Stability Of The Generalized 2-Variable Quadratic Functional Equation

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Abstract: In this paper, we derive the stability of the 2-variable quadratic functional equation (0.01) \( (x + y, z + t) + f(x + \sigma(y), z + \sigma(t)) = 2f(x, z) + 2f(y, t) \). and the stability of the 2-variable quadratic functional equation (0.02) \( f(x + y, z + t) + g(x + \sigma(y), z + \sigma(t)) = h(x, z) + k(y, t) \) for all \( x, y, z, t \in G \), where \( G \) is a semigroup and \( \sigma \) is a homomorphism of \( G \) such that \( \sigma \circ \sigma = I \).

1. INTRODUCTION AND PRELIMINARIES
Let \( X, Y \) be real vector spaces. For the mapping \( f : X \times X \rightarrow Y \), consider the 2-variable quadratic functional equation
\[
(1.0.3) \quad f(x + y, z + t) + f(x + \sigma(y), z + \sigma(t)) = 2f(x, z) + 2f(y, t).
\]
we define
\[
(1.0.4) \quad Df(x, y, z, t) := f(x + y, z + t) + f(x + \sigma(y), z + \sigma(t)) - 2f(x, z) - 2f(y, t).
\]
\[
D_{\mu_1,\mu_2}f(x, y, z, t) := f(\mu_1(x + y), \mu_3(z + t)) + f(\mu_4(x + \sigma(y)), \mu_5(z + \sigma(t))) - 2\mu_1\mu_2f(x, z) - 2\mu_3\mu_5f(y, t).
\]
for all \( \mu_1, \mu_2 \in T^1 := \{ \lambda \in C : |\lambda| = 1 \} \) and all \( x, y, z, t \in X \).

Let \( X \) be a set. A function \( d : X \times X \rightarrow [0, \infty) \) is called a generalized metric on \( X \) if and only if \( d \) satisfies
(\mu_1) \( d(x, y) = 0 \) if and only if \( x = y \)
(\mu_2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \)
(\mu_3) \( d(x, z) \leq d(x, y) + d(y, z) \)

Theorem 1.0.1 Let \((X,d)\) be a generalized complete metric space. Assume that \( A : X \rightarrow X \) is a strictly contracting operator with the lipstick constant < 1. If there exists a nonnegative integer \( n \) such that \( d(A^{k+1}f, A^k f) < \infty \) for some \( f \in X \), then the following are true.
(a) The sequence \( \{A^nf\} \) converges to a fixed point \( F \) of \( A \);
(b) \( F \) is the unique fixed point of \( A \) in
\[
(1.0.5) \quad X^* = \{ g \in X : d(A^k f, g) < \infty \};
\]
(c) If \( h \in X^* \), then
\[
(1.0.6) \quad d(h, F) \leq \frac{1}{1-\lambda} d(h, h).
\]

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2. MAIN RESULT

Theorem 2.0.2 Let \( \sigma \) be an homomorphism of the semigroup \( G \) such that \( \sigma \circ \sigma = I \) and \( Y \) is a Banach space. Suppose that \( f : G \times G \rightarrow Y \) satisfies the inequality
\[
(2.0.7) \quad \|D_{\mu_1,\mu_2}f(x, y, z, t)\| \leq \delta
\]
for all \( \mu_1, \mu_2 \in T^1 \) and for all \( x, y, z, t \in G \) and for some \( \delta \in [0, \infty) \). Then there exists a unique 2-variable quadratic mapping \( F : G \times G \rightarrow Y \) such that
\[
(2.0.8) \quad \|f(x, z) - F(x, z)\| \leq \delta
\]
for all \( x, y, z, t \in G \).

