there is a neighborhood V of $\alpha \in K$ and a neighborhood $U \in \tau$ of x in such that V ×U ∈ W.A topological vector space is necessarily a topological group: the definition ensures that the group operation (vector addition) is continuous, and the inverse operation is the same as multiplication by −1, and so is also continuous.

Examples 1.2 [5]. (1). Any real or complex normed space is a topological vector space when equipped with the topology induced by the norm. (2). Any vector space is a topological vector space when equipped with the indiscrete topology. Of course, this will fail to be a separated topological vector space unless it is the zero-dimensional space {0}. (3). Most frequently given examples of topological vector space R^n, C^n . (4) C[0, 1] in the topology given by the metric d(f , g) = sup$_{t \in [0,1]}$|f (t) − g(t)|.

1.2 Generalized Topological Vector Space

In this section, we will discuss the generalized topological vector space. Since every topological space is generalized topological space but

the converse is not generally true. We will give some examples of generalized topological vector space which is not a topological vector space by introducing the generalized topology based upon the generalized semi open sets. We will further look into the structure of generalized topological vector space containing the additive abelian group and draw some nice results accordingly.

Definition 1.2.1[1] Let (X, G_1) and (X, G_2) are the given generalized topological spaces, f from (X, G_1) to (X, G_2) is (G_1, G_2)-continuous (briefly generalized continuous) if $f^{-1}(U) \in G_1$ for all $U \in G_2$.

Definition 1.2.2 [2] Now let $K \neq \emptyset$ be an index set. $X_k \neq \emptyset$ for $k \in K$, and $X = \prod_{k \in K} X_k$ the Cartesian product of the sets X_k. We denote the projection $p_k : X \longrightarrow X_k$. Supposing that the μ_k is the generalized topology over X_k. Now consider all the sets of the form $M_k \in \mu_k$ and with the exception of finite number of $k \in K$. Such that $M_k = Z_k = M_{\mu k}$. We denote β the base of collection of all these sets. We call μ as the product of topologies of μ_k and denote it by $P_{k \in K} \mu_k$. If each μ_k is a topology then clearly μ is the product topology of the factor μ_k.

Definition1.2.3 A vector space V over the field K (usually R or C) is said to be generalized topological vector space if V is endowed with the generalized topology μ. Then (V, μ) is called generalized topological vector space if the following two axioms are satisfied:
(i) $m : V \times V \longrightarrow V$ is a generalized continuous where m is defined by,
$$m(v_1, v_2) = v_1 + v_2 \quad \forall \, v_1, v_2 \in V.$$
(ii) $M : K \times V \longrightarrow V$ is a generalized continuous where M is defined by, $M(\alpha, v) = \alpha v \quad \forall \, v \in V \text{ and } \alpha \in K$.

Surely X will forms a discrete generalized topological space and therefore $X \times X$ will be product of Generalized topological space which is itself a Generalized topological space [2].

Every topological vector space is a generalized topological vector space where as generalized topological vector space is not topological vector space since the generalized continuity may not be continuum. In the following example we will discuss the generalized topology which is not a usual topology. In the next example we will give the example of generalized topological vector space which is not a topological space.

Example1.2.3. Let (X, ψ) be a generalized topological space. And $A \subseteq X$ is called generalized semi open in (X, ψ) if there exist a generalized open set O such that $O \subseteq A \subseteq cl(O)$ where $A \subset X$ and $O \in \psi$. The notion of Generalized topological space and generalized semi open sets were introduced

by Császár [3].

Corollary 1.2.4. (a). Every open set is semi-open. (b).Union of any number of semi open set is semi open. (c). Intersection of two or more semi open sets need not to be a semi open.

Example 1.2.3. Let X be the vector space and any collection of semi open subsets of X satisfies the condition of semi continuity overaddition and scalar multiplication[6].

2.0 GENERALIZED TOPOLOGICAL VECTOR SUBSPACES

Definition 2.1. Let V be the vector space over the field K (R or C) and S is the subset of V such that S itself satisfies to be vector space over K with respect to addition and scalar multiplication.

Definition 2.2. A vector subspace of the vector space V over K is said to be the Generalized topological vector subspace of Generalized topological vector space if V endows the Generalized topology (like Def. 1.2.2) and U inherits the Generalized topology (i.e., by relative generalized topology, induced generalized topology or trace generalized topology).

Theorem 2.3. In a generalized topological vector subspace, the closure of generalized topological vector subspace is a generalized topological subspace.

Proof. Let U be the generalized topological vector subspace of generalized topological vector space V. According to Definition 1.2.2, the addition and scalar multiplication's generalized continuity will be satisfied by the U, i.e.,
$$\text{'}m\text{'} : U \times U \longrightarrow U$$
$$\text{'}M\text{'} : R \times U \longrightarrow U$$
And for closure of U, we can stat that:
$$cl(U \times U) = cl(U) \times cl(U) \longrightarrow cl(U)$$

Similarly we can express the closure over the scalar multiplication as, $cl(R \times U) = cl(R) \times cl(U) \longrightarrow cl(U)$ this completes the proof.

Theorem 2.4. Every proper generalized topological vector subspace U of generalized topological vector space V has empty interior.

Proof. Let $x \in \text{int}(U)$, means that there exists g-open set O such that $x \in O \subseteq U$. Since $x \in U$ and $U - \{x\}$ is contained in 'V' that contains the origin. For $x \in V$, we consider $f_x : K \longrightarrow K \times V$ defined by $f_x(\lambda) = K\lambda$. Since the generalized continuity of f_x is confirmed by the fact that K is itself a generalized topological vector space. As the generalized continuity has been established, we will proceed by considering that for

any vector $v \in V$ we will define f_v in the same way as f_x. Then $f_v^{-1}(U - \{x\})$ is g-open set. Thus making the value $\lambda \in f_v^{-1}(U - \{x\})$, we see that $\lambda v \in U - \{x\}$, i.e, $U - \{x\}$ is contained in V and $\lambda v \in V$ therefore $v \in V$. This proves that $V = U$ i.e, U is not proper and which is the reason that interior of U is empty.

