On Some Partial Differential Equations with Operator Coefficients and Non-local Conditions

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Abstract: In this paper we are going to study the partial differential equation $L\frac{\partial^k u}{\partial t^k} = \sum_{j=1}^{k} L_{ij}\frac{\partial^{k-j} u}{\partial t^{k-j}}$. With the non-local condition $D_t^j u(x,0) = f_j(x) \quad ; j = 0,1,...,k-1$

Where;

- L is an elliptic partial differential operator,
- $L_{ij}; j = 1,...,k$ is a family of partial differential operator with bounded operator coefficient in a suitable functional space, and

$$f_o(x) = \phi_o(x) + \sum_{i=1}^{p}\alpha_{0i}u(t_i)$$

$$f_1(x) = \phi_1(x) + \sum_{m=1}^{q}\alpha_{1m}u(t_m)$$

$$\vdots$$

$$f_{k-1}(x) = \phi_{k-1}(x) + \sum_{l=1}^{s}\alpha_{k-1l}\, u(t_l)$$

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1. Introduction

Consider the equation

$$\sum_{|q|=2m} a_q(t)D^q D_t^k u = \sum_{j=1}^{k}\sum_{|q|=2m} A_{q,j}(t)D^q D_t^{k-j}u, \qquad (1)$$

with the non-local conditions

$$D_t^j u(x,t)|_{t=0} = f_j(x) \quad ; j = 0,1,...,k-1 \qquad (2)$$

where;

- $q = (q_1,...,q_n)$ is an n-tuple of non negative integers,
- $|q| = q_1 + ... + q_n$,
- $D^q = \frac{\partial^{|q|}}{\partial x_1^{q_1}...\partial x_n^{q_n}}$, $D_t = \frac{\partial}{\partial t}$,
- m, k are positive integers.

Let us suppose that:

- $a_q(t); |q| = 2m;$ are continuous functions of $t \in [0,1]$,
- For every $t \in [0,1]$, $\sum_{|q|=2m} a_q(t)D^q$ is an elliptic operator.
- $A_{q,j}; |q| = 2m, j = 1,...,k;$ are linear bounded operators from $L_2(E_n)$ into itself, for every $t \in [0,1]$.
- $A_{q,j}; |q| = 2m, j = 1,...,k;$ are strongly continuous in $t \in [0,1]$.

Assume that; $W^{2m}(E_n)$ is the space of all functions $f \in L_2(E_n)$ such that the "Distributional derivatives" $D^q f$ with $|q| < 2m$ all belong to $L_2(E_n)$ [1].

In the present work we are going to find a solution u of (1)(2),that mean

- $u \in W^{2m}(E_n)$, for every $t \in (0,1)$,
- $D_t^j u; j = 0,1,..., k;$ exist for every $t \in (0,1)$ and belong to $W^{2m}(E_n)$.
- u satisfies (1) and the initial condition (2).

Also, the uniqueness of the solution is proved.

2. Theorem

If $\phi_j \in W^{2m}(E_n); j = 0,1,...,k-1; 4m > n;$ there exist a unique solution u of the non-local Cauchy problem (1), (2) in the space $W^{2m}(E_n)$.

Proof. The differential operator $D^q; |q| = 2m;$ can be written as [2]

$$D^q f = R^q \nabla^{2m} f \quad ; f \in W^{2m}(E_n) \tag{3}$$

where;

$\nabla^2 = D_1^2 + ... + D_n^2$,

$R^q = R_1^{q_1}...R_n^{q_n}$; R_j are the Riesz transform defined by

$$R_j f = -i\pi^{\frac{-(n+1)}{2}} \Gamma(\frac{n+1}{2}) \int_{E_n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)dy,$$

Γ is the gamma function and $|x| = x_1^2 + ... + x_n^2$.

from (4) at (1)

$$\sum_{|q|=2m} a_q(t) R^q \nabla^{2m} D_t^k u = \sum_{j=1}^{k} \sum_{|q|=2m} A_{q,j}(t) R^q \nabla^{2m} D_t^{k-j} u \tag{4}$$

Let

$\nabla^{2m} u = v, \qquad \nabla^{2m} f_j = g_j$

$\sum_{|q|=2m} a_q(t) R^q = H_o(t), \quad \sum_{|q|=2m} A_{q,j}(t) R^q = H_j(t).$