**Proof.** Letting \( \mu_x, \mu_z = 1 \) and for all \( x, y, z, t \in G \), we have

\[
\| f(2x, 2z) + f(x + \sigma(x), z + \sigma(z)) - 4f(x, z) \| \leq \delta.
\]

for all \( x, y, z, t \in G \). Then we obtain

\[
\| \frac{f(2x, 2z) + f(x + \sigma(x), z + \sigma(z))}{4} - f(x, z) \| \leq \delta.
\]

for all \( x, y, z, t \in G \). Now we set \( X = \{ h \mid h : G \times G \to Y \text{ is a function} \} \) and introduce a generalized metric on \( X \) as follows:

\[
d(g, h) = \inf \{ \delta \in [0, \infty) \mid \| g(x, y) - h(x, y) \| \leq \delta \}.
\]

First, we will verify that \((X, d)\) is a complete space. Let \( \{ g_n \} \) be a Cauchy sequence in \((X, d)\). According to definition Cauchy sequence, for any \( \varepsilon > 0 \) there exists a positive integer \( N_\varepsilon \) such that \( d(g_m, g_n) \leq \varepsilon \) for all \( m, n \geq N_\varepsilon \). From the definition of the generalized metric \( d \), it follows that

\[
\forall \varepsilon > 0 \exists N_\varepsilon \in N \forall m, n \geq N_\varepsilon \| g_m(x, y) - g_n(x, y) \| \leq \varepsilon.
\]

This implies that \( \{ g_n(x, y) \} \) is a Cauchy sequence in \( Y \). Since \( Y \) is a complete space, \( \{ g_n(x, y) \} \) converges in \( Y \) for each \( x, y \in G \). Hence we can define a function \( g : G \times G \to Y \) by

\[
g(x, y) = \lim_{n \to \infty} g_n(x).
\]

If we let \( m \) increase to infinity, it follows from (2.0.13) that for any \( \varepsilon > 0 \), there exists a positive integer \( N_\varepsilon \) with \( \| g_n(x, y) - g(x, y) \| \leq \varepsilon \) for all \( n \geq N_\varepsilon \), that is, for any \( \varepsilon > 0 \), there exists a positive integer \( N_\varepsilon \) such that \( d(g_m, g_n) \leq \varepsilon \) for any \( m, n \geq N_\varepsilon \). This fact leads us to the conclusion that \( \{ g_n \} \) converges in \((X, d)\). Hence \((X, d)\) is a complete space. Now we define an operator \( A : X \to X \) such that

\[
(Af)(x, z) := \frac{f(2x, 2z) + f(x + \sigma(x), z + \sigma(z))}{4}.
\]

We assert that \( A \) is strictly contractive on \( X \). Given \( h \in X \), let \( \delta \in [0, \infty) \) be an arbitrary constant with \( d(g, h) \leq \delta \), that is,

\[
\| g(x, y) - h(x, y) \| \leq \delta.
\]

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It then follows from (2.0.15) that

\[
\| (Ag)(x, z) - (Ah)(x, z) \| = \| \frac{g(2x, 2z) + g(x + \sigma(x), z + \sigma(z))}{4} - \frac{h(2x, 2z) + h(x + \sigma(x), z + \sigma(z))}{4} \|
\]

\[
\leq \| \frac{g(2x, 2z)}{4} - \frac{h(2x, 2z)}{4} \| + \| \frac{g(x + \sigma(x), z + \sigma(z))}{4} - \frac{h(x + \sigma(x), z + \sigma(z))}{4} \|
\]

\[
\leq \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2} + \frac{\delta}{2} = \frac{\delta}{2} + \frac{\delta}{2} \leq \frac{\delta}{2} \leq \| g(x, z) - h(x, z) \|
\]

That is, \( d(Ag, Ah) \leq \frac{\delta}{2} d(g, h) \), for any \( g, h \in X \). Hence \( A \) is a strictly contractive function. It easily follows that

\[
(A^2 f)(x, z) = \frac{f(2x, 2z) + 2f(x + \sigma(x), z + \sigma(z))}{2^{2x}}.
\]

And by direct computation, we obtain

\[
(A^n f)(x, z) = \frac{f(2^n x, 2^n z) + (2^n - 1)f(2^{n-1} x + 2^{n-1} z + 2^{n-1} \sigma(x), 2^{n-1} z + 2^{n-1} \sigma(z))}{2^{2^n}}.
\]

