On the Existence, Uniqueness and Stability Behavior of a Random Solution to a Non local Perturbed Stochastic Fractional Integro-Differential Equation

Mahmoud M. El-Borai[1,*], M.A.Abdou [2], Mohamed Ibrahim M. Youssef [2]

[1] Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt
[2] Department of Mathematics, Faculty of Education, Alexandria University, Alexandria, Egypt
m_m_elborai@yahoo.com; abdella_777@yahoo.com; m_ibrahim283@yahoo.com

Abstract: In this paper, we prove the existence and uniqueness of a nonlinear perturbed stochastic fractional integro-differential equation of Volterra-Itô type involving nonlocal initial condition by using the theory of admissibility of integral operator and Banach fixed-point principle. Also the stability and boundedness of the second moments of the stochastic solution are studied. In addition, an application to fractional stochastic feedback system is presented.

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1. Introduction

Integro-differential equations arise quite naturally in the study of many physical phenomena in life science and engineering, for example, equations of this form occur in the formulation of problems in reactor dynamics, in the study of the growth of biological population models and in the theory of automatic systems resulting in the delay-differential equations, see for more details [1]. Many investigations have been carried out concerning the existence and uniqueness of solution of deterministic and stochastic integro-differential equations of Volterra type, see [2-5]. However, due to the complex nature of the problems being characterized by such equations, many authors, in the last few decades, pointed out that fractional stochastic models are very suitable for the description of properties of various real materials, e.g. polymers. It has been shown that new models are more adequate than previously integer-order models [6-8]. In many cases, it is better to have more initial information to obtain a good description of the evolution of a physical system. The local initial condition is replaced then by a nonlocal condition, which gives better effect than the initial condition, since the measurement given by a nonlocal condition is usually more precise than the only one measurement given by a local condition, see [9]. Therefore, in this paper we shall be concerned with extending the results in El-Borai et al. [10], William J. padgett and Chris P. Tsokos [5]. That is, we shall consider a nonlinear stochastic perturbed factional integro-differential equation of Volterra-Itô type of the form:

$$\frac{\partial^\alpha x(t;\omega)}{\partial t^\alpha} = h\big(t, x(t;\omega)\big) + \int_0^t k_1(t,\tau;\omega)\, f_1\big(\tau, x(\tau;\omega)\big)d\tau + \int_0^t k_2(t,\tau;\omega)\, f_2\big(\tau, x(\tau;\omega)\big)d\beta(\tau) \qquad (1.1)$$

with the nonlocal condition

$$x(0;\omega) + \sum_{i=1}^{p} c_i x(t_i,\omega) = x_0(\omega) \qquad (1.2)$$

where $0 < \alpha \le 1, t \in R_+ = [0,\infty), 0 < t_1 < \ldots < t_p < \infty$. The fractional derivative is provided by the Caputo derivative and

(i) $\omega \in \Omega$, the supporting set of a probability measure space $(\Omega, \mathcal{A}, P)$;
(ii) $x(t;\omega)$ is the unknown stochastic process for $t \in R_+$;
(iii) $h(t,x)$ is called the stochastic perturbing term and it is a scalar function of $t \in R_+$ and $x \in R$;
(iv) $k_1(t,\tau;\omega)$, $k_2(t,\tau;\omega)$ are scalar stochastic kernels defined for t and τ satisfying $0 \le \tau \le t < \infty$;
(v) $f_1(t,x), f_2(t,x)$ are scalar functions of $t \in R_+, x \in R$ to be specified later;
(vi) $\beta(t)$ is a stochastic process to be defined later.

The purpose of this paper is to obtain the conditions which guarantee the existence and uniqueness of random solution $x(t;\omega)$ of the problem (1.1), (1.2) and to investigate the asymptotic moment behavior of such a random solution. In addition, the usefulness of the results will be illustrated with an application to fractional stochastic feedback system. Equations (1.1), (1.2) generalize the results of El-Borai et al. [10], and the results of Padgett and Tsokos [5]. The considered nonlocal Cauchy problem (1.1), (1.2) consists of two parts, the first integral being a lebesgue integral and the second a stochastic integral of the Itô-Doob type. In our work we shall utilize the spaces of functions and admissibility theory which were introduced into the study of stochastic integral equations by Tsokos [11]. The nonlocal Cauchy problem (1.1), (1.2) has applications in many fields such as electromagnetic theory, viscoelasticity, and fluid mechanics [12-13].

2. Preliminaries

Let $(\Omega, \mathcal{A}, P)$ denotes a probability measure space, that is Ω is a nonempty set known as the sample space, $\mathcal{A}$ is a sigma-algebra of subsets of Ω, and P is a complete probability measure on $\mathcal{A}$. Let $L_2(\Omega, \mathcal{A}, P)$ be the space of all random variables $x(t;\omega)$, $t \in R_+$, which have a second moment with respect to P-measure for each $t \in R_+$. That is: $E\{|x(t;\omega)|^2\} = \int_\Omega |x(t;\omega)|^2 \, dp(\omega) < \infty$.

The norm of $x(t;\omega)$ in $L_2(\Omega, \mathcal{A}, P)$ is defined for each $t \in R_+$ by: $\|x(t;\omega)\| = [E\{|x(t;\omega)|^2\}]^{1/2}$.

Let $L_\infty(\Omega, \mathcal{A}, P)$ be the space of all measurable and P-essentially bounded random variables of $\omega \in \Omega$. The norm of $k(t,\tau;\omega)$ in $L_\infty(\Omega, \mathcal{A}, P)$ will be defined by: $|\|k(t,\tau;\omega)\|| = P - ess\, sup_{\omega\in\Omega} |k(t,\tau;\omega)|$. With respect to the random process $\beta(t)$, we shall assume that $\beta(t)$ is adapted to the filtration $(\mathcal{A}_t)_{t\geq 0}$ which is an increasing family of sub sigma-algebras $\mathcal{A}_t \subset \mathcal{A}$. furthermore, we shall assume that:

(i) The process $\{\beta(t), \mathcal{A}_t, 0 \leq t < \infty\}$ is a real martingale.

(ii) There is a continuous monotone nondecreasing function $F(t)$ on R_+, such that, if $s < t$, then $E\{|\beta(t;\omega) - \beta(s;\omega)|^2\} = E\{|\beta(t;\omega) - \beta(s;\omega)|^2 \setminus \mathcal{A}_s\} = F(t) - F(s) \quad P - a.e.$

Note that:

If $F(t) = ct$, c is a constant, with almost all its sample functions are continuous, then $\beta(t)$ is a Brownian motion process, (see [14], pp. 436-437), and this is the most important special case.