Thus, formally we have

$$H_o(t) D_t^k v = \sum_{j=1}^{k} H_j(t) D_t^{k-j} v \tag{5}$$

But, $\sum_{|q|=2m} a_q(t) D^q$ is an elliptic operator that is $H_o(t)$ has a unique bounded inverse $H_o^{-1}(t)$ from $L_2(E_n)$ to itself for every $t \in [0,1]$. Applying $H_o^{-1}(t)$ to both sides of (5)

$$D_t^k v = \sum_{j=1}^{k} H_o^{-1}(t) H_j(t) D_t^{k-j} v \tag{6}$$

Also, since R_j are bounded in $L_2(E_n)$ then $H_j(t); j = 1,...,k;$ are bounded operator in $L_2(E_n)$ for each $t \in [0,1]$.

Now, consider the square matrix

$$A(t) = \begin{bmatrix} H_1^*(t) & H_2^*(t) & \cdots & H_{k-1}^*(t) & H_k^*(t) \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}$$

where; $H_j^*(t) = H_o^{-1} H_j(t) \quad ; j = 1,2,\cdots,k$,

I is the identity operator.

Let

$$V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_k \end{bmatrix} \tag{7}$$

where; $V_j = D_t^{k-j} v \quad ; v = \nabla^{2m} u$.

Thus, equation (6) can be written as follows

$$\frac{dV(t)}{dt} = A(t)V(t) \tag{8}$$

and the initial conditions

$D_t^j v|_{t=0} = g_j(x)$

i.e,

$V_{k-j}(0) = g_j$

thus

$$V(0) = \begin{bmatrix} g_{k-1} \\ g_{k-2} \\ \vdots \\ g_0 \end{bmatrix} = G \tag{9}$$

where;

$$g_o = \nabla^{2m} f_o = \nabla^{2m}[\phi_o(x) + \sum_{i=1}^{p} \alpha_{0i} u(t_i)]$$

$$\vdots$$

$$g_{k-1} = \nabla^{2m} f_{k-1} = \nabla^{2m}[\phi_{k-1}(x) + \sum_{l=1}^{s} \alpha_{k-1l}\, u(t_l)]$$

Assume that B is the space of column vectors V with norm

$$\|V\| = \sum_{i=1}^{k} \|V_j\|_{L_2(E_n)}; \|f\|_{L_2(E_n)} = (\int_{E_n} (f(x))^2 dx)^{1/2}$$

i.e,

$$\|V\| = \sum_{j=1}^{k} (\int_{E_n} V_j^2(x) dx)^{1/2}$$

So, it will be easy to prove that B is a banach space and $A(t)$ is a linear bounded operator B from B into itself for each $t \in [0,1]$.

Also, from the conditions on $a_q(t)$ and $A_{q,j}(t)$ we can show that $A(t)$ is strictly continuous on [0,1]. Notice that $g_j \in L_2(E_n)$ which implies $G \in B$.

As in [3], the cauchy problem (8) and (9) has the solution

$$V(t) = Q(t)G \quad ; \forall t \in (0,1) \tag{10}$$

Where;

$Q(t)$ is a unique bounded operator in the Banach space B

$$Q(t)=\begin{bmatrix} Q_{11}(t) & Q_{12}(t) & \cdots & Q_{1k} \\ Q_{21}(t) & Q_{22}(t) & \cdots & Q_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{k1}(t) & Q_{k2}(t) & \cdots & Q_{kk} \end{bmatrix} \qquad (11)$$

$Q_{rs}(t); r=1,\cdots,k$, $s=1,\cdots,k;$ are bounded operators in the space $L_2(E_n)$ for every $t\in[0,1]$.

Since $V(0)=Q(0)G=G$. Then, we can say that

$$Q_{rs}(0)=\begin{cases} I & ;r=s \\ 0 & ;r\neq s \end{cases} \qquad (12)$$

Now, let us re-write the initial condition as follow

$$\begin{aligned} f_o(x) &= \phi_o(x)+\sum_{j=n_o+1}^{n_1} c_j u_j \\ f_1(x) &= \phi_1(x)+\sum_{j=n_1+1}^{n_2} c_j u_j \\ &\vdots \\ f_{k-1}(x) &= \phi_{k-1}(x)+\sum_{j=n_{k-1}+1}^{n_k} c_j u_j \end{aligned} \qquad (13)$$

Where;