Now we obtain
\[
\| (A^{n+1}f)(x, z) - (A^n f)(x, z) \| \leq \frac{1}{2^{2(n+1)}} \| f(2^{n+1}x, 2^{n+1}z) \|
+ \frac{1}{2^{2(n+1)}} \| 2(2^n - 1) f(2^n x + 2^n \sigma(x), 2^n z + 2^n \sigma(z)) \| \\
\leq \frac{\delta}{2^{2(n+1)}} + \frac{(2^n - 1)\delta}{2^{2(n+1)}} = \frac{2^n \delta}{2^{2(n+1)}} = \frac{\delta}{2^{n+2}} < \infty.
\]

Hence, if set \( n = 0 \) then \( d(Af, f) \leq \frac{\delta}{4} < \infty \). Thus Theorem (1.0.1)(a) implies that there exists a function \( F \in X \), which is a fixed point of \( A \), such that \( F(x, z) : = \lim_{n \to \infty} (A^n f)(x, z) \) for any \( x, z \in G \). One can verify \( F \) satisfies of (1.0.3). Indeed,

\[
\| (A^n f)(x + y, z + t) + (A^n f)(x + \sigma(y), z + \sigma(t)) - 2(A^n f)(x, z) - 2(A^n f)(y, t) \|
= \frac{1}{2^n} \| f(2^n (x + y), 2^n (z + t)) + (2^n - 1) f(2^{n-1} (x + y) + 2^{n-1} \sigma(x + y), 2^{n-1}(z + t) + 2^{n-1} \sigma(z + t)) \\
+ f(2^{n-1}(x + \sigma(y)), 2^{n-1}(z + \sigma(t))) \|
\leq \frac{\delta}{2^n} + \frac{(2^n - 1)\delta}{2^{2n}} = \frac{2^n \delta}{2^{2n}} = \frac{\delta}{2^n}
\]

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Letting \( n \to \infty \), we see that \( F \) satisfies (1.0.3).

In Theorem (1.0.1) let \( k = 0 \). Since \( f \in X^* = \{ f \in X | d(f, g) < \infty \} \) in Theorem (1.0.1), by Theorem (1.0.1)(c) and (3.0.31), we have

(2.0.19) \[ d(f, F) \leq \frac{1}{1-k} d(Af, f) \leq \frac{2 \delta}{4} = \frac{\delta}{2} \leq \delta. \]

Therefor (2.0.8) is true. One can verify \( F \) satisfies of (2.0.10). Indeed,

\[
\| D_{\mu_1, \mu_2} f(x, y, z, t) \|
= \lim_{n \to \infty} \left\| \frac{D_{\mu_1, \mu_2} f(2^n x, 2^n y)}{2^{2n}} \right\| \\
+ \lim_{n \to \infty} \left\| \frac{D_{\mu_1, \mu_2} f(2^n x + 2^n \sigma(x), 2^n z + 2^n \sigma(z))}{2^{2n}} \right\| \leq \lim_{n \to \infty} \frac{2^n \delta}{2^{2n}} = \lim_{n \to \infty} \frac{\delta}{2^n} = 0
\]

for all \( x, y, z, t \in G \). So \( D_{\mu_1, \mu_2} f(x, y, z, t) = 0 \) for all \( \mu_1, \mu_2 \in T^1 \) and for all \( x, y, z, t \in G \). Assume that there exists another mapping \( H : G \times G \to Y \) which satisfies (1.0.3) and (2.0.8). we obtain

\[
\| (A^n f)(x, z) - H(x, z) \| \leq \| (A^n f)(x, z) - F(x, z) \| + \| F(x, z) - f(x, z) \| + \| f(x, z) - H(x, z) \|
\]

\[ \infty + \delta + \delta < \infty \]

Thus \( d(A^n f, H) < \infty \), and so \( H \in X^* \). On the other hand \( \Delta H(x, z) = H(x, z) \) for all \( x, z \in G \). Hence by Theorem(1.0.1)(b), we get \( F = H \). This completes the proof of the theorem. □

3. STABILITY OF EQUATION (0.0.2) IN ABELIAN SEMIGROUPS

In this section we investigate the stability of the 2-variable quadratic functional equation

(3.0.20) \[ f(x + y, z + t) + g(x + \sigma(y), z + \sigma(t)) = h(x, z) + k(y, t) \]

for all \( x, y, z, t \in G \), where \( G \) is an abelian semigroup and \( \sigma \) is a homomorphism of \( G \) such that \( \sigma \circ \sigma = I \).