Definition 2.1

We define the space $C_c = C_c(R_+, L_2(\Omega, \mathcal{A}, P))$ to be the space of all continuous functions $x(t;\omega)$ from R_+ into $L_2(\Omega, \mathcal{A}, P)$, such that for each $t \in R_+$, $x(t;\omega)$ is $\mathcal{A}_t$-measurable.

We define a topology in the space C_c by means of the following family of seminorms:

$$\|x(t;\omega)\|_n = sup_{0\leq t\leq n}\{\| x(t;\omega)\|\},$$
$$n = 1,2,3, \ldots$$

It is known that this topology is metrizable and the space C_c is Frechet space.

Definition 2.2

We define the space $C_g = C_g(R_+, L_2(\Omega, \mathcal{A}, P))$ to be the space of all continuous functions from R_+ into $L_2(\Omega, \mathcal{A}, P)$, such that there exist a constant $a > 0$ and a positive continuous function $g(t)$ on R_+ satisfying $\| x(t;\omega)\| \leq a\, g(t)$. The norm in $C_g(R_+, L_2(\Omega, \mathcal{A}, P))$ will be defined by:

$$\|x(t;\omega)\|_{C_g} = sup_{t\in R_+}\left\{\frac{\|x(t;\omega)\|}{g(t)}\right\}.$$

Definition 2.3

We define the space $C = C(R_+, L_2(\Omega, \mathcal{A}, P))$ to be the space of all continuous and bounded functions on R_+ with values in $L_2(\Omega, \mathcal{A}, P)$, that is, C is the space of all second order stochastic processes on R_+ which are bounded and continuous in mean square. The norm in C is defined by:

$$\|x(t;\omega)\|_C = sup_{t\in R_+}\{\|x(t;\omega)\|\} < \infty$$

It is clear that C, C_g are Banach spaces and the following inclusion hold: $C \subset C_g \subset C_c$.

Definition 2.4

The pair of Banach spaces (B, D) with $B, D \subset C_c$ is said to be admissible with respect to the operator $T: C_c \to C_c$ if and only if $T(B) \subset D$.

Definition 2.5

The Banach space B is said to be stronger than C_c, if every convergent sequence in B, with respect to its norm, will also converge in C_c. (but the converse is not true in general).

Definition 2.6

We call $x(t;\omega)$ a random solution of the equation (1.1) if $x(t;\omega) \in C_c$ for each $t \in R_+$, satisfies the equation (1.1) for every $t > 0$ and satisfies the nonlocal initial condition almost surely.

We now state the following lemma which is given by Tsokos [4].

Lemma 2.1

Let T be a continuous linear operator from $C_c(R_+, L_2(\Omega, \mathcal{A}, P))$ into itself, if B and D are Banach spaces stronger than C_c and if (B, D) is admissible with respect to T, then T is a continuous linear operator from B into D.

3. Main results

Using the definitions of the fractional derivatives and integrals, it is suitable to rewrite the considered problem in the form:

$$x(t;\omega) = x(0;\omega) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h(\tau, x(\tau;\omega))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s (t-s)^{\alpha-1} k_1(s,\tau;\omega) f_1(\tau, x(\tau;\omega))d\tau ds + \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s (t-s)^{\alpha-1} k_2(s,\tau;\omega) f_2(\tau, x(\tau;\omega))d\beta(\tau)ds$$

Changing the order of integration (note that: the assumptions on the functions k_2 and f_2 permit this operation on the last integral and the proof is essential the same as the one given in ([14], pp.430-431)

$$x(t;\omega) = x(0;\omega) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h(\tau, x(\tau;\omega))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t \boldsymbol{K_1}(t,\tau;\omega) f_1(\tau, x(\tau;\omega))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t \boldsymbol{K_2}(t,\tau;\omega) f_2(\tau, x(\tau;\omega))d\beta(\tau) \quad (3.1)$$

Where

$$\boldsymbol{K_1}(t,\tau;\omega) = \int_\tau^t (t-s)^{\alpha-1} k_1(s,\tau;\omega)ds \quad (3.2)$$

$$\boldsymbol{K_2}(t,\tau;\omega) = \int_\tau^t (t-s)^{\alpha-1} k_2(s,\tau;\omega)ds \quad (3.3)$$

Now define the integral operators T_1, T_2 and T_3 on $C_c(R_+, L_2(\Omega, \mathcal{A}, P))$ as follows:

$$(T_1x)(t;\omega) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} x(\tau;\omega)d\tau \quad (3.4)$$

$$(T_2x)(t;\omega) = \frac{1}{\Gamma(\alpha)}\int_0^t \boldsymbol{K_1}(t,\tau;\omega) x(\tau;\omega)d\tau \quad (3.5)$$

$$(T_3x)(t;\omega) = \frac{1}{\Gamma(\alpha)}\int_0^t \boldsymbol{K_2}(t,\tau;\omega) x(\tau;\omega)d\beta(\tau) \quad (3.6)$$

In lemma (3.1) in [10], El-Borai et al. proved that T_1 is continuous operator from C_c into C_c. Now we shall prove a lemma concerning the continuity of T_2 and T_3 as mappings from $C_c(R_+, L_2(\Omega, \mathcal{A}, P))$ into itself.

Lemma 3.1

Suppose that

(i) The functions $k_1(t,\tau;\omega)$ and $k_2(t,\tau;\omega)$ are $\mathcal{A}_\tau$ measurable and P-ess bounded for each t,τ satisfying $0 \le \tau \le t < \infty$;

(ii) $k_1(t,\tau;\omega)$ and $k_2(t,\tau;\omega)$ are continuous as maps from $\Delta = \{(t,\tau): 0 \le \tau \le t < \infty\}$ into $L_\infty(\Omega, \mathcal{A}, P)$.

Then the operators T_2 and T_3 defined by the equation (3.5) and (3.6) are continuous mappings from the space $C_c(R_+, L_2(\Omega, \mathcal{A}, P))$ into itself.

Proof:

The assertion about T_2 follows from lemma (3.2) in El-Borai et al. [10]. Hence, we shall only prove the assertion regarding the operator T_3.

Step1, we shall show that $T_3: C_c \to C_c$.

