On sufficient condition for Sakaguchi type spiral-like functions of order $\beta$

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ABSTRACT: In our present investigation, motivated from Goyal and Goswami work, we obtain a sufficient condition for Sakaguchi type spiral-like function of order $\beta$. Some interesting consequences of our main result are also given. [Arif M., Khan W.A, Ayaz M, Islam S. On sufficient condition for Sakaguchi type spiral-like functions of order $\beta$. Life Sci J 2013;10(5s):253-253] (ISSN:1097-8135). http://www.lifesciencesite.com. 45

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1. INTRODUCTION

Let the class of all functions $f(z) = z + a_{n+1}z^{n+1} + \cdots$ which are analytic in $E = \{z: |z| < 1\}$ be denoted by $A_n$ and let $A_1 = A$. A function $f(z) \in A_n$ is said to be in the class $S_n^\beta(n, \beta, t)$, if

$$\text{Re} \ e^{i\lambda} \left(1 - t\right)zf'(z) > \beta \cos \lambda, \ t \in [-1, 1),$$

for all $z \in E$, $0 \leq \beta < 1$ and $\lambda$ is real with $|\lambda| < \frac{\pi}{2}$.

For $\lambda = 0$, this class reduces to $S_n^\beta(\beta, -1)$ (see, [6]) and for $n = 1, \lambda = 0$, we obtain the class $S(\beta, t)$ studied by Owa et.al [9] and Goyal et.al [5]. The class $S_n^\beta(1, 0, -1)$ was introduced by Sakaguchi [11].

Therefore, a function $f(z) \in S_n^\beta(1, \beta, -1)$ is called Sakaguchi functions of order $\beta$ (see, [4]). Also we note that for $n = 1, t = 0, \beta = 0$, the class $S_n^\beta(n, \beta, t)$ reduces to the class of spiral-like functions introduced by Spaczek [12] in 1933.

Sufficient conditions for different classes were studied by various authors, see [1,2,3].

In this paper, we obtain a sufficient condition for a function $f(z) \in S_n^\beta(n, \beta, t)$. To prove our main result, we need the following Lemma proved in [8].

**Lemma 1.1.** Let $\Omega$ be a set in the complex plane $\mathbb{C}$ and suppose that $\phi$ is a mapping from $\mathbb{C}^2 \times E$ to $\mathbb{C}$ which satisfies $\phi(ix, y; z) \notin \Omega$ for $z \in E$, and for all real $x, y$ such that $y \leq -n(1 + x^2)/2$. If $p(z) = 1 + c_nz^n + \cdots$ is analytic in $E$ and $\phi(p(z),zp'(z); z) \notin \Omega$ for all $z \in E$, then $p(z) > 0$.

2. MAIN RESULTS

**Theorem 2.1.** If $f(z) \in A_n$, satisfies

$$\text{Re} \left(1 - t\right)zf'(z) > \frac{M}{2}\left(\frac{M^2}{4L} + N\right),$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta < 1$, $t \in [-1, 1)$, $\lambda$ is real with $|\lambda| < \frac{\pi}{2}$ and $L = \alpha(1 - \beta)\left(\frac{M}{2}\left(1 - t\right) + (1 - \beta)\cos \lambda\right)\cos \lambda$ (2.1)

$$M = -(1 - \beta)^2\sin 2\lambda\cos \lambda$$

$$N = \alpha\beta\left((\beta + 2\alpha \sin^2 \lambda)\cos^2 \lambda + \frac{\pi}{2} - \cos \lambda\right)(1 - t) - \frac{\alpha \sin 2\lambda}{2} + \left(\beta \cos \lambda - \frac{\pi}{2}\right)(1 - t),$$

then $f(z) \in S_n^\beta(n, \beta, t)$.

**Proof.** Set

$$e^{i\lambda} \left(1 - t\right)zf'(z) = q(z) = \cos \lambda \left[(1 - \beta)p(z) + \beta\right] + i \sin \lambda.$$

Then $p(z)$ and $q(z)$ are analytic in $E$ with $p(0) = 1$ and $q(0) = 1$.