First we will establish some results which will be instrumental in proving our main results.

In the following lemma, we will present a stability result for Jensen’s functional equation :

\[
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\]
Lemma 3.0.3. Let \( \sigma \) be an homomorphism of the abelian semigroup \( G \) such that \( \sigma \circ \sigma = I \). Let \( Y \) be a Banach space. Suppose that \( f : G \times G \to Y \) satisfies the inequality
\[
\| f(\mu_1(x+y), \mu_2(z+t)) + f(\mu_1(x+\sigma(y)), \mu_2(z+\sigma(t))) - 2\mu_1\mu_2 f(x,z) \| \leq \delta.
\]
for all \( \mu_1, \mu_2 \in T^1 \), for all \( x, y, z, t \in G \) and for some \( \delta \geq 0 \). Then there exists a 2-variable quadratic mapping \( F : G \times G \to Y \) such that
\[
\| f(x,z) - F(x,z) \| \leq \delta.
\]

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and
\[
F(\mu_1(x+y), \mu_2(z+t)) + F(\mu_1(x+\sigma(y)), \mu_2(z+\sigma(t))) - 2\mu_1\mu_2 F(x,z) = 0.
\]
for all \( \mu_1, \mu_2 \in T^1 \) and for all \( x, y, z, t \in G \).

Proof. Letting \( \mu_1, \mu_2 = 1 \), \( y = x \) and \( z = t \) in (3.0.33), we have
\[
\| f(2x,2z) + f(x+\sigma(x),z+\sigma(z)) - 2f(x,z) \| \leq \delta.
\]
for all \( x, z \in G \). Then we obtain
\[
\| f(2x,2z) + f(\mu_1(x+\sigma(y)), \mu_2(z+\sigma(t))) - 2\mu_1\mu_2 f(x,z) \| \leq \delta.
\]
for all \( x, z \in G \). Now we set \( X = \{ h : h : G \times G \to Y \text{ is a linear function} \} \) and introduce a generalized metric on \( X \) as follows :
\[
d(g,h) = \inf_{\delta \in [0,\infty)} \| g(x,y) - h(x,y) \| \leq \delta.
\]
By Theorem (2.0.2), follows that \( (X,d) \) is a complete space. Now we define an operator \( J_{\frac{1}{2}} A : X \to X \) such that
\[
J_{\frac{1}{2}} A f(\frac{1}{2}) = \frac{1}{2} \left[ \frac{f(2x,2z) + f(x+\sigma(x),z+\sigma(z))}{2} \right]
\]
We assert that \( J_{\frac{1}{2}} A \) is strictly contractive on \( X \). Given \( g, h \in X \), let \( \delta \in (0,\infty) \) be an arbitrary constant with
\[
d(g,h) \leq \delta,
\]
that is,
\[
\| g(x,y) - h(x,y) \| \leq \delta
\]
It then follows from (3.0.28) that
\[
\| J_{\frac{1}{2}} A g(x,z) - J_{\frac{1}{2}} A h(x,z) \| = \frac{1}{2} \left[ \| g(2x,2z) + g(x+\sigma(x),z+\sigma(z)) - h(2x,2z) + h(x+\sigma(x),z+\sigma(z)) \| \right]
\]
\[
\leq \frac{1}{2} \left[ \| g(2x,2z) - h(2x,2z) \| + \| g(x+\sigma(x),z+\sigma(z)) - h(x+\sigma(x),z+\sigma(z)) \| \right]
\]
\[
\leq \frac{1}{2} \left[ \frac{\delta + \delta}{2} \right] = \frac{1}{2} \| g(x,y) - h(x,y) \|
\]
That is, \( d(J_{\frac{1}{2}} A g, J_{\frac{1}{2}} A h) \leq \frac{1}{2} d(g,h) \), for any \( g, h \in X \). Hence \( J_{\frac{1}{2}} A \) is a strictly contractive function. It easily follows that
\[
J_{\frac{1}{2}} A f(\frac{1}{2}) = \frac{1}{2} \left[ \frac{f(2x,2z) + f(x+\sigma(x),z+\sigma(z))}{2} \right]
\]
And by direct computation, we obtain
\[
J_{\frac{1}{2}} A f(x,z) = \frac{1}{2} \left[ \frac{f(2x,2z) + f(x+\sigma(x),z+\sigma(z))}{2} \right]
\]
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\[
\| \left( J^{n+1} \frac{1}{n} A f \right) (x, z) - \left( J^{n} \frac{1}{n} A f \right) (x, z) \| \leq \frac{1}{2^{n+1}} \frac{1}{2} \| f(2^{n+1} x, 2^{n+1} z) + f(2^n x + 2^n \sigma(x), 2^n z + 2^n \sigma(z)) - 4f(2^n x, 2^n z) \|
+ \frac{1}{2^{n+1}} \| 2(2^n - 1) f(2^n x + 2^n \sigma(x), 2^n z + 2^n \sigma(z)) - 4(2^n - 1) f(2^{n-1} x + 2^{n-1} \sigma(x), 2^{n-1} z + 2^{n-1} \sigma(z)) \| \|
\leq \frac{1}{2^{n+1}} \left[ \frac{\delta}{2^{n+1}} + \frac{(2^n - 1)\delta}{2^{n+1}} \right] = \frac{1}{2^{n+1}} \frac{2^n \delta}{2^n+1} = \frac{\delta}{2^{n+1}}.
\]