We need to prove that $(T_3x)(t;\omega) \in L_2(\Omega, \mathcal{A}, P)$ and is continuous function in mean square sense for each $t \in R_+$. By the same way which was used in lemma (3.2) in El-Borai et al. [10], it is easy to prove that the assumptions (i),(ii) on $k_2(t,\tau;\omega)$ imply that $\boldsymbol{K_2}(t,\tau;\omega) \in L_\infty(\Omega, \mathcal{A}, P)$, also for each $(t,\tau) \in \Delta$, $\boldsymbol{K_2}(t,\tau;\omega)$ is $\mathcal{A}_\tau$ measurable and is a continuous map from Δ into $L_\infty(\Omega, \mathcal{A}, P)$, hence for each $x(t;\omega) \in C_c$ and for each t, the function $\boldsymbol{K_2}(t,\tau;\omega)x(\tau;\omega)$ is $d\tau dP$ measurable and

$$\|(T_3x)(t;\omega)\|^2 = \frac{1}{[\Gamma(\alpha)]^2}\left\|\int_0^t \boldsymbol{K_2}(t,\tau;\omega)x(\tau;\omega)d\beta(\tau)\right\|^2 \le \frac{1}{[\Gamma(\alpha)]^2}\int_0^t \||\boldsymbol{K_2}(t,\tau;\omega)|\|^2 \|x(\tau;\omega)\|^2 dF(\tau) < \infty$$

Thus, the stochastic kernel in (3.6) is well defined, and $(T_3x)(t;\omega) \in L_2(\Omega, \mathcal{A}, P)$. Now it remains only to prove that T_3 is continuous in the mean square sense for each $t \in R_+$ as follow:

Let $x(t;\omega) \in C_c$, $0 \le t_1 < t_2$, $t_1, t_2 \in [0,n] \subset R_+$, then

$$\|(T_3x)(t_2;\omega) - (T_3x)(t_1;\omega)\|^2 \le \frac{1}{[\Gamma(\alpha)]^2}\left\|\int_0^{t_2} \boldsymbol{K_2}(t_2,\tau;\omega)x(\tau;\omega)d\beta(\tau) - \int_0^{t_1} \boldsymbol{K_2}(t_1,\tau;\omega)x(\tau;\omega)d\beta(\tau)\right\|^2$$

$$= \frac{1}{[\Gamma(\alpha)]^2}\left\|\int_0^{t_1} [\boldsymbol{K_2}(t_2,\tau;\omega) - \boldsymbol{K_2}(t_1,\tau;\omega)]x(\tau;\omega)d\beta(\tau) + \int_{t_1}^{t_2} \boldsymbol{K_2}(t_2,\tau;\omega)x(\tau;\omega)d\beta(\tau)\right\|^2$$

Using the inequality $(A+B)^2 \le 2(A^2+B^2)$, yields

$$\|(T_3x)(t_2;\omega)-(T_3x)(t_1;\omega)\|^2 \le \frac{2}{[\Gamma(\alpha)]^2}\left\|\int_0^{t_1}[\boldsymbol{K_2}(t_2,\tau;\omega) - \boldsymbol{K_2}(t_1,\tau;\omega)]x(\tau;\omega)d\beta(\tau)\right\|^2 + \frac{2}{[\Gamma(\alpha)]^2}\left\|\int_{t_1}^{t_2}\boldsymbol{K_2}(t_2,\tau;\omega)x(\tau;\omega)d\beta(\tau)\right\|^2$$

Appling Itô-Doob Isometry, yields

$$\le \frac{2}{[\Gamma(\alpha)]^2}\int_0^{t_1}|||\boldsymbol{K_2}(t_2,\tau;\omega) - \boldsymbol{K_2}(t_1,\tau;\omega)|||^2.\|x(\tau;\omega)\|^2dF(\tau) + \frac{2}{[\Gamma(\alpha)]^2}\int_{t_1}^{t_2}|||\boldsymbol{K_2}(t_2,\tau;\omega)|||^2.\|x(\tau;\omega)\|^2dF(\tau)$$

$$\le \frac{2}{[\Gamma(\alpha)]^2}\|x(t;\omega)\|_n^2\int_0^{t_1}|||\boldsymbol{K_2}(t_2,\tau;\omega) - \boldsymbol{K_2}(t_1,\tau;\omega)|||^2dF(\tau) + \frac{2}{[\Gamma(\alpha)]^2}\|x(t;\omega)\|_n^2\int_{t_1}^{t_2}|||\boldsymbol{K_2}(t_2,\tau;\omega)|||^2\,dF(\tau) \to 0, \text{as } t_1\to t_2$$

Since $\boldsymbol{K_2}$ is continuous from $\Delta=\{(t,\tau):0\le\tau\le t<\infty\}$ into $L_\infty(\Omega,\mathcal{A},P)$, and F is continuous, then T_3 is continuous in the mean square sense for each $t\in R_+$, and hence, $T_3: C_c\to C_c$.

Step2, we shall show that $T_3: C_c\to C_c$ is a continuous operator as follow:

Let $x(t;\omega)\in C_c$, then

$$\|(T_3x)(t;\omega)\|^2 = \frac{1}{[\Gamma(\alpha)]^2}\int_\Omega\left\{\int_0^t\boldsymbol{K_2}(t,\tau;\omega)x(\tau;\omega)d\beta(\tau)\right\}^2dP$$

Appling Itô-Doob Isometry, yields

$$\le \frac{1}{[\Gamma(\alpha)]^2}\int_\Omega\left(\int_0^t|\boldsymbol{K_2}(t,\tau;\omega)x(\tau;\omega)|^2dF(\tau)\right)dP$$

$$\le \frac{1}{[\Gamma(\alpha)]^2}\int_0^t\left(\int_\Omega|\boldsymbol{K_2}(t,\tau;\omega)x(\tau;\omega)|^2\,dP\right)dF(\tau)$$

$$\le \frac{1}{[\Gamma(\alpha)]^2}\int_0^t|||\boldsymbol{K_2}(t,\tau;\omega)|||^2.\|x(\tau;\omega)\|^2dF(\tau)$$

$$\le \frac{1}{[\Gamma(\alpha)]^2}\|x(t;\omega)\|_n^2\int_0^t|||\boldsymbol{K_2}(t,\tau;\omega)|||^2\,dF(\tau)$$

Now,

$$\|(T_3x)(t;\omega)\| \le \frac{1}{\Gamma(\alpha)}\|x(t;\omega)\|_n\left[\int_0^t|||\boldsymbol{K_2}(t,\tau;\omega)|||^2\,dF(\tau)\right]^{\frac{1}{2}}$$

Thus,

$$\|T_3x((t;\omega))\|_n = sup_{0\le t\le n}\|T_3x((t;\omega))\|$$

$$\le \frac{1}{\Gamma(\alpha)}\|x(t;\omega)\|_n\left\{sup_{0\le t\le n}\left[\int_0^t|||\boldsymbol{K_2}(t,\tau;\omega)|||^2\,dF(\tau)\right]^{\frac{1}{2}}\right\}$$

$$\le N_1\|x(t;\omega)\|_n$$

where N_1 is a constant depends upon n and α. Since $|||\boldsymbol{K_2}(t,\tau;\omega)|||$ is continuous, it follows that T_3 is continuous operator from C_c into C_c, (see [15] p. 42). Hence the required results.