Taking logarithmic differentiation of (2.2), we have

$$zf''(z) + tzf'(z)$$

$$f'(z) f(z) - f'(tz)$$

$$= (1 - t)q' + e^{-i\lambda}q(z) - (1 - t)q(z)$$

and hence

$$e^{i\lambda} \left(1 - t\right)zf'(z) \left(\frac{f''(z)}{f'(z)} + \frac{atf'(z)}{f'(z) - f(tz)}\right)$$

$$= \frac{A(zp'(z) + Bp^2(z) + Cp(z) + D)}{\phi(p(z),zp'(z); z)},$$

with

$$A = \alpha(1 - \beta)\cos \lambda,$$

$$B = \alpha\beta e^{-i\lambda}\cos^2 \lambda,$$

$$C = (1 - \beta)(2\alpha\beta e^{-i\lambda}\cos^2 \lambda + i\alpha e^{-i\lambda}\sin 2\lambda) + (1 - \alpha)(1 - t)\cos \lambda,$$

$$D = \alpha\beta^2 \cos^2 \lambda - \sin^2 \lambda + i\beta \sin 2\lambda + (1 - t)(1 - \alpha)(\beta \cos \lambda + i \sin \lambda).$$

Now
\[ \phi(r, s; t) = As + Br^2 + Cr + D. \]

For all real \( x \) and \( y \) satisfying \( y \leq -n(1 + x^2)/2 \), we have
\[ \phi(ix, y; z) = Ay - Bx^2 + iCx + D. \]

Taking real part on both sides and then by simple computation, we obtain
\[
\text{Re} \left( \phi(ix, y; z) \right) \leq -Lx^2 + Mx + N
\]
\[ = -\left[ \sqrt{Lx + \frac{M}{2\sqrt{L}}} \right]^2 + \frac{M^2}{4L} + N
\]
\[ < \frac{M^2}{4L} + N, \]

where \( L, M \) and \( N \) are given by (2.1).

Let \( \Omega = \left\{ \omega; \text{Re} \omega \geq \frac{M^2}{4L} + N \right\}. \) Then
\[ \phi(p(z), z'; z) \in \Omega \text{ and } \phi((x, y; z) \notin \Omega, \text{ for all real } x \text{ and } y \leq -n(1 + x^2)/2, z \in E. \]

Now by using Lemma 1.1, we obtain the required result.

On taking \( \lambda = 0 \), in Theorem 2.1, we have the following result proved in [7].

**Corollary 2.2.** If \( f(z) \in A_{n1} \), satisfies
\[
\text{Re} \left( \frac{(1 - t)^{\frac{1}{2}}zf''(z)}{f(z) - f(tz)} \right) > 0
\]
\[ \left( \frac{\alpha zf'(x)}{f'(x)} + \frac{\alpha zf'(tx)}{f'(tx)} + 1 \right)
\]
where \( 0 \leq \alpha \leq 1, 0 \leq \beta < 1, t \in [-1, 1] \) and
\[ \xi_1 = \alpha \beta \left( N(1 - t) - (1 - t) + \beta \right)
\]
\[ + (1 - t) \left( \beta - \frac{n\alpha}{2} \right), \]

Then \( f(z) \in S_1^2(n, \beta, t). \)

If we put \( t = 0, n = 1 \) and \( \alpha = 0 \) in Theorem 2.1, we obtain

**Corollary 2.3.** If \( f(z) \in A \), satisfies
\[ \text{Re} \left( e^{i\alpha} \frac{zf'(z)}{f(z)} \right) > \beta \cos \lambda \]
then \( f(z) \in S_1^2(1, \beta, 0). \)

If we take \( \lambda = 0, \ t = -1 \) and \( \beta = 0 \) in Theorem 2.1, we get

**Corollary 2.4.** If \( f(z) \in A_{n1} \), satisfies
\[ \text{Re} \left( \frac{zf''(z)}{f(z) - f(-z)} \right) > -\frac{n\alpha}{4}
\]
then \( f(z) \in S_1^2(n, 0, -1). \)

For \( t = 0 = \lambda, \) in Theorem 2.1, we have the following result proved in [10].

**Corollary 2.5.** If \( f(z) \in A_{n1} \), satisfies
\[ \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > \alpha \beta \left( \beta + \frac{n}{2} - 1 \right) + \left\{ \beta - \frac{n\alpha}{2} \right\}, \]

Then \( f(z) \in S_1^2(n, \beta, 0). \)

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**References**