Hence, \( \left( J^{n} \frac{1}{n} A f \right) (x, z) \) is a Cauchy sequence. Since \( Y \) is a complete space, implies that there exists a linear function \( F \), such that \( F(x, z) := \lim_{n \to \infty} \left( J^{n} \frac{1}{n} A f \right) (x, z) \) for any \( x, z \in G \). one can see by Theorem (2.0.2) that \( F \) satisfies of (3.0.21), (3.0.23) and (3.0.24).

By using the proof of preceding lemma, we get the stability of the Jensen’s function equation

\[(3.0.32) \quad f(y + x, t + z) + f(\sigma(y) + x, \sigma(t) + z) = 2f(x, z).\]

**Corollary 3.0.4.** Let \( \sigma \) be an homomorphism of the abelian semigroup \( G \) such that \( \sigma 0 = I \). Let \( Y \) be a Banach space. Suppose that \( f : G \times G \to Y \) satisfies the inequality

\[(3.0.33) \quad \| f(\mu_1(y + x), \mu_2(t + z)) + f(\mu_1(\sigma(y) + x), \mu_2(\sigma(t) + z)) - 2\mu_1\mu_2 f(x, z) \| \leq \delta.\]

for all \( \mu_1, \mu_2 \in T^1 \), for all \( x, y, z, t \in G \) and for some \( \delta \geq 0 \). Then there exists a 2-variable quadratic mapping \( F : G \times G \to Y \) such that

\[(3.0.34) \quad \| f(x, z) - F(x, z) \| \leq \delta.\]

and

\[(3.0.35) \quad F(\mu_1(y + x), \mu_2(t + z)) + F(\mu_1(\sigma(y) + x), \mu_2(\sigma(t) + z)) = 2\mu_1\mu_2 F(x, z) = 0.\]

for all \( \mu_1, \mu_2 \in T^1 \) and for all \( x, y, z, t \in G \).

In the following lemma, we obtain a partial stability theorem for the 2-variable quadratic functional equation

\[(3.0.36) \quad f(x + y, z + t) + g(x + \sigma(y), z + \sigma(t)) = h(x, z) + k(y, t).\]

for all \( x, y, z, t \in G \).