Now let the operators T_1, T_2 and T_3 be defined by equations (3.4), (3.5) and (3.6) respectively, and let $B, D\subset C_c(R_+, L_2(\Omega,\mathcal{A},P))$ be Banach spaces stronger than C_c , such that (B,D) is admissible with respect to each of the operators T_1, T_2 and T_3. Then, It follows from lemma (2.1), that T_1, T_2 and T_3 are continuous from B into D, hence there exist constants M_1, M_2 and M_3 such that

$$\|(T_ix)(t;\omega)\|_D \le M_i\|x(t;\omega)\|_B \qquad i=1,2,3$$

The infimum of such constants M_1, M_2, and M_3 is called the norm of the operators T_1, T_2 and T_3 respectively.

In what follow we shall assume that f_1 and f_2 are maps from C_c into C_c and that $k_1(t,\tau;\omega)$ and $k_2(t,\tau;\omega)$ satisfy the conditions of lemma (3.1).

Lemma 3.2

Assume that $\sum_{i=1}^p c_i \ne -1$, then the nonlocal Cauchy problem (1.1), (1.2) is equivalent to the following integral equation

$$x(t;\omega) = Ax_0(\omega) - A\left(\sum_{i=1}^p c_i\,[(T_{1i}hx)(t_i,\omega) + (T_{2i}f_1x)(t_i,\omega) + (T_{3i}f_2x)(t_i,\omega)]\right) + (T_1hx)(t;\omega) + (T_2f_1x)(t;\omega) + (T_3f_2x)(t;\omega) \quad (3.7)$$

where

$$A = \left[1+\sum_{i=1}^p c_i\right]^{-1}$$

T_1, T_2 and T_3 are defined by (3.4), (3.5) and (3.6) respectively and

$$(T_{1i}x)(t_i;\omega) = \frac{1}{\Gamma(\alpha)}\int_0^{t_i}(t_i-\tau)^{\alpha-1}x(\tau;\omega)d\tau$$

$$(T_{2i}x)(t_i;\omega) = \frac{1}{\Gamma(\alpha)}\int_0^{t_i}\boldsymbol{K_1}(t_i,\tau;\omega)x(\tau;\omega)d\tau$$

$$(T_{3i}x)(t;\omega) = \frac{1}{\Gamma(\alpha)}\int_0^{t_i}\boldsymbol{K_2}(t_i,\tau;\omega)x(\tau;\omega)d\beta(\tau)$$

$$i = 1,2,\dots\dots.p$$

The proof is analogous to that of lemma (3.3) in El-Borai et al. [10] and hence omitted.

We now go to the following existence theorem.

Theorem 3.1

Suppose the integral equation (1.1) satisfies the following conditions:

(i) B and D are Banach spaces stronger than C_c and the pair (B, D) is admissible with respect to each of the operators, T_1, T_2 and T_3 defined by (3.4), (3.5) and (3.6) respectivly;

(ii) $x(t;\omega) \to h(t, x(t;\omega))$ is an operator on $S = \{x(t;\omega) \in D: \|x(t;\omega)\|_D \le \rho\}$, with values in B satisfying:

$$\left\|h(t,x(t;\omega)) - h(t,y(t;\omega))\right\|_B \le \lambda_1\|x(t;\omega) - y(t;\omega)\|_D$$

for $x(t;\omega), y(t;\omega) \in S$ and $\rho > 0, \lambda_1 > 0$ are constants;

(iii) $x(t;\omega) \to f_1(t, x(t;\omega))$ is an operator on S with values in B satisfying:

$$\left\|f_1(t,x(t;\omega)) - f_1(t,y(t;\omega))\right\|_B \le \lambda_2\|x(t;\omega) - y(t;\omega)\|_D$$

for $x(t;\omega), y(t;\omega) \in S$ and $\lambda_2 > 0$ constant;

(iv) $x(t;\omega) \to f_2(t, x(t;\omega))$ is an operator on S with values in B satisfying:

$$\left\|f_2(t,x(t;\omega)) - f_2(t,y(t;\omega))\right\|_B \le \lambda_3\|x(t;\omega) - y(t;\omega)\|_D$$

for $x(t;\omega), y(t;\omega) \in S$ and $\lambda_3 > 0$ constant;

(v) $k_1(t,\tau;\omega)$ and $k_2(t,\tau;\omega)$ satisfy the conditions of lemma (3.1)

(vi) $x_0(\omega) \in D$.

Then there exists a unique random solution $x(t;\omega) \in S$ of equation (1.1), provided that:

$$\left[(M_1\lambda_1 + M_2\lambda_2 + M_3\lambda_3)\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right)\right] < 1$$

$$|\mathrm{A}|\|x_0(\omega)\|_D + M_1\|h(t,0)\|_B\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right) +$$

$$(M_2\|f_1(t,0)\|_B + M_3\|f_2(t,0)\|_B)\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right)$$

$$\le \rho\left(1 - (M_1\lambda_1 + M_2\lambda_2 + M_3\lambda_3)\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right)\right)$$

where M_1 and M_2 and M_3 are the norms of T_1, T_2 and T_3, respectively.

Proof:

By condition (i) and lemmas (3.1) in [10], (2.1), and (3.1), T_1, T_2 and T_3 are continuous from B into D. Hence, their norms M_1 and M_2 and M_3 exist.

Define the operator $U: S \to D$ by

$$(Ux)(t;\omega) = \mathrm{A}x_0(\omega) + (T_1hx)(t;\omega) + (T_2f_1x)(t;\omega) + (T_3f_2x)(t;\omega) - \mathrm{A}\sum_{i=1}^{p}c_i\,[(T_{1i}hx)(t_i,\omega) + (T_{2i}f_1x)(t_i,\omega) + (T_{3i}f_2x)(t_i,\omega)] \tag{3.8}$$

We must show that $U(S) \subset S$ and that the operator U is a contraction operator on S. Then, we may apply Banach's fixed-point theorem to obtain the existence of a unique random solution.

