Developable Ruled Surfaces with Darboux Frame in Minkowski 3-Space

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Abstract: In this paper, we obtain the distribution parameter of a ruled surface generated by a straight line in Darboux trihedron moving along two different curves with the same parameter. Besides, we give necessary and suficient conditions for this ruled surface to become developable in Minkowski 3-space.
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1. Introduction

Ruled surfaces are the surfaces generated by a continuously moving of a straight line in the space. These surfaces are one of the most important topics of differential geometry. Ruled surfaces, particularly developable surfaces, have been widely studied and applied in mathematics and engineering. The generation and machining of ruled surfaces play an important role in design and manufacturing of products and many other areas. Because of this position of ruled surfaces, many geometers have studied on these surfaces in both Euclidean space and Minkowski space [6, 7, 8, 10, 11]. A ruled surface in Minkowski 3-space E_1^3 is (locally) the map $M_{(\alpha,\delta)} : I \times \mathrm{P} \to E_1^3$ defined by

$$M_{(\alpha,\delta)}(s,v) = \alpha(s) + v\delta(s)$$

where $\alpha : I \to E_1^3$, $\delta : I \to E_1^3 \setminus \{0\}$ are smooth mappings and I is an open interval or the unit circle S^1, we call α the base curve and δ the director curve. The straight lines $v \to \alpha(s) + v\delta(s)$ are called rullings. The ruled surface $M_{(\alpha,\delta)}$ is called developable if the Gaussian curvature of the regular part of $M_{(\alpha,\delta)}$ vanishes. This is equivalent to the fact that $M_{(\alpha,\delta)}$ is developable if and only if the distinguished parameter

$$P_\delta = \frac{\det(\alpha', \delta, \delta')}{\langle \delta', \delta' \rangle} = 0.$$

In[2], S. Izumiya and N. Takeuchi[8] studied a special type of ruled surface with Darboux vector. They called the ruled surface rectifying developable surface of the space curve. Turgut and Hac salihoglu[2] have studied timelike ruled surfaces in Minkowski 3-space and given some properties of these surfaces. Y. Yayl and S. Saracoglu[10] study developable ruled surfaces in Minkowski 3-space and give necessary and sufficient condition ruled surface to become developable. Besides, Kızıltuğ and Yaylı [9] investigated curves on tubular surface by using Darboux frame. In this paper, making use of method in paper of Y. Yayl and S. Saracoglu, we study developable ruled surfaces with Darboux Frame.

2. Preliminaries

Let E_1^3 be a Minkowski 3-space with natural Lorentz Metric

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. According to this metric, in E_1^3 an arbitrary vector $\vec{v} = (v_1, v_2, v_3)$ can have one of three Lorentzian causal characters; it can be spacelike if

$\langle \vec{v}, \vec{v} \rangle > 0$ or $\vec{v} = 0$, timelike if $\langle \vec{v}, \vec{v} \rangle < 0$ and null (lightlike) if $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ is spacelike, timelike or null (lightlike), if all of its velocity vectors $\vec{\alpha}'(s)$ are spacelike, timelike or null (lightlike), respectively[3]. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. For any vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in E_1^3, the vector product of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \times \vec{y} = (x_2 y_3 - x_3 y_2, x_1 y_3 - x_3 y_1, x_2 y_1 - x_1 y_2).$$

A surface in the Minkowski 3-space E_1^3 is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector[3].

Let S be an oriented surface in E_1^3 and let consider a non-null curve $\alpha(s)$ lying fully on S. Since the curve $\alpha(s)$ is also in the space, there exists a Frenet frame $\{T, N, B\}$ along the curve where T is unit tangent vector, N is principal normal vector and B is binormal vector, respectively. Moreover, since the curve $\alpha(s)$ lies on the surface S there exists another frame along the curve $\alpha(s)$. This frame is called Darboux frame and denoted by $\{T, Y, Z\}$ which gives us an opportunity to investigate the properties of the curve according to the surface. In this frame T is the unit tangent of the curve, Z is the unit normal of the surface S along $\alpha(s)$ and Y is a unit vector given by $Y = \pm Z \times T$. Since the unit tangent T is common in both Frenet frame and Darboux frame, the vectors N, B, Y and Z lie on the same plane. So that the relations between these frames can be given as follows:

If the surface M is an oriented timelike surface, the relations between the frames can be given as follows

If the curve $\alpha(s)$ is timelike

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

If the curve $\alpha(s)$ is spacelike

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

If the surface M is an oriented spacelike surface, then the curve $\alpha(s)$ lying on M is a spacelike curve. So, the relations between the frames can be given as follows

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

In all cases, θ is the angle between the vectors Y and N [3, 5].