**Lemma 3.0.5.** Let \( \sigma \) be an homomorphism of the abelian semigroup \( G \) such that \( \sigma 0 = I \). Let \( Y \) be a Banach space. Suppose that \( f, g, h, k : G \times G \to Y \) satisfies the inequality

\[(3.0.37) \quad \| f(\mu_1(x + y), \mu_2(x + t)) + g(\mu_1(x + \sigma(y)), \mu_2(z + \sigma(t))) - \mu_1 h(x, z) - \mu_2 k(y, t) \| \leq \delta.\]

for all \( \mu_1, \mu_2 \in T^1 \), for all \( x, y, z, t \in G \) and for some \( \delta \geq 0 \). Then there exists a unique 2-variable quadratic mapping \( Q : G \times G \to Y \) a solution of (1.0.3). Also there exists a solution \( J_1, J_2 \) of Jensen’s functional equation (3.0.21) and (3.0.32) such that

\[(3.0.38) \quad \| h(x, z) - J_2(x, z) - Q(x, z) - h(e, e) \| \leq 16\delta.
\]

\[(3.0.39) \quad \| k(x, z) - J_1(x, z) - Q(x, z) - k(e, e) \| \leq 16\delta.\]

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\[(3.0.40) \quad \| f^e(x, z) + g^e(x, z) - Q(x, z) - \frac{1}{2} f(e, e) - \frac{1}{2} g(e, e) \| \leq 12\delta.\]

\[(3.0.41) \quad \| f^e - g^e(x + y, z + t) - (f^e - g^e)(x + \sigma(y), z + \sigma(t)) \| \leq 12\delta.
\]

and

\[(3.0.42) \quad \| f^e(x, z) - \frac{1}{2} J_1(x, z) - \frac{1}{2} J_2(x, z) \| \leq 10\delta.\]

and

\[(3.0.43) \quad \| g^e(x, z) - \frac{1}{2} J_2(x, z) + \frac{1}{2} J_1(x, z) \| \leq 10\delta.\]

and

\[(3.0.44) \quad D_{\mu_1, \mu_2} Q(x, y, z, t) = 0.\]
\[(3.0.45) f_1(x+y, z+t) + f_1(x+\sigma(y), z+\sigma(t)) - 2\mu_1\mu_2 f_1(x, z) = 0.\]

\[(3.0.46) f_2(x+y, z+t) + f_2(\sigma(x)+y, \sigma(z)+t) - 2\mu_1\mu_2 f_2(x, z) = 0.\]

for all \(\mu_1, \mu_2 \in T^1\) and for all \(x, y, z, t \in G\).

**Proof.** Letting \(\mu_1, \mu_2 = 1\), Let us denote by \(f_0(x, y, z, t) = f(x+y, z+t) + f(\sigma(x)+y, \sigma(z)+t) - 2 f_1(x, z) - f_2(x, z)\). By putting \(x = y = z = t = e\) in (3.0.37), we get

\[(3.0.47) \|f_0(x, y, z, t)\| \leq \delta.\]

Consequently, if we subtract the inequality (3.0.37) from the new inequality (3.0.47), we obtain

\[(3.0.48) \|f_1(x+y, z+t) - f_2(x+y, z+t) - f_1(\sigma(x)+y, \sigma(z)+t) + f_2(\sigma(x)+y, \sigma(z)+t)\| \leq 2\delta.\]

Now by replacing \(x\) by \(\sigma(x)\) and \(y\) by \(\sigma(y)\) and \(z\) by \(\sigma(z)\) and \(t\) by \(\sigma(t)\) in (3.0.48) and we add the inequality obtained in (3.0.48), we deducethat

\[(3.0.49) \|F_1(x+y, z+t) + F_2(x+y, z+t) - F_1(\sigma(x)+y, \sigma(z)+t) - F_2(\sigma(x)+y, \sigma(z)+t)\| \leq 2\delta.\]

and

\[(3.0.50) \|F_1(x+y, z+t) + F_2(x+y, z+t) - F_1(\sigma(x)+y, \sigma(z)+t) - F_2(\sigma(x)+y, \sigma(z)+t)\| \leq 2\delta.\]

for all. Hence, if we replace \(y, t\) by \(e\) and \(x, z\) by \(e\) respectively in (3.0.49), we get

\[(3.0.51) \|F_1(x, z) + F_2(x, z) - F_1(\sigma(x)+y, \sigma(z)+t) - F_2(\sigma(x)+y, \sigma(z)+t)\| \leq 2\delta.\]

and

\[(3.0.52) \|F_1(x, y) + F_2(x, y) - F_1(\sigma(x)+y, \sigma(z)+t) - F_2(\sigma(x)+y, \sigma(z)+t)\| \leq 2\delta.\]