Let $x(t;\omega) \in S$, then take the norm of (3.8), we get

$$\|(Ux)(t;\omega)\|_D \le \|\mathrm{A}x_0(\omega)\|_D + \|(T_1hx)(t;\omega)\|_D + \|(T_2f_1x)(t;\omega)\|_D + \|(T_3f_2x)(t;\omega)\|_D + \left\|-\mathrm{A}\sum_{i=1}^{p}c_i[(T_{1i}hx)(t_i;\omega) + (T_{2i}fx)(t_i;\omega) + (T_{3i}f_2x)(t_i,\omega)]\right\|_D$$

$$\le |\mathrm{A}|\|x_0(\omega)\|_D + \|(T_1hx)(t;\omega)\|_D + \|(T_2f_1x)(t;\omega)\|_D + \|(T_3f_2x)(t;\omega)\|_D$$
$$+|\mathrm{A}|\sum_{i=1}^{p}|c_i|\;\|(T_{1i}hx)(t_i;\omega)\|_D$$
$$+|\mathrm{A}|\sum_{i=1}^{p}|c_i|\;\|(T_{2i}f_1x)(t_i;\omega)\|_D$$
$$+|\mathrm{A}|\sum_{i=1}^{p}|c_i|\;\|(T_{3i}f_2x)(t_i,\omega)\|_D$$
$$\le |\mathrm{A}|\|x_0(\omega)\|_D + M_1\left\|h(t,x(t;\omega))\right\|_B$$
$$+|\mathrm{A}|\sum_{i=1}^{p}|c_i|M_1\left\|h(t_i,x(t_i;\omega))\right\|_B$$
$$+|\mathrm{A}|\sum_{i=1}^{p}|c_i|M_2\left\|f_1(t_i,x(t_i;\omega))\right\|_B$$
$$+|\mathrm{A}|\sum_{i=1}^{p}|c_i|M_3\left\|f_2(t_i,x(t_i;\omega))\right\|_B$$
$$+M_2\left\|f_1(t,x(t;\omega))\right\|_B + M_3\left\|f_2(t,x(t;\omega))\right\|_B$$
$$\le |\mathrm{A}|\|x_0(\omega)\|_D + M_1[\lambda_1\|x(t;\omega)\|_D + \|h(t,0)\|_B]$$
$$+|\mathrm{A}|\sum_{i=1}^{p}|c_i|M_1[\lambda_1\|x(t_i;\omega)\|_D + \|\,h(t_i,0)\|_B]$$

$$+|A|\sum_{i=1}^{p}|c_i|\,M_2[\lambda_2\|x(t_i;\omega)\|_D+\|\,f_1(t_i,0)\|_B]$$
$$+|A|\sum_{i=1}^{p}|c_i|M_3\,[\lambda_3\|x(t_i;\omega)\|_D+\|f_2(t_i,0)\|_B]$$
$$+M_2[\lambda_2\|x(t;\omega)\|_D+\|f_1(t,0)\|_B]$$
$$+M_3[\lambda_3\|x(t;\omega)\|_D+\|f_2(t,0)\|_B]$$
$$\le|A|\|x_0(\omega)\|_D+M_1\|h(t,0)\|_B\left(1+|A|\sum_{i=1}^{p}|c_i|\right)$$
$$+\rho(M_1\lambda_1+M_2\lambda_2+M_3\lambda_3)\left(1+|A|\sum_{i=1}^{p}|c_i|\right)$$
$$+(M_2\|\,f_1(t,0)\|_B+M_3\|\,f_2(t,0)\|_B)\left(1+|A|\sum_{i=1}^{p}|c_i|\right)$$
$$\le\rho$$

Thus, $U(S)\subset S$, by the last condition of the theorem.

Let $y(t;\omega)$ be another element of S, from the assumptions, it is clear that: $[(Ux)(t;\omega)-(Uy)(t;\omega)]\in D$, since the difference of two elements of a Banach space is in the Banach space.

$$\|(Ux)(t;\omega)-(Uy)(t;\omega)\|_D$$
$$\le\left\|-A\left(\sum_{i=1}^{p}c_i\,[(T_{1i}hx)(t_i;\omega)-(T_{1i}hy)(t_i;\omega)]\right)\right\|_D$$
$$+\left\|-A\left(\sum_{i=1}^{p}c_i\,[(T_{2i}f_1x)(t_i;\omega)-(T_{2i}f_1y)(t_i;\omega)]\right)\right\|_D$$
$$+\left\|-A\left(\sum_{i=1}^{p}c_i\,[(T_{3i}f_2x)(t_i;\omega)-(T_{3i}f_2y)(t_i;\omega)]\right)\right\|_D$$
$$+\|(T_1hx)(t;\omega)-(T_1hy)(t;\omega)\|_D$$
$$+\|(T_2f_1x)(t;\omega)-(T_2f_1y)(t;\omega)\|_D$$
$$+\|(T_3f_2x)(t;\omega)-(T_3f_2y)(t;\omega)\|_D$$
$$\le|A|\sum_{i=1}^{p}|c_i|\,M_1\,\left\|h\big(t_i,x(t_i;\omega)\big)-h\big(t_i,y(t_i;\omega)\big)\right\|_B$$
$$+|A|\sum_{i=1}^{p}|c_i|\,M_2\left\|f_1\big(t_i,x(t_i;\omega)\big)-f_1\big(t_i,y(t_i;\omega)\big)\right\|_B$$
$$+|A|\sum_{i=1}^{p}|c_i|\,M_3\left\|f_2\big(t_i,x(t_i;\omega)\big)-f_2\big(t_i,y(t_i;\omega)\big)\right\|_B$$
$$+M_1\left\|h\big(t,x(t;\omega)\big)-h\big(t,y(t;\omega)\big)\right\|_B$$
$$+M_2\left\|f_1\big(t,x(t;\omega)\big)-f_1\big(t,y(t;\omega)\big)\right\|_B$$
$$+M_3\left\|f_2\big(t,x(t;\omega)\big)-f_2\big(t,y(t;\omega)\big)\right\|_B$$
$$\le|A|\sum_{i=1}^{p}|c_i|\,M_1\lambda_1\|x(t_i;\omega)-y(t_i;\omega)\|_D$$
$$+|A|\sum_{i=1}^{p}|c_i|\,M_2\lambda_2\|x(t_i;\omega)-y(t_i;\omega)\|_D$$
$$+|A|\sum_{i=1}^{p}|c_i|\,M_3\lambda_3\|x(t_i;\omega)-y(t_i;\omega)\|_D$$
$$+M_1\lambda_1\|x(t;\omega)-y(t;\omega)\|_D$$
$$+M_2\lambda_2\|x(t;\omega)-y(t;\omega)\|_D$$
$$+M_3\lambda_3\|x(t;\omega)-y(t;\omega)\|_D$$
$$\le\left[(M_1\lambda_1+M_2\lambda_2+M_3\lambda_3)\left(1+|A|\sum_{i=1}^{p}|c_i|\right)\right]\|x(t;\omega)-y(t;\omega)\|_D$$

Since by hypothesis: $\left[(M_1\lambda_1+M_2\lambda_2+M_3\lambda_3)\left(1+|A|\sum_{i=1}^{p}|c_i|\right)\right]<1$, then U is a contraction operator on S. Applying Banach's fixed-point theorem, there exists a unique element of S so that $(Ux)(t;\omega)=x(t;\omega)$. That is, there is a unique random solution of the random equation (1.1), completing the proof.