According to the Lorentzian causal characters of the surface M and the curve $\alpha(s)$ lying on M, the derivative formulae of the Darboux frame can be changed as follows:

i) If the surface M is a timelike surface, then the curve $\alpha(s)$ lying on M can be a spacelike or a timelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by

$$\begin{bmatrix} T' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & k_g & -\varepsilon k_n \\ k_g & 0 & \varepsilon t_r \\ k_n & t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix},$$

$$\langle T,T\rangle = \varepsilon = \pm 1, \langle Y,Y\rangle = -\varepsilon, \langle Z,Z\rangle = 1. \tag{1}$$

ii) If the surface M is a spacelike surface, then the curve $\alpha(s)$ lying on M is a spacelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by

$$\begin{bmatrix} T' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & t_r \\ k_n & t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix},$$

$$\langle T,T\rangle = 1, \langle Y,Y\rangle = 1, \langle Z,Z\rangle = -1. \tag{2}$$

In these formulae k_g, k_n and t_r are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively[1]. The relations between geodesic curvature, normal curvature,geodesic torsion and κ, τ are given as follows

if both M and $\alpha(s)$ are timelike or spacelike

$$k_g = \kappa\cos\theta, k_n = \kappa\sin\theta, t_r = \tau + \theta', \tag{3}$$

if M timelike and $\alpha(s)$ is spacelike

$$k_g = \kappa\cosh\theta, k_n = \kappa\sinh\theta, t_r = \tau + \theta' \tag{4}$$

where $\theta = (N\hat{,}Z)$ the angle function is between the unit normal and binormal to $\alpha(s)$.

In the differential geometry of surfaces, for a curve $\alpha(s)$ lying on a surface M the followings are well-known[1],

i) $\alpha(s)$ is a geodesic curve $\Leftrightarrow k_g = 0$,

ii) $\alpha(s)$ is an asymptotic line $\Leftrightarrow$ $k_n = 0$,

iii) $\alpha(s)$ is a principal line $\Leftrightarrow t_r = 0$.

3. Developable Ruled Surfaces with Darboux Frame in Minkowski 3-Space

Let $\alpha : I \to E_1^3$ be a curve in Minkowski 3-space and $\{T, Y, Z\}$ be a Darboux frame, where In this frame T is the unit tangent of the curve, Z is the unit normal of the surface M along $\alpha(s)$ and Y is a unit vector given by $Y = \pm Z \times T$. As we have said above, with the assistance of α, we can define curve β.

$$\beta : I \to E_1^3$$

with the same parameter of the curve $\alpha(s)$ and such that

$$\beta' = aT + bY + cZ. \tag{5}$$

And also, we can get the ruled surface that produced during the curve $\beta(s)$ with each fixed line δ of the moving space H as:

$$M_{(\beta,\delta)}(s,v)=\beta(s)+v\delta(s).$$

Let δ be a fixed unit vector. Thus

$$\delta \in Sp\{T, Y, Z\} \text{ and } \delta = \lambda_1 T + \lambda_2 Y + \lambda_3 Z \tag{6}$$

Then the distribution parameter of the ruled surface for the curve β can be given as:

$$P_\delta = \frac{\det(\beta', \delta, \delta')}{\langle \delta', \delta' \rangle}. \tag{7}$$

We can obtain the distribution parameter of the ruled surface generated by line δ of the moving space H. Different cases can be investigated as following:

3.1. The Ruled Surface $M_{(\beta,\delta)}$ and the curve β is timelike

Let the ruled surface be timelike and β be timelike. By taking derivative of (6) with respect to s and by using Darboux formulas (1) we have

$$\delta' = (\lambda_2 k_g + \lambda_3 k_n)T + (\lambda_1 k_g + \lambda_3 t_r)Y + (\lambda_1 k_n - \lambda_2 t_r)Z. \tag{8}$$