So, in view of (3.0.49), (3.0.51) and (3.0.52), we obtain

\[(3.0.53) \|F_1(x+y, z+t) + F_2(x+y, z+t) - (F_1(\sigma(x)+y, \sigma(z)+t) + F_2(\sigma(x)+y, \sigma(z)+t))\| \leq 6\delta.\]

If we add the inequality above to (3.0.53), we get

\[(3.0.54) \|F_1(x+y, z+t) + F_2(x+y, z+t) - (F_1(\sigma(x)+y, \sigma(z)+t) + F_2(\sigma(x)+y, \sigma(z)+t))\| \leq 12\delta,\]

for all \(x, y, z, t \in G\). Hence, in view of Theorem (2.0.2), there exists a unique function \(Q\), a solution of equation (1.0.3) such that

\[(3.0.55) \|F_1(x+y, z+t) - Q(x+y, z+t)\| \leq 12\delta,\]

Consequently, from (3.0.54) and (3.0.55), we deduce that

\[(3.0.56) \|F_1(x+y, z+t) - Q(x+y, z+t)\| \leq 16\delta,\]

and

\[(3.0.57) \|F_1(x+y, z+t) - Q(x+y, z+t)\| \leq 16\delta,\]

for all \(x, y, z, t \in G\). On the other hand, from (3.0.50) we get

\[(3.0.58) \|F_1(x+y, z+t) - F_1(\sigma(x)+y, \sigma(z)+t) - F_2(x+y, z+t)\| \leq 2\delta,\]

and

\[(3.0.59) \|F_1(x+y, z+t) - F_1(\sigma(x)+y, \sigma(z)+t) - F_2(x+y, z+t)\| \leq 2\delta,\]

for all \(x, y, z, t \in G\). Hence, we obtain
\text{(3.0.60)} \quad \|2F_{1}^{q}(x,z) - F_{1}^{q}(x,z) - F_{1}^{q}(x,z)\| \leq 4\delta.
\text{and}\n\text{(3.0.61)} \quad \|2F_{2}^{q}(x,z) - F_{2}^{q}(x,z) + F_{2}^{q}(x,z)\| \leq 4\delta.

for all \(x,z \in G\) and Consequently, we have

\[\|F_{2}^{q}(x + y, z + t) + F_{2}^{q}(x + \sigma(y), z + \sigma(t)) - 2F_{2}^{q}(x, z)\| \leq \|F_{2}^{q}(x + y, z + t)\| - \|F_{2}^{q}(x + \sigma(y), z + \sigma(t)) - 2F_{2}^{q}(x, z)\|\]

and

\[\|F_{2}^{q}(y + x, t + z) + F_{2}^{q}(\sigma(y) + x, \sigma(t) + z) - 2F_{2}^{q}(x, z)\| \leq \|F_{2}^{q}(y + x, t + z)\| - \|F_{2}^{q}(\sigma(y) + x, \sigma(t) + z) - 2F_{2}^{q}(x, z)\|\]

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for all \(x, y, z, t \in G\). Now from lemma (3.0.3) and corollary (3.0.4) there exists two solution of Jensen’s functional equation \(J_{1}, J_{2} : G \times G \rightarrow Y\) such that

\text{(3.0.62)} \quad \|F_{1}^{q}(x,z) - J_{1}^{q}(x,z)\| \leq 8\delta.
\text{and}\n\text{(3.0.63)} \quad \|F_{2}^{q}(x,z) - J_{2}^{q}(x,z)\| \leq 8\delta.

for all \(x, z \in G\). Now, by small computations, we obtain the rest of the proof. □

REFERENCES