We now state the following corollary when the stochastic perturbing term $h\big(t,x(t;\omega)\big)$ is zero which is a generalization of the integro-differential equation studied by Tsokos [5] and El-Borai et al. [10].

Corollary 3.1

If the stochastic fractional integro-differential equation

$$\frac{\partial^\alpha x(t;\omega)}{\partial t^\alpha}=\int_0^t k_1(t,\tau;\omega)\,f_1\big(\tau,x(\tau;\omega)\big)d\tau+\int_0^t k_2(t,\tau;\omega)\,f_2\big(\tau,x(\tau;\omega)\big)d\beta(\tau)\quad(3.9)$$

with the nonlocal condition

$$x(0;\omega)+\sum_{i=1}^{p}c_ix(t_i,\omega)=x_0(\omega)\quad(3.10)$$

satisfies the following conditions:

(i) B and D are Banach spaces stronger than C_c and the pair (B,D) is admissible with respect the operator T_2 and T_3 defined by (3.5), (3.6);

(ii) $x(t;\omega)\to f_1\big(t,x(t;\omega)\big)$ is an operator on $S=\{x(t;\omega)\in D:\|x(t;\omega)\|_D\le\rho\}$, with values in B satisfying:

$$\left\|f_1\big(t,x(t;\omega)\big)-f_1\big(t,y(t;\omega)\big)\right\|_B\le\lambda_2\|x(t;\omega)-y(t;\omega)\|_D$$

for $x(t;\omega),y(t;\omega)\in S$ and $\rho>0,\lambda_2>0$ are constants;

(iii) $x(t;\omega) \to f_2(t, x(t;\omega))$ is an operator on S with values in B satisfying:

$$\|f_2(t,x(t;\omega)) - f_2(t,y(t;\omega))\|_B \leq \lambda_3 \|x(t;\omega) - y(t;\omega)\|_D$$

for $x(t;\omega), y(t;\omega) \in S$ and $\lambda_3 > 0$ constant;

(vii) $k_1(t,\tau;\omega)$ and $k_2(t,\tau;\omega)$ satisfy the conditions of lemma (3.1), and

(iv) $x_0(\omega) \in D$.

Then there exists a unique random solution $x(t;\omega) \in S$ of equation(3.9), provided that:

$$\left[(\lambda_2 M_2 + \lambda_3 M_3)\left(1 + |\mathrm{A}| \sum_{i=1}^{p} |c_i|\right)\right] < 1$$

$$|\mathrm{A}| \|x_0(\omega)\|_D + (M_2 \|f_1(t,0)\|_B + M_3 \|f_2(t,0)\|_B)\left(1 + |\mathrm{A}| \sum_{i=1}^{p} |c_i|\right) \leq \rho\left(1 - (\lambda_2 M_2 + \lambda_3 M_3)\left(1 + |\mathrm{A}| \sum_{i=1}^{p} |c_i|\right)\right)$$

Where M_2, and M_3 are the norm of T_2 and T_3 respectively.

Since (3.9) is the equivalent of (3.7) with $h(t,x)$ equal to zero, the proof follows from that theorem (3.1) with T_1 being the null operator.

4. Boundedness and Asymptotic Behavior of Random Solution.

Using the spaces C_g and C, we now give some results concerning the asymptotic behavior of the random solution of (1.1). We first consider the unperturbed case (3.9).

Theorem 4.1

Suppose that equation (3.9) satisfies the following conditions:

(i) $\||k_1(s,\tau;\omega)|\| \leq \Lambda_1 \, e^{-\gamma(t-\tau)}$ for some constants $\Lambda_1 > 0$ and $\gamma > 0$, $0 \leq \tau \leq s \leq t$;

(ii) $\int_\tau^t \||k_2(s,\tau;\omega)|\|^2 ds \leq \Lambda_2$ for some constant $\Lambda_2 > 0$ and $0 \leq \tau \leq s \leq t$;

(iii) $\int_0^t (t-\tau)^{2\alpha-1} \, e^{-2\beta\tau} dF(\tau) \leq \Lambda_3$ for some constant $\Lambda_3 > 0$;

(iv) $x(t;\omega) \to f_1(t, x(t;\omega))$ satisfies $\|f_1(t,x(t;\omega))\| \leq \Lambda_4 e^{-\beta t}, t \geq 0$, for some $\Lambda_4 > 0$, $\gamma > \beta > 0$, and

$$\|f_1(t,x(t;\omega)) - f_1(t,y(t;\omega))\| \leq \lambda_2 e^{-\beta t} \|x(t;\omega) - y(t;\omega)\|$$

for $\|x(t;\omega)\|$ and $\|y(t;\omega)\| \leq \rho$ at each $t \geq 0$ and $\lambda_2 > 0$ constant;

(v) $x(t;\omega) \to f_2(t, x(t;\omega))$ satisfies $\|f_2(t,x(t;\omega))\| \leq \Lambda_5 e^{-\beta t}, t \geq 0$, for some $\Lambda_5 > 0, \gamma > \beta > 0$, and

$$\|f_2(t,x(t;\omega)) - f_2(t,y(t;\omega))\| \leq \lambda_3 e^{-\beta t} \|x(t;\omega) - y(t;\omega)\|$$

for $\|x(t;\omega)\|$ and $\|y(t;\omega)\| \leq \rho$ at each $t \geq 0$ and $\lambda_3 > 0$ constant; and

(vi) $x_0(\omega) = 0 \; P - a.e.$

Then there exists a unique random solution to equation (3.9) such that

$$\sup_{t\geq 0} \|x(t;\omega)\| = \sup_{t\geq 0} \{E[|x(t;\omega)|^2]\}^{\frac{1}{2}} \leq \rho \,,$$

where $E[\cdot]$ is the mathematical expectation, provided that: λ_2 , λ_3, $\| f_1(t,0)\|_{C_g}$ and $\| f_2(t,0)\|_{C_g}$ are small enough.

Proof:

It is sufficient to show that conditions (i) , (ii) and (iii) implies the admissibility of the pair of spaces (C_g, C) with respect to the operators T_2 and T_3 defined by (3.5), (3.6), and that conditions (iv) and (v) are equivalent to condition (ii) and (iii) of corollary (3.1) with $B = C_g$, $D = C$, $g(t) = e^{-\beta t}, \beta > 0$.