Besides,

$$\langle \delta', \delta' \rangle = -(\lambda_2 k_g + \lambda_3 k_n)^2 + (\lambda_1 k_g + \lambda_3 t_r)^2 + (\lambda_1 k_n - \lambda_2 t_r)^2. \tag{9}$$

So, From (5), (8) and (9) we obtain P_δ

$$P_\delta = \frac{a(-\lambda_2^2 t_r - \lambda_3^2 t_r - \lambda_1\lambda_3 k_g + \lambda_1\lambda_2 k_n) + b(-\lambda_1^2 k_n + \lambda_3^2 k_n + \lambda_1\lambda_2 t_r + \lambda_2\lambda_3 k_g) + c(\lambda_1^2 k_g - \lambda_2^2 k_g - \lambda_2\lambda_3 k_n + \lambda_1\lambda_3 t_r)}{-(\lambda_2 k_g + \lambda_3 k_n)^2 + (\lambda_1 k_g + \lambda_3 t_r)^2 + (\lambda_1 k_n - \lambda_2 t_r)^2}.$$

For $a=1,\ b=c=0,$ the distribution parameter P_δ is given as follows

$$P_\delta = \frac{-\lambda_2^2 t_r - \lambda_3^2 t_r - \lambda_1\lambda_3 k_g + \lambda_1\lambda_2 k_n}{-(\lambda_2 k_g + \lambda_3 k_n)^2 + (\lambda_1 k_g + \lambda_3 t_r)^2 + (\lambda_1 k_n - \lambda_2 t_r)^2}. \tag{10}$$

For $b=1,\ a=c=0,$ the distribution parameter P_δ is given as follows

$$P_\delta = \frac{-\lambda_1^2 k_n + \lambda_3^2 k_n + \lambda_1\lambda_2 t_r + \lambda_2\lambda_3 k_g}{-(\lambda_2 k_g + \lambda_3 k_n)^2 + (\lambda_1 k_g + \lambda_3 t_r)^2 + (\lambda_1 k_n - \lambda_2 t_r)^2}. \tag{11}$$

For $c=1,\ a=b=0,$ the distribution parameter P_δ is given as follows

$$P_\delta = \frac{\lambda_1^2 k_g - \lambda_2^2 k_g - \lambda_2\lambda_3 k_n + \lambda_1\lambda_3 t_r}{-(\lambda_2 k_g + \lambda_3 k_n)^2 + (\lambda_1 k_g + \lambda_3 t_r)^2 + (\lambda_1 k_n - \lambda_2 t_r)^2}. \tag{12}$$

Now we investigate some different cases for $c=1,\ a=b=0.$
Firstly, from (12) we can give the following lemma:

Lemma 1: The ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed line δ of the moving
space H is developable if and only if

$$\frac{t_r}{\kappa} = \frac{\cos\theta(\lambda_2^2 - \lambda_2^2) - \sin\theta(\lambda_2\lambda_3)}{-\lambda_1\lambda_3}.$$

Proof: The ruled surface $M_{(\beta,\delta)}$ is developable,Then

$$P_\delta = \frac{\lambda_1^2 k_g - \lambda_2^2 k_g - \lambda_2\lambda_3 k_n + \lambda_1\lambda_3 t_r}{-(\lambda_2 k_g + \lambda_3 k_n)^2 + (\lambda_1 k_g + \lambda_3 t_r)^2 + (\lambda_1 k_n - \lambda_2 t_r)^2} = 0 \quad (13)$$

from (13), we get

$$\lambda_1^2 k_g - \lambda_2^2 k_g - \lambda_2\lambda_3 k_n = -\lambda_1\lambda_3 t_r. \quad (14)$$

If we replace $k_g = \kappa\cos\theta$ and $k_n = \kappa\sin\theta$ in Eq. (14), we get

$$\frac{t_r}{\kappa} = \frac{\cos\theta(\lambda_2^2 - \lambda_2^2) - \sin\theta(\lambda_2\lambda_3)}{-\lambda_1\lambda_3}.$$

3.1.1. Special Cases

1. The Case: $\delta = T$

In this case, $\lambda_1 = 1,\ \lambda_2 = \lambda_3 = 0$. Thus, from Eq. (12)

$$P_\delta = \frac{\lambda_1^2 k_g}{(\lambda_1 k_g)^2 + (\lambda_1 k_n)^2}.$$

If $P_\delta = 0$, then $k_g = 0.$ Thus, $\alpha(s)$ and also $\beta(s)$ geodesic curve.Hence the following proposition holds:

Proposition 2: Let $\delta = T$. Then the ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed line δ of the moving space H is developable if and only if $\alpha(s)$ and also $\beta(s)$ geodesic curve.