Theorem 2.5. The product of family of generalized topological vector subspaces is a generalized topological vector subspace under the point wise algebraic operations and the product generalized topology.

Proof. Let $\{(X_i, \mu_i) : i \in I\}$ be the family of generalized topological subspaces and let $X = \prod_{i \in I} X_i$ and $\mu = \prod_{i \in I} \mu_i$. We will show only that addition and multiplication on X is continuous. Let $(x_i^f) \xrightarrow[f]{\mu} (x_i)$ and $(y_i^g) \xrightarrow[g]{\mu} (y_i)$ in X. Then $x_i^f \xrightarrow[f]{\mu} x_i$ and $y_i^g \xrightarrow[g]{\mu} y_i$ in X_i for each i, so also $x_i^f + y_i^g \xrightarrow[f,g]{\mu_i} x_i + y_i$ in X_i for each 'i'. Since the product generalized topology of point wise convergence, we see that

$$(x_i^f) + (y_i^g) \, \Box \, (x_i^f + y_i^g) \xrightarrow[f,g]{\mu} (x_i + y_i) = (x_i) + (y_i)$$

Defininition 2.6. For a subset K of X, let $\hat{K}_f = \{x \in X : f_K(x) = f(x)\}$. Now we have the following characterizations of a f-proximinal subspace K of X.

Theorem 2.7. *If K is a subspace of X and f is e real function, then K is f-proximinal if and only if*

$X = K + \hat{K}_f$.

Proof. If K is proximinal and $x \in X$, then there exists $k_0 \in K$ such that $f_K(x) = f(x - k_0)$. Hence $x - k_0 \in \hat{K} f$. Thus $x = k_0 + (x - k_0) \in K + \hat{K}_f$. Therefore $X = K + \hat{K}_f$. For converse let $x \in X$. Then there exists $k_1 \in K$ and $k_2 \in \hat{K} f$ such that $x = k_1 + k_2$. Thus $x - k_1 = k_2 \in \hat{K}_f$. Hence $f_K(x - k_1) = f(x - k_1)$. Since K is subspace $f(x - k_1) = f_K(x)$. Thus $k_1 \in Pf.K(x)$ and K is f-proximinal.

Definition 2.8 A subset K of a topological space X is called f-closed if for all sequence $\{k_n\}$ of K and $x \in X$ such that $f(x - k_n) \to 0$, we have $x \in K$.

2.0 Quotient Generalized Topological Vector Space

Let M be the Generalized topological vector subspace of Generalized topological vector space V over the field K (R or C). Then the quotient generalized topological vector space is defined as V/M over K (R or C) which satisfies generalized continuity with respect to addition and scalar multiplication. That is

'+' : $V/M \times V/M \to V/M$

'.' : $R \times V/M \to V/M$

For understanding the structure of generalized topological vector subspace, we should take a look at the structure of cosets. $V + M = \{v + m : m \in M\}$ is called the cosets of generalized topological vector subspace M over generalized topological space V. "The collection of all cosets for the structure of quotient generalized topological vector space." Therefore for $a, b \in V$

$$\Rightarrow a + M, b + M \in V/M \qquad \text{and}$$

$$\Rightarrow (a + M) + (b + M) = (a + b) + M \qquad \text{and}$$

$$\alpha(a + M) = \alpha a + M.$$

Corollary 3.1: Let M be the closed subspace of V then quotient map $f : V \to V/M$ is generalized open, that is, carries g-open set to g-open set.

Proof. Let U be g-open set in V. then

$$f^{-1}(f(U)) = f^{-1}(U + M) = U + M = U + \bigcup_{m \in M} m$$

Whereas union of g-open set is g-open.

Definition 3.2. Let M and N are Generalized topological vector subspaces of Generalized topological vector space V such that $M \cap N = \{0\}$. The direct sum of M and N is the set of vector $M \oplus N = \{m + n : m \in M \text{ and } n \in N\}$.

Theorem 3.3. Let V be a generalized topological vector space and W be generalized topological subspace. Then V is isomorphic to $W \oplus V/W$.

Proof. Let β be the base of V/W, and $f : \beta \to V$ such that $f(b) \in V$ holds for $b \in \beta$. We will show that $\{f(b) : b \in \beta\}$ is linearly independent.

Let's define $g : V/W \to V$ by

$$g\left(\sum_n \lambda_n b\right) = \sum_n \lambda_n f(b)$$

Consider,

$T : W \oplus V/W \to V$ defined by $T(w, x) = w + g(x)$. Obviously T is linearly. Let's show T is injective. This is to show that $T(w,x) = 0$ implies $w = 0$ and $x = 0$.

Let $x = \sum_n \lambda_n f(b)$, $0 = T(w,x)$ implies that

$g(x) \in W$.

Hence, $0 = x = \sum_n \lambda_n f(b) = \sum_n \lambda_n b = x$.   Thus

$x=0$, $g(x)=0$ and $w=0$. Let $y \in V$, and $[y] = \sum_n \lambda_n b$

then we have, $= [y] - \sum_n \lambda_n f(b) = [y - \sum_n \lambda_n f(b)] = [y] -$

$\sum_n \lambda_n b = 0$.

Thus $y \square \sum_n \lambda_n f(b)$, we get

$T(y - \sum_n \lambda_n f(b)),[y]) = y$.   Hence   T   is

injective.

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