- $n_o=0, n_1=p, n_2=n_1+q,\cdots,n_k=n_{k-1}+s.$
- $c_j=\begin{cases} \alpha_{1i} & ;n_o+1\le j\le n_1 & ;i=j-n_o \\ \alpha_{2m} & ;n_1+1\le j\le n_2 & ;m=j-n_1 \\ \vdots & & \\ \alpha_{k-1l} & ;n_{k-1}+1\le j\le n_k & ;l=j-n_{k-1} \end{cases}$
- $t_j=\begin{cases} t_i & ;n_o+1\le j\le n_1 & ;i=j-n_o \\ t_m & ;n_1+1\le j\le n_2 & ;m=j-n_1 \\ \vdots & & \\ t_l & ;n_{k-1}+1\le j\le n_k & ;l=j-n_{k-1} \end{cases}$

From (7) we have

$$V_r(x,t)=D_t^{k-r}\nabla^{2m}u(x,t).$$

From (10) and (11)

$$V_r(x,t)=\sum_{s=1}^{k}Q_{rs}(t)g_{k-s} =\sum_{s=1}^{k}Q_{rs}(t)\nabla^{2m}f_{k-s}.$$

Thus;

$$\begin{aligned} D_t^{k-r}\nabla^{2m}u(x,t) &= \sum_{s=1}^{k}Q_{rs}(t)\nabla^{2m}f_{k-s} \\ \nabla^{2m}u(x,t) &= \sum_{s=1}^{k}Q_{ks}(t)\nabla^{2m}f_{k-s} \end{aligned} \qquad (14)$$

From (14)

$$\nabla^{2m}u(x,t)=\sum_{s=1}^{k}Q_{ks}(t)\nabla^{2m}\ [\,\phi_{k-s}+Q_{sk}(0)\sum_{j=n_o+1}^{n_1}c_ju_j +Q_{s\,k-1}(0)\sum_{j=n_1+1}^{n_2}c_ju_j+\cdots+Q_{s1}(0)\sum_{j=n_{k-1}+1}^{n_k}c_ju_j\,]$$

This can be written as

$$\nabla^{2m}u(x,t)=\sum_{s=1}^{k}Q_{ks}(t)\nabla^{2m}\,[\,\phi_{k-s} +\sum_{i=0}^{k-1}Q_{s\,k-i}(0)\sum_{j=n_i+1}^{n_{i+1}}c_ju_j\,] \qquad (15)$$

$$\nabla^{2m}u(t_l)=\sum_{s=1}^{k}Q_{ks}(t_l)\nabla^{2m}\,[\,\phi_{k-s}+\sum_{i=0}^{k-1}Q_{s\,k-i}(0)\sum_{j=n_i+1}^{n_{i+1}}c_ju_j\,]$$

$$\nabla^{2m}c_lu_l=c_l\sum_{s=1}^{k}Q_{ks}(t_l)\nabla^{2m}\,[\,\phi_{k-s}+\sum_{i=0}^{k-1}Q_{s\,k-i}(0)\sum_{j=n_i+1}^{n_{i+1}}c_ju_j\,]$$

$$\sum_{l=n_i+1}^{n_{i+1}}\nabla^{2m}c_lu_l=\sum_{l=n_i+1}^{n_{i+1}}c_l\sum_{s=1}^{k}Q_{ks}(t_l)\nabla^{2m}\phi_{k-s} +\sum_{l=n_i+1}^{n_{i+1}}c_l\sum_{s=1}^{k}Q_{ks}(t_l)\nabla^{2m}\sum_{i=0}^{k-1}Q_{s\,k-i}(0)\sum_{j=n_i+1}^{n_{i+1}}c_ju_j.$$

Now, since ∇^{2m} is linear operator

$$\nabla^{2m}(\sum_{l=n_i+1}^{n_{i+1}}c_lu_l)=\sum_{l=n_i+1}^{n_{i+1}}c_l\sum_{s=1}^{k}Q_{ks}(t_l)\nabla^{2m}\phi_{k-s} +\sum_{l=n_i+1}^{n_{i+1}}c_l\sum_{s=1}^{k}Q_{ks}(t_l)\nabla^{2m}\sum_{i=0}^{k-1}Q_{s\,k-i}(0)\sum_{j=n_i+1}^{n_{i+1}}c_ju_j.$$