In [10], El-Borai et al. proved that (C_g, C) is admissible with respect to the operators T_2. Now let us consider T_3 , let $x(t;\omega) \in C_g(R_+, L_2(\Omega, \mathcal{A}, P))$, taking the norm in $L_2(\Omega, \mathcal{A}, P)$ of (3.6), we obtain

$$\|(T_3 \, x)(t;\omega)\|^2 \leq \left\| \frac{1}{\Gamma(\alpha)} \int_0^t \boldsymbol{K_2}(t,\tau;\omega) x(\tau;\omega) d\beta(\tau) \right\|^2$$

$$\leq \frac{1}{[\Gamma(\alpha)]^2} \int_0^t \|\boldsymbol{K_2}(t,\tau;\omega) x(\tau;\omega)\|^2 dF(\tau)$$

$$\leq \frac{1}{[\Gamma(\alpha)]^2} \int_0^t \||\boldsymbol{K_2}(t,\tau;\omega)|\|^2 . \|x(\tau;\omega)\|^2 dF(\tau)$$

$$\leq \frac{\Lambda_2 \, \Lambda_3}{(2\alpha - 1)\,[\Gamma(\alpha)]^2} \|x(t;\omega)\|_{C_g}^2$$

This implies that $sup_{t\geq 0} \|(T_3 \, x)(t;\omega)\|$ is bounded, which implies $(T_3 \, x)(t;\omega) \in C$, and thus, (C_g, C) is admissible with respect to the operators T_3. Now we will show that conditions (iv) and (v) are equivalent to condition (ii) and (iii) of corollary (3.1), let $f_1(t, x(t;\omega)), f_1(t, y(t;\omega)) \in C_g$, then

$$\|f_1(t,x(t;\omega)) - f_1(t,y(t;\omega))\|_{C_g} = sup_{t\geq 0} \left\{ \frac{\|f_1(t,x(t;\omega)) - f_1(t,y(t;\omega))\|}{e^{-\beta t}} \right\}$$

$$\le sup_{t\ge 0}\left\{\frac{\lambda_2 e^{-\beta t}\|x(t;\omega)-y(t;\omega)\|}{e^{-\beta t}}\right\}$$
$$=\lambda_2\|x(t;\omega)-y(t;\omega)\|_C$$

and similarly for the other condition, applying corollary (3.1), we get on the required result.

Now we state the results concerning the perturbed equation (1.1)

Theorem 4.2

Assume that equation (1.1) satisfies the following conditions:

(i) $|||k_1(s,\tau;\omega)||| \le \Lambda_1$ for some constant $\Lambda_1 > 0$, $0 \le \tau \le s \le t$;

(ii) $\int_\tau^t |||k_2(s,\tau;\omega)|||^2 ds \le \Lambda_2$ for some $\Lambda_2 > 0$, $0 \le \tau \le s \le t$;

(iii) $\int_0^t (t-\tau)^{2\alpha-1}\, dF(\tau) \le \Lambda_3$, for some $\Lambda_3 > 0$;

(iv) $x(t;\omega) \to h(t,x(t;\omega))$ satisfies, for some $\Lambda_4 > 0$ and $\beta > 0$, $\|h(t,x(t;\omega))\| \le \Lambda_4, t \ge 0$, and

$$\|h(t,x(t;\omega)) - h(t,y(t;\omega))\| \le \lambda_1 \|x(t;\omega)-y(t;\omega)\|$$

for $\|x(t;\omega)\|$ and $\|y(t;\omega)\| \le \rho, t \ge 0$ and $\lambda_1 > 0$ constant;

(v) $x(t;\omega) \to f_1(t,x(t;\omega))$ satisfies, for some constant $\Lambda_5 > 0$, $\|f_1(t,x(t;\omega))\| \le \Lambda_5,\ t \ge 0$, and

$$\|f_1(t,x(t;\omega)) - f_1(t,y(t;\omega))\| \le \lambda_2\|x(t;\omega)-y(t;\omega)\|$$

for $\|x(t;\omega)\|$ and $\|y(t;\omega)\| \le \rho$ at each $t \ge 0$ and $\lambda_2 > 0$ constant;

(vi) $x(t;\omega) \to f_2(t,x(t;\omega))$ satisfies, for some constant $\Lambda_6 > 0$, $\|f_2(t,x(t;\omega))\| \le \Lambda_6, t \ge 0$, and

$$\|f_2(t,x(t;\omega)) - f_2(t,y(t;\omega))\| \le \lambda_3\|x(t;\omega)-y(t;\omega)\|$$

for $\|x(t;\omega)\|$ and $\|y(t;\omega)\| \le \rho$ at each $t \ge 0$ and $\lambda_3 > 0$ constant;

(vii) $x_0(\omega) \in C$

Then, there exists a unique random of solution of (1.1) satisfying

$$\sup_{t\ge 0}\|x(t;\omega)\| = \sup_{t\ge 0}\{E[|x(t;\omega)|^2]\}^{1/2} \le \rho,\quad t \ge 0,$$

Provided that: λ_1, λ_2, $\lambda_3, \|x_0(\omega)\|_C, \| h(t,0)\|_C \| f_1(t,0)\|_C$, and $\| f_2(t,0)\|_C$ are sufficiently small.

Proof:

It will suffice to show that the pair of spaces (C,C) is admissible with respect to the integral operators defined by (3.4), (3.5), (3.6) under conditions (i), (ii) and (iii). In [10], El-Borai et al. proved that (C,C) is admissible with respect to T_1, T_2, so we need to prove that (C,C) is admissible with respect to T_3 as follow: Let $x(t;\omega) \in C$. then from (3.6) we have that

$$\|(T_3\, x)(t;\omega)\|^2 \le \left\|\frac{1}{\Gamma(\alpha)}\int_0^t \boldsymbol{K_2}(t,\tau;\omega)x(\tau;\omega)d\beta(\tau)\right\|^2$$
$$\le \frac{1}{[\Gamma(\alpha)]^2}\int_0^t \|\boldsymbol{K_2}(t,\tau;\omega)x(\tau;\omega)\|^2 dF(\tau)$$
$$\le \frac{1}{[\Gamma(\alpha)]^2}\int_0^t |||\boldsymbol{K_2}(t,\tau;\omega)|||^2 . \|x(\tau;\omega)\|^2 dF(\tau)$$
$$\le \frac{\Lambda_2\Lambda_3}{(2\alpha-1)\,[\Gamma(\alpha)]^2}\,\|x(t;\omega)\|_C^2 < \infty.$$

This implies that $sup_{t\ge 0}\|(T_3\, x)(t;\omega)\|$ is bounded, which implies $(T_3\, x)(t;\omega) \in C$ and thus, (C,C) is admissible with respect to the operators T_3. Therefore, the conditions of theorem (3.1) hold with $B = C, g(t) = 1$, and $D = C$, and then, there exists a unique random solution $x(t;\omega)$ of (1.1), (1.2), which is bounded in the mean square by ρ for all $t \in R_+$. and hence, $\sup_{t\ge 0}\|x(t;\omega)\| \le \rho$.

