Three-step Method for Finding Multiple Root of Non-linear Equation

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Abstract In this paper, we present a three-step method of tenth order for determining multiple roots of a nonlinear equation. This method requires six evaluations per iteration, three evaluations of function and three evaluations of first derivative. Numerical examples are tested to compare our method with some other methods in the literature. We observe that our method is either comparable or has better performance in some cases than the other methods and preserves the order of convergence.

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1. Introduction

One of the most important and interesting problem in the field of numerical analysis, engineering, optimization theory and applied mathematics is determining the multiple roots of a nonlinear equation. A function *g(x)* is said to have a multiple root of a non-linear equation

$$g(x)=0\,, \tag{1.1}$$

iff *g(x)* can be written as

$$g(x)=(x-t)^m h(x)\,,$$

where $\lim_{x\to t} h(t)\neq 0$ and m is the known multiplicity.

The modified Newton's method (E.Schroder, 1870) is the most popular method for finding multiple roots of a nonlinear equation and is given as :

$$x_{n+1}=x_n-m\frac{g(x_n)}{g(x_n)}. \tag{1.2}$$

This method has second order of convergence. In recent years, many variants of modified Newton's method (N. A. Mir and Naila Rafiq 2007 and Li, et al 2010,) have been developed and analyzed by using various techniques, which have either equal or improved order of convergence, efficiency and show better performance than the Newton's methods

2. The Iterative Method

Thukral (2010) proposed a third order convergent method for finding multiple roots of non-linear equation:

$$\begin{cases} y_n=x_n-\frac{g(x_n)}{g(x_n)} \\ x_{n+1}=x_n-\left(\frac{ug(x_n)^2}{ug(x_n)g(x_n)-vg(x_n)g(y_n)}\right), \\ \text{where } u=(m-1)^{m-1},\ v=m^{m-1},\ n\in N. \end{cases} \tag{2.1}$$

Li, et. al. (2010) developed fourth order method given as:

$$\begin{cases} y_n=x_n-\left(\frac{2m}{m+2}\right)\frac{g(x_n)}{g(x_n)}, \\ z_n=x_n-2\left(\frac{m}{m+2}\right)^m\frac{g(x_n)}{g(y_n)}\,, \\ x_{n+1}=x_n-\left(\frac{2m}{m+2}\right)\left(\frac{g(x_n)}{a_1g(x_n)+a_2g(y_n)+a_3g(z_n)}\right), \end{cases} \tag{2.2}$$

where

$$a_1=\frac{m^6-m^5-14m^4+12m^3+48m^2-80m+32}{8m(m^3+2m^2-8m+4)},$$

$$a_2=\left(\frac{-m}{16}\right)\left(\frac{3m^4-6m^3-20m^2+40m-16}{a_4(m^3+2m^2-8m+4)}\right),$$

$$a_3=\left(\frac{1}{16}\right)\left(\frac{m^3(m^2-4)}{a_4(m^3+2m^2-8m+4)}\right),$$

$$a_4=\left(\frac{m}{m=2}\right)^m.$$

Thukral (2012) proposed a fifth order covergent method:

$$\begin{cases} y_n=x_n-m\frac{g(x_n)}{g(x_n)}\,, \\ x_{n+1}=y_n-m\left(1+z^{\frac{2}{m}}\right)\frac{g(y_n)}{g(y_n)}\,, \end{cases} \tag{2.3}$$

where $z=g(y_n)g(x_n)^{-1}$.

Thukral (2010) proposed a three-step method:

$$\begin{cases} y_n=x_n-m\frac{g(x_n)}{g(x_n)}, \\ z_n=y_n-mp^s\left(\frac{1+bp^s}{1+(b-2)p^s}\right)\frac{g(x_n)}{g(x_n)}, \\ x_{n+1}=z_n-m\left(u(p^s)+\left(\frac{g(z_n)}{g(y_n)-ag(z_n)}\right)^s+4\left(\frac{g(z_n)}{g(x_n)}\right)^s\left(\frac{g(x_n)}{g(x_n)}\right)\left(\frac{g(z_n)}{g(x_n)}\right)^s\,, \end{cases} \tag{2.4}$$