2. The Case: $\delta = Y$

In this case, $\lambda_2 = 1,\ \lambda_1 = \lambda_3 = 0$. Thus, from Eq. (12)

$$P_\delta = \frac{-\lambda_2^2 k_g}{-(\lambda_2 k_g)^2 + (\lambda_2 k_n)^2}.$$

If $P_\delta = 0$, then $k_g = 0$. Thus, $\alpha(s)$ and also $\beta(s)$ geodesic curve.Hence the following proposition holds

Proposition 3: Let $\delta = Y$. Then the ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed
line δ of the moving space H is developable if and only if $\alpha(s)$ and also $\beta(s)$ geodesic curve.

3. The Case: $\delta = Z$

In this case, $\lambda_3 = 1,\ \lambda_1 = \lambda_2 = 0$. Thus, from Eq. (12)

$$P_\delta = 0.$$

Thus we can easily see the ruled surface $M_{(\beta,\delta)}$ is developable

Proposition 4: Let $\delta = Z.$ Then the ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed line δ of the moving space H is developable.

4. The Case: δ is in the $T-Y$ plane

In this case, $\lambda_3 = 0$. Thus, from Eq. (12)

$$P_\delta = \frac{\lambda_1^2 k_g - \lambda_2^2 k_g}{-(\lambda_2 k_g)^2 + (\lambda_1 k_g)^2 + (\lambda_1 k_n - \lambda_2 t_r)^2}.$$

If $P_\delta = 0,$ then $k_g = 0$. Thus, $\alpha(s)$ and also $\beta(s)$ geodesic curve.Hence the following proposition holds

Proposition 5: The ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed line δ in the $T-Y$ plane of the moving space H is developable if and only if $\alpha(s)$ and also $\beta(s)$ geodesic.

5. The Case: δ is in the $T-Z$ plane

In this case, $\lambda_2 = 0$. Thus, from Eq. (12)

$$P_\delta = \frac{\lambda_1^2 k_g + \lambda_1 \lambda_3 t_r}{-(\lambda_3 k_n)^2 + (\lambda_1 k_g + \lambda_3 t_r)^2 + (\lambda_1 k_n)^2}.$$

If $P_\delta = 0,$ then $k_g = t_r = 0.$ Thus, $\alpha(s)$ and also $\beta(s)$ both geodesic curve and principal line.Hence the following proposition holds:

Proposition 6: The ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed line δ in the $T-Z$ plane of the moving space H is developable if and only if $\alpha(s)$ and also $\beta(s)$ both geodesic curve and principal line.

6. The Case: δ is in the $Y-Z$ plane

In this case, $\lambda_1 = 0$. Thus, from Eq. (12)

$$P_\delta = \frac{-\lambda_2^2 k_g - \lambda_2 \lambda_3 k_n}{-(\lambda_2 k_g + \lambda_3 k_n)^2 + (\lambda_3 t_r)^2 + (\lambda_2 t_r)^2}.$$

If $P_\delta = 0,$ then $k_g = k_n = 0$. Thus, $\alpha(s)$ and also $\beta(s)$ both geodesic curve and asymptotic curve. Hence the following proposition holds.

Proposition 7: The ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed line δ in the $Y-Z$ plane of the moving space H is developable if and only if $\alpha(s)$ and also $\beta(s)$ both geodesic curve and asymptotic curve.