Since, $Q_{s\,k-i}$ is either the identity or the zero operator. i.e,

$$\nabla^{2m}Q_{s\,k-i}(0)=Q_{s\,k-i}(0)\nabla^{2m}$$

Then,

$$\nabla^{2m}(\sum_{j=n_i+1}^{n_{i+1}}c_ju_j)=\sum_{l=n_i+1}^{n_{i+1}}\sum_{s=1}^{k}c_lQ_{ks}(t_l)\nabla^{2m}\phi_{k-s} +\sum_{l=n_i+1}^{n_{i+1}}\sum_{s=1}^{k}c_lQ_{ks}(t_l)\sum_{i=0}^{k-1}Q_{s\,k-i}(0)\nabla^{2m}\sum_{j=n_i+1}^{n_{i+1}}c_ju_j$$

$$[I - \sum_{l=n_i+1}^{n_{i+1}} \sum_{s=1}^{k} c_l Q_{ks}(t_l) \sum_{i=0}^{k-1} Q_{s\,k-i}(0)] \nabla^{2m} (\sum_{j=n_i+1}^{n_{i+1}} c_j u_j) = \sum_{l=n_i+1}^{n_{i+1}} \sum_{s=1}^{k} c_l Q_{ks}(t_l) \nabla^{2m} \phi_{k-s} .$$

Set

$$\Lambda = I - \sum_{l=n_i+1}^{n_{i+1}} \sum_{s=1}^{k} c_l Q_{ks}(t_l) \sum_{i=0}^{k-1} Q_{s\,k-i}(0)$$

$$\Lambda \nabla^{2m} \sum_{j=n_i+1}^{n_{i+1}} c_j u_j = \sum_{l=n_i+1}^{n_{i+1}} \sum_{s=1}^{k} c_l Q_{ks}(t_l) \nabla^{2m} \phi_{k-s} \qquad (16)$$

We can easily prove that $\sum_{s=1}^{k} Q_{ks}(t_l) \sum_{i=0}^{k-1} Q_{s\,k-i}(0)$ is bounded operator (see [4]). i.e,

$$\exists M > 0 \qquad s.t. \quad \| \sum_{s=1}^{k} Q_{ks}(t_l) Q_{sk}(0) \| < M . \qquad (17)$$

Assume that

$$CM < 1$$

where;

$$C = \sum_{l=n_i+1}^{n_{i+1}} | c_l | \qquad (18)$$

By using (17) and (18) we find that the inverse operator Λ^{-1} exist. Applying Λ^{-1} on (17)

$$\nabla^{2m} \sum_{j=n_i+1}^{n_{i+1}} c_j u_j = \Lambda^{-1} [\sum_{l=n_i+1}^{n_{i+1}} \sum_{s=1}^{k} c_l Q_{ks}(t_l) \nabla^{2m} \phi_{k-s}]$$

$$\sum_{j=n_i+1}^{n_{i+1}} c_j u_j = (\nabla^{2m})^{-1} \Lambda^{-1} [\sum_{l=n_i+1}^{n_{i+1}} \sum_{s=1}^{k} c_l Q_{ks}(t_l) \nabla^{2m} \phi_{k-s}] . \qquad (19)$$

Where;

$(\nabla^{2m})^{-1}$ is a closed operator defined on $L_2(E_n)$ and representing the inverse of ∇^{2m}.

From (20) at (16)

$$\nabla^{2m} u(x,t) = \sum_{s=1}^{k} Q_{ks}(t) \nabla^{2m} [\phi_{k-s} + \sum_{i=0}^{k-1} Q_{s\,k-i}(0) (\nabla^{2m})^{-1} \Lambda^{-1} [\sum_{l=n_i+1}^{n_{i+1}} \sum_{s=1}^{k} c_l Q_{ks}(t_l) \nabla^{2m} \phi_{k-s}]]$$

Thus;

$$u(x,t) = (\nabla^{2m})^{-1} \sum_{s=1}^{k} Q_{ks}(t) \nabla^{2m} [\phi_{k-s} + \sum_{i=0}^{k-1} Q_{s\,k-i}(0) (\nabla^{2m})^{-1} \Lambda^{-1} [\sum_{l=n_i+1}^{n_{i+1}} \sum_{s=1}^{k} c_l Q_{ks}(t_l) \nabla^{2m} \phi_{k-s}]] \qquad (20)$$

Now, we are going to prove that the formula (20) which we have obtained in a formal way is in fact the required solution of the problem (1)and (2) in the space $W^{2m}(E_n)$.