where

$$u_1(p^s)=1+2p^s+(5-2b)p^{2s}+(12-12b+2b^2)p^{3s},$$

$$u_2(p^s)=\frac{5-2b-(2-8b+2b^2)p^s+(1+4b)p^{2s}}{5-2b-(12-12b+2b^2)p^s},$$

$$p=\frac{g(y_n)}{g(x_n)}.$$

This method has eighth order of convergence. We suggest the following three-step method for finding multiple roots which is the modification of the method (2.3) for multiple roots and is given as:

$$\begin{cases} y_n = x_n - m\frac{g(x_n)}{g'(x_n)}\ , \\ z_n = y_n - m(1+a^{\frac{2}{m}})\frac{g(y_n)}{g'(y_n)}\ , \\ x_{n+1} = z_n - m(1+b^{\frac{2}{m}})\frac{g(z_n)}{g'(z_n)}\ , \end{cases} \qquad (2.5)$$

where $a = g(y_n)g(x_n)^{-1}$ and $b = g(z_n)g(y_n)^{-1}$.

3. Convergence Analysis

Now, we prove a theorem for finding the order of convergence of method (2.5).

Theorem: *Let t be the multiple root of a nonlinear equation (1.1) with multiplicity m , where the function g(x) is sufficiently differentiable in an open interval I=(a,b). Let* x_0 *be an initial guess sufficiently close to t . Then the method (2.5) has tenth order of convergence and its error equation is given by:*

$$e_{n+1} = \left(\frac{c_2^9}{m^7} + 4\frac{c_2^5c_3^2}{m^7} + 6\frac{c_2^9}{m^8} - 12\frac{c_2^7c_3}{m^8} - 4\frac{c_2^7c_3}{m^7} + 9\frac{c_2^9}{m^9}\right)e_n^{10} + o(e_n^{11}).$$

Proof:

Let $e_n = x_n - t.$ Using the Taylor expansion of $g(x_n), g'(x_n)$ about t , we get:

$$g(x_n) = \left(\frac{g^{(m)}(t)}{m}\right)e_n^m\left[1 + c_1e_n + c_2e_n^2 + ...\right],$$

$$g'(x_n) = \left(\frac{g^{(m)}(t)}{m-1}\right)e_n^{m-1}\left[1 + \left(\frac{m+1}{m}\right)c_1e_n + \left(\frac{m+2}{m}\right)c_2e_n^2 + ...\right],$$

where $c_k = \frac{mg^{(m+k)}(t)}{(m+k)g^{(m)}(t)}, \quad k = 1, 2, ...$.

Putting the values of $g(x_n)$ and $g'(x_n)$ in (2.5) and after simplification, we have:

$$y_n = t + \frac{c_2}{m}e_n^2 + o(e_n^3).$$

Also

$$g(y_n) = 1 + \frac{c_2^2e_n^2}{m} + \left(2\frac{c_2c_3}{m} - \frac{c_2^3}{m^2} - \frac{c_2^3}{m}\right)e_n^3 + o(e_n^4)\ ,$$

$$g(y_n) = 1 + (\frac{1}{m^2}c_2^2 + \frac{c_2^2}{m})e_n^2 + \left(2\frac{c_2^3}{m^2} + 2\frac{c_2c_3}{m^2} + 2\frac{c_2c_3}{m}\right)e_n^3 + o(e_n^4).$$

Now, putting the values of y_n, $g(y_n)$ and $g'(y_n)$ in (2.5), we have:

$$z_n = t + \left(3\frac{c_2^4}{m^4} - 2\frac{c_2^2c_3}{m^3} + \frac{c_2^4}{m^3}\right)e_n^5 + o(e_n^6)\ .$$

Again, by Taylor expansion, we get:

$$g(z_n) = 1 + \left(3\frac{c_2^5}{m^4} - 2\frac{c_2^3c_3}{m^3} + \frac{c_2^5}{m^3}\right)e_n^5 + o(e_n^6),$$

$$g'(z_n) = \left(3\frac{c_2^4}{m^4} - 2\frac{c_2^2c_3}{m^3} + \frac{c_2^4}{m^3}\right)e_n^5 + \left(11\frac{c_2^3c_3}{m^3} - 11\frac{c_2^5}{m^4} - 8\frac{c_2c_3^2}{m^3} - 19\frac{c_2^5}{m^5} - 3\frac{c_2^2c_4}{m^3} - 3\frac{c_2^5}{m^3} + 28\frac{c_2^3c_3}{m^4}\right)e_n^6 + o(e_n^7)$$