3.2. The Ruled Surface $M_{(\beta,\delta)}$ and the curve β is spacelike

Let the Ruled surface be spacelike and β be spacelike. By taking derivative of (6) with respect to s and by using Darboux formulas (1) we have

$$\delta' = \left(-\lambda_2 k_g + \lambda_3 k_n\right)T + \left(\lambda_1 k_g + \lambda_3 t_r\right)Y + \left(\lambda_1 k_n + \lambda_2 t_r\right)Z. \tag{15}$$

Besides,

$$\langle \delta', \delta' \rangle = \left(-\lambda_2 k_g + \lambda_3 k_n\right)^2 + \left(\lambda_1 k_g + \lambda_3 t_r\right)^2 - \left(\lambda_1 k_n + \lambda_2 t_r\right)^2. \tag{16}$$

So, from (5), (15) and (16) we obtain P_δ

$$P_\delta = \frac{a\left(\lambda_2^2 t_r - \lambda_3^2 t_r - \lambda_1\lambda_3 k_g + \lambda_1\lambda_2 k_n\right) + b\left(-\lambda_1^2 k_n + \lambda_3^2 k_n - \lambda_1\lambda_2 t_r - \lambda_2\lambda_3 k_g\right) + c\left(\lambda_1^2 k_g + \lambda_2^2 k_g - \lambda_2\lambda_3 k_n + \lambda_1\lambda_3 t_r\right)}{\left(-\lambda_2 k_g + \lambda_3 k_n\right)^2 + \left(\lambda_1 k_g + \lambda_3 t_r\right)^2 - \left(\lambda_1 k_n + \lambda_2 t_r\right)^2}.$$

For $a = 1, \quad b = c = 0,$ the distribution parameter P_δ is given as follows

$$P_\delta = \frac{\lambda_2^2 t_r - \lambda_3^2 t_r - \lambda_1\lambda_3 k_g + \lambda_1\lambda_2 k_n}{\left(-\lambda_2 k_g + \lambda_3 k_n\right)^2 + \left(\lambda_1 k_g + \lambda_3 t_r\right)^2 - \left(\lambda_1 k_n + \lambda_2 t_r\right)^2}. \tag{17}$$

For $b = 1, \quad a = c = 0,$ the distribution parameter P_δ is given as follows

$$P_\delta = \frac{-\lambda_1^2 k_n + \lambda_3^2 k_n - \lambda_1\lambda_2 t_r - \lambda_2\lambda_3 k_g}{\left(-\lambda_2 k_g + \lambda_3 k_n\right)^2 + \left(\lambda_1 k_g + \lambda_3 t_r\right)^2 - \left(\lambda_1 k_n + \lambda_2 t_r\right)^2}. \tag{18}$$

For $c = 1, \quad a = b = 0,$ the distribution parameter P_δ is given as follows

$$P_\delta = \frac{\lambda_1^2 k_g + \lambda_2^2 k_g - \lambda_2\lambda_3 k_n + \lambda_1\lambda_3 t_r}{\left(-\lambda_2 k_g + \lambda_3 k_n\right)^2 + \left(\lambda_1 k_g + \lambda_3 t_r\right)^2 - \left(\lambda_1 k_n + \lambda_2 t_r\right)^2}. \tag{19}$$

Now we investigate some different cases for $c = 1, \quad a = b = 0.$
Firstly, from (19) we can give the following lemma

Lemma 8: The ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed line δ of the moving space H is developable if and only if

$$\frac{t_r}{\kappa} = \frac{\sin\theta\left(-\lambda_1^2 + \lambda_3^2\right) - \cos\theta\left(\lambda_2\lambda_3\right)}{-\lambda_1\lambda_2}.$$

Proof: The ruled surface $M_{(\beta,\delta)}$ is developable, Then

$$P_\delta = \frac{-\lambda_1^2 k_n + \lambda_3^2 k_n - \lambda_1\lambda_2 t_r - \lambda_2\lambda_3 k_g}{\left(-\lambda_2 k_g + \lambda_3 k_n\right)^2 + \left(\lambda_1 k_g + \lambda_3 t_r\right)^2 - \left(\lambda_1 k_n + \lambda_2 t_r\right)^2} = 0. \tag{20}$$

From (20), we get

$$-\lambda_1^2 k_n + \lambda_3^2 k_n - \lambda_2\lambda_3 k_g = \lambda_1\lambda_2 t_r. \tag{21}$$

If we replace $k_g = \kappa\cos\theta$ and $k_n = \kappa\sin\theta$ in Eq. (21), we get

$$\frac{t_r}{\kappa} = \frac{\sin\theta\left(-\lambda_1^2 + \lambda_3^2\right) - \cos\theta\left(\lambda_2\lambda_3\right)}{-\lambda_1\lambda_2}.$$