Since $(\nabla^{2m})^{-1}$ is a closed operator from $L_2(E_n)$ onto $W^{2m}(E_n)$, it follows immediately from (20) that $u \in W^{2m}(E_n) \quad \forall t \in [0,1]$. Notice that, the differential operator $\frac{d}{dt}$ in (8) is the abstract derivative with respect to t in the space $L_2(E_n)$.

Also, since $\frac{d}{dt} (\nabla^{2m})^{-1} \frac{d}{dt} f_t \quad ; f_t \in L_2(E_n)$ then from (20) we have

$$\frac{d^{k-r}}{dt^{k-r}} u = (\nabla^{2m})^{-1} \sum_{s=1}^{k} \frac{d^{k-r}}{dt^{k-r}} Q_{ks}(t) \nabla^{2m} [\phi_{k-s} + \sum_{i=0}^{k-1} Q_{s\,k-i}(0) (\nabla^{2m})^{-1} \Lambda^{-1} [\sum_{l=n_i+1}^{n_{i+1}} \sum_{s=1}^{k} c_l Q_{ks}(t_l) \nabla^{2m} \phi_{k-s}]]$$

From (14)

$$\frac{d^{k-r}}{dt^{k-r}} u = (\nabla^{2m})^{-1} \sum_{s=1}^{k} Q_{rs}(t) \nabla^{2m} [\phi_{k-s} + \sum_{i=0}^{k-1} Q_{s\,k-i}(0) (\nabla^{2m})^{-1} \Lambda^{-1} [\sum_{l=n_i+1}^{n_{i+1}} \sum_{s=1}^{k} c_l Q_{ks}(t_l) \nabla^{2m} \phi_{k-s}]] \qquad (21)$$

Which prove that

$$\frac{d^{k} - r}{dt^{k-r}} u \in W^{2m}(E_n) \qquad ; r = 1, \cdots, k \quad \text{forall} \quad t \in (0,1)$$

In [5], it is proved that if $u, \frac{du}{dt} \in W^{2m}(E_n)$ and $\frac{d}{dt} D^q u \in L_2(E_n), |q| = 2m, 4m > n$. Then, the partial derivatives $D_t u$ exists in the usual sense and that it is identical to the corresponding abstract derivative.

Although, since these conditions are valid by u in (10). By the same way we can find that the partial derivatives $D_t^j u;\ j = 1,2,\cdots,k;$ exist in the usual sense for all $t \in [0,1], x \in E_n$ and that they are identical to the corresponding abstract derivatives.

From (20) we can see that

$$\nabla^{2m} u(x,0) = \sum_{s=1}^{k} Q_{ks}(0) \nabla^{2m} [\phi_{k-s} + \sum_{i=0}^{k-1} Q_{s\,k-i}(0) (\nabla^{2m})^{-1} \Lambda^{-1} [\sum_{l=n_i+1}^{n_{i+1}} \sum_{s=1}^{k} c_l Q_{ks}(t_l) \nabla^{2m} \phi_{k-s}]]$$

By using of (12)

$$\nabla^{2m} u(x,0) = \nabla^{2m} [\, \phi_o + \sum_{i=0}^{k-1} Q_{k\,k-i}(0)(\nabla^{2m})^{-1} \Lambda^{-1} \sum_{l=n_i+1}^{n_{i+1}} \sum_{s=1}^{k} c_l\, Q_{ks}(t_l) \nabla^{2m} \phi_{k-s} \,]$$

Thus;

$$\nabla^{2m} u(x,0) = \nabla^{2m} [\, \phi_o + (\nabla^{2m})^{-1} \Lambda^{-1} \sum_{l=n_o+1}^{n_1} \sum_{s=1}^{k} c_l\, Q_{ks}(t_l) \nabla^{2m} \phi_{k-s} \,] \quad (22)$$

From (16)

$$u(x,0) = \phi_o + \sum_{j=n_0+1}^{n_1} c_j u_j$$

Similarly, we can prove that

$$D_t u(x,0) = \phi_1 + \sum_{j=n_1+1}^{n_2} c_j u_j$$

$$D_t^{k-1} u(x,0) = \phi_{k-1} + \sum_{j=n_{k-1}+1}^{n_k} c_j u_j \quad .$$

Which complete the proof, (see [6] [7] [8] [9] [10] [11] [12] [13]).

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