Finally, putting the values of z_n, $g(z_n)$ and $g'(z_n)$ in (2.5) and after simplification, we have

$$x_{n+1} = t + \left(\frac{c_2^9}{m^7} + 4\frac{c_2^5c_3^2}{m^7} + 6\frac{c_2^9}{m^8} - 12\frac{c_2^7c_3}{m^8} - 4\frac{c_2^7c_3}{m^7} + 9\frac{c_2^9}{m^9}\right)e_n^{10} + o(e_n^{11})\ ,$$

or $e_{n+1} = \left(\frac{c_2^9}{m^7} + 4\frac{c_2^5c_3^2}{m^7} + 6\frac{c_2^9}{m^8} - 12\frac{c_2^7c_3}{m^8} - 4\frac{c_2^7c_3}{m^7} + 9\frac{c_2^9}{m^9}\right)e_n^{10} + o(e_n^{11})\ ,$

which shows that the method defined in (2.5) has tenth order of convergence.

4. Comparison of Method(2.5)

Here, we take some numerical examples from Thukral (2012) and compare our method with some other methods in the literature. The test functions, their roots and their initial guesses are displayed in Table 1, while Table 2 shows the absolute error $|e_n| = |x_n - t|$ for each method. Each method is compared on fourth iteration taking the following stopping criterion:

$$|x_n - t| < \varepsilon, \text{ where } \varepsilon = 10^{-300}.$$

All the computations reported here are done using 1000 significant digits. MNM represents Modified Newton's Method, BNM represents Li's method of fourth order given in (2.2) and NM represents the new method defined in (2.5) .We can observe that the present method is well behaved.

Table 1. Test functions, their roots, and their initial guesses

Functions	m	x_0	Roots
$g_1(x) = ((x-1)^{10} - 1)^4$	4	0.01	0
$g_2(x) = (e^x + x - 20)^3$	3	3.0	2.842438
$g_3(x) = \left(x^2 - e^x - 3x + 2\right)^{12}$	12	0.2	0.257530
$g_4(x) = (\mathrm{Sin}^2 x - x^2 + 1)^8$	8	1.5	1.404491
$g_5(x) = (Cos x + x)^5$	5	-1	-0.739085
$g_6(x) = (\tan x - e^x - 1)^5$	5	1.4	1.371045

Table 2. Comparison with various methods

g	$\lvert e_n\rvert = \lvert x_n - t\rvert$				
	NM	Thukral5	BNM	Thukra3	MNM
g_1	$0.955e-614$	0.577e-153	$0.453e-86$	0.193e-41	0.432e-11
g_2	0.561e-531	0.591e-130	0.310e-72	0.151e-35	0.570e-2
g_3	0.254e-586	0.207e-252	0.356e-143	0.340e-51	0.933e-17
g_4	0.734e-525	0.103e-127	0.182e-73	0.117e-31	0.205e-2
g_5	0.128e-569	0.515e-137	0.180e-82	0.341e-29	0.120e-3
g_6	0.126e-370	0.453e-90	0.764e-51	0.133e-30	0.882e-7

Now we take few numerical examples from (Thukral,2012) and will compare new method (2.5) with method Thukral 8 defined in (Thukral, 2012).

Table 3. Test functions, initial guesses and roots

Functions	x_0	Roots
$g_1 = (e^x + x - 20)^{95}$	3.5	2.842438
$g_2 = (\sin x^2 - x^2 + 1)^{200}$	1.7	1.404491
$g_3 = (\tan x - e^x - 1)^{11}$	1.4	1.371045
$g_4 = (xe^{x^2} - \sin x^2 + 3\cos x + 5)^4$	-1.2	-1.207647

Table 4. Comparison of Method NM

Functions	$\lvert e_n\rvert = \lvert x_n - t\rvert$	
	NM	Thukral
g_1	0.100*e* - 998	0.273*e* - 291
g_2	0.133*e* - 324	0.214*e* - 194
g_3	0.125*e* - 370	0.128*e* - 513
g_4	0.770*e* - 930	0.946*e* - 201

Conclusions

In this paper, we proposed a new method, namely NM (2.5) for finding multiple roots of a nonlinear equation. We discussed the convergence analysis of new method and proved that this method has tenth order. At the end, we compared our method with some other methods. We observed that our method is comparable with other methods or has better performance than the other existing methods given in the Tables 2 and 4.

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