3.2.1. Special Cases

1. The Case: $\delta = T$

In this case, $\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = 0.$ Thus from Eq. (19),

$$P_\delta = \frac{-\lambda_1^2 k_n}{(\lambda_1 k_g)^2 - (\lambda_1 k_n)^2}$$

If $P_\delta = 0,$ then $k_n = 0.$ Thus, $\alpha(s)$ and also $\beta(s)$ asymptotic curve. Hence the following proposition holds:

Proposition 9: Let $\delta = T.$ Then the ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed line δ of the moving space H is developable if and only if $\alpha(s)$ and also $\beta(s)$ asymptotic curve.

2. The Case: $\delta = Y$

In this case, $\lambda_2 = 1, \ \lambda_1 = \lambda_3 = 0.$ Thus, from Eq. (19)

$$P_\delta = 0.$$

Thus, we can easily see the ruled surface $M_{(\beta,\delta)}$ is developable

3. The Case: $\delta = Z$

In this case, $\lambda_3 = 1, \ \lambda_1 = \lambda_2 = 0.$ Thus, from Eq. (19)

$$P_\delta = \frac{\lambda_3^2 k_n}{(\lambda_3 k_n)^2 + (\lambda_3 t_r)^2}.$$

If $P_\delta = 0,$ then $k_n = 0$. Thus, $\alpha(s)$ and also $\beta(s)$ asymptotic curve. Hence the following proposition holds:

Proposition 10: Let $\delta = Z.$ Then the ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed line δ of the moving space H is developable if and only if $\alpha(s)$ and also $\beta(s)$ asymptotic curve.

4. The Case: δ is in the $T-Y$ plane

In this case, $\lambda_3 = 0.$ Thus, from Eq. (19)

$$P_\delta = \frac{-\lambda_1^2 k_n - \lambda_1\lambda_2 t_r}{(-\lambda_2 k_g)^2 + (\lambda_1 k_g)^2 - (\lambda_1 k_n + \lambda_2 t_r)^2}.$$

If $P_\delta = 0,$ then $k_n = t_r = 0$. Thus, $\alpha(s)$ and also $\beta(s)$ both asymptotic curve and principal line.Hence the following proposition holds.

Proposition 11: The ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed line δ in the $T-Y$ plane of the moving space H is developable if and only if $\alpha(s)$ and also $\beta(s)$ asymptotic curve and principal line.

5. The Case: δ is in the $T-Z$ plane

In this case, $\lambda_2 = 0.$ Thus, from Eq. (19)

$$P_\delta = \frac{-\lambda_1^2 k_n + \lambda_3^2 k_n}{(\lambda_3 k_n)^2 + (\lambda_1 k_g + \lambda_3 t_r)^2 - (\lambda_1 k_n)^2}.$$

If $P_\delta = 0,$ then $k_n = 0$. Thus, $\alpha(s)$ and also $\beta(s)$ asymptotic curve.Hence the following proposition holds:

Proposition 12: The ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed line δ in the $T-Y$ plane of the moving space H is developable if and only if $\alpha(s)$ and also $\beta(s)$ asymptotic curve.

6. The Case: δ is in the $Y-Z$ plane

In this case, $\lambda_1 = 0.$ Thus, from Eq. (19)

$$P_\delta = \frac{\lambda_3^2 k_n - \lambda_2 \lambda_3 k_g}{\left(-\lambda_2 k_g + \lambda_3 k_n\right)^2 + \left(\lambda_1 k_g + \lambda_3 t_r\right)^2 - \left(\lambda_1 k_n + \lambda_2 t_r\right)^2}$$

If $P_\delta = 0,$ then $k_g = k_n = 0.$ Thus, $\alpha(s)$ and also $\beta(s)$ both geodesic curve and asymptotic curve.Hence the following proposition holds:

Proposition 13: The ruled surface $M_{(\beta,\delta)}$ that produced during the curve $\beta(s)$ with each fixed line δ in the $Y-Z$ plane of the moving space H is developable if and only if $\alpha(s)$ and also $\beta(s)$ both geodesic curve and asymptotic curve

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