5. Application to a Stochastic Fractional Feedback System.

Consider the following nonlinear stochastic fractional differential system:

$$dy(t;\omega) = \Pi(\omega)y(t;\omega) + b_1(\omega)\Phi_1(t,\sigma(t;\omega))dt + b_2(\omega)\Phi_2(t,\sigma(t;\omega))d\beta(t) \quad (5.1)$$

$$\frac{\partial^\alpha \sigma(t;\omega)}{\partial t^\alpha} = C^T(t;\omega)y(t;\omega), \quad (5.2)$$

with the following initial conditions

$$\sigma(0;\omega) + \sum_{i=1}^{p} c_i\sigma(t_i,\omega) = \sigma_0(\omega),$$
$$y(0;\omega) = y_0(\omega)$$

where $0 < \alpha \le 1, t \in R_+ = [0,\infty), 0 < t_1 < \ldots < t_p < \infty$. The fractional derivative is provided by the Caputo derivative. $\Pi(\omega)$ is an $n \times n$ matrix of measurable functions, $x(t;\omega)$ and $C(t;\omega)$ are $n \times 1$ vectors of random variables for each $t \in R_+$, $b_i(\omega), i = 1,2$, is an $n \times 1$ vector of random variables, $\sigma(t;\omega)$ is a scalar random variable for each $t \in R_+$, $\Phi_i(t,\sigma), i = 1,2$, is a scalar function for each $t \in R_+$, and T denotes the transpose of a matrix. $\beta(t)$ is a standard Brownian motion.

Now integrating (5.1), we have

$$y(t;\omega) = e^{\Pi(\omega)t}y(0;\omega) + \int_0^t e^{\Pi(\omega)(t-\tau)}b_1(\omega)\Phi_1(\tau,\sigma(\tau;\omega))d\tau + \int_0^t e^{\Pi(\omega)(t-\tau)}b_2(\omega)\Phi_2(\tau,\sigma(\tau;\omega))d\beta(\tau) \quad (5.3)$$

Substituting from (5.3) into (5.2), we obtain

$$\frac{\partial^\alpha \sigma(t;\omega)}{\partial t^\alpha} = c^T(t;\omega)e^{\Pi(\omega)t}\, y_0(\omega)$$

$$+\int_0^t e^{\Pi(\omega)(t-\tau)}c^T(t;\omega)\mathrm{b}_1(\omega)\Phi_1(\tau,\sigma(\tau;\omega))d\tau$$
$$+\int_0^t e^{\Pi(\omega)(t-\tau)}c^T(t;\omega)\,\mathrm{b}_2(\omega)\Phi_2(\tau,\sigma(\tau;\omega))d\beta(\tau) \qquad (5.4)$$

Now assume that $|||c^T(t;\omega)||| \le K_1$ for all $t \ge 0$ and $K_1 > 0$ a constant. Also, let $y_0(\omega) \in C$, and $\mathrm{b_i}(\omega) \in L_\infty(\Omega,\mathcal{A},P\,), i = 1,2$, if we assume that the matrix $\Pi(\omega)$ is stochastically stable, that is, there exist an $\propto\, > 0$ such that

$$P\{\omega: Re\,\psi_k(\omega) < -\propto, \qquad k = 1,2,\dots\dots\dots,n\} = 1,$$

where $\psi_k(\omega)$, $k = 1,2,\dots\dots\dots,n$, are the characteristic roots of the matrix $\Pi(\omega)$, then it has been shown by Morozan [16] that

$$\left|\left|\left|e^{\Pi(\omega)t}\right|\right|\right| \le K_2 e^{-\propto t} < K_2$$

for some constant $K_2 > 0$. We also let $\Phi_\mathrm{i}(t,\sigma(t;\omega)) \in C_c(R_+, L_2(\Omega,\mathcal{A},P\,)), i = 1,2$ for each $t \in R_+$, and

$$\|\Phi_\mathrm{i}(t,\sigma_1(t;\omega)) - \Phi_\mathrm{i}(t,\sigma_2(t;\omega))\| \le \lambda_i\|\sigma_1(t;\omega) - \sigma_2(t;\omega)\|.$$

where $\lambda_i > 0$, $i = 1,2$, is a constant, also let

$$\mathrm{h}(t,\sigma(t;\omega)) = c^T(t;\omega)e^{\Pi(\omega)t}\,y_0(\omega)$$

then

$$\|\mathrm{h}(t,\sigma(t;\omega))\| \le |||c^T(t;\omega)|||.\left|\left|\left|e^{\Pi(\omega)t}\right|\right|\right|.\|y_0(\omega)\| \le ZK_1K_2e^{-\propto t} \le ZK_1K_2$$

where $Z > 0$ is a constant, since $y_0(\omega) \in C$.

Thus, by definition $\mathrm{h}(t,\sigma(t;\omega)) \in C$, also, since h does not depend on σ, then

$$\|\mathrm{h}(t,\sigma_1(t;\omega)) - \mathrm{h}(t,\sigma_2(t;\omega))\| = 0$$

that is, it satisfies a Lipschitz condition.

Now, by the assumptions on $c^T(t;\omega)$, $\mathrm{b}(\omega)$, and $\Pi(\omega)$, we have

$$k_1(s,\tau;\omega) = e^{\Pi(\omega)(s-\tau)}c^T(s;\omega)\mathrm{b}_1(\omega)$$

Satisfying

$$|||k_1(s,\tau;\omega)||| \le \left|\left|\left|e^{\Pi(\omega)(s-\tau)}\right|\right|\right|\,|||c^T(s;\omega)|||\,|||\mathrm{b}_1(\omega)||| \le K_1K_2e^{-\propto(s-\tau)}\,|||\mathrm{b}_1(\omega)||| \le K_1K_2|||\mathrm{b}_1(\omega)|||$$

and

$$k_2(s,\tau;\omega) = e^{\Pi(\omega)(s-\tau)}c^T(s;\omega)\,\mathrm{b}_2(\omega)$$

satisfying

$$\int_\tau^t |||k_2(s,\tau;\omega)|||^2 ds \le K_1^2\,K_2^2|||\mathrm{b}_2(\omega)|||^2\int_\tau^t ds = K_1^2\,K_2^2|||\mathrm{b}_2(\omega)|||^2(t-\tau) < \infty$$

Moreover,

$$\int_0^t (t-\tau)^{2\alpha-1}\,dF(\tau) = a\int_0^t (t-\tau)^{2\alpha-1}\,d\tau = \frac{a}{2\alpha}\,t^{2\alpha} < \infty.$$

Therefore, all conditions of theorem (4.2) are satisfied, and hence, there exists a unique random solution of the system (5.1), (5.2) which is bounded in the mean square on R_+.

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