On some nonlocal perturbed random fractional integro-differential equations

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Abstract: Some classes of stochastic fractional integro-differential equations involving nonlocal initial condition are investigated. The theory of admissibility of integral operator and Banach fixed-point principle are used to establish the existence and uniqueness of stochastic solution. The boundedness and asymptotic behavior of the stochastic solution as $t \to \infty$ are also studied. In addition, an application to fractional differential systems with random parameters is presented.
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1. Introduction

Many investigations have been carried out concerning the existence and uniqueness of solution of deterministic and stochastic integro-differential equations of Volterra type. See [1]-[5].However, in the last few decades, many authors pointed out that fractional models are very suitable for the description of properties of various real materials, e.g. polymers. It has been shown that new fractional-order models are more adequate than previously integer-order models. See [6]-[7]. In many cases it is better to have more initial information to obtain a good description of the evolution of a physical system. The local initial condition is replaced then by a nonlocal condition, which gives better effect than the initial condition, since the measurement given by a nonlocal condition is usually more precise than the only one measurement given by a local condition, see [8]-[9]. Therefore, in this paper we shall be concerned with extending the results in William J.padgett and Chris P. Tsokos [5].That is we shall consider a nonlinear random perturbed factional integro-differential equation of Volterra type of the form:

$$\frac{\partial^\alpha x(t;\omega)}{\partial t^\alpha} = h\big(t, x(t;\omega)\big) + \int_0^t k(t,\tau;\omega)\, f\big(\tau, x(\tau;\omega)\big)d\tau$$

With the nonlocal condition

$$x(0;\omega) + \sum_{i=1}^{p} c_i x(t_i,\omega) = x_0(\omega)$$

Where

$0 < \alpha \le 1,\ t \in R_+ = [0,\infty),\ 0 < t_1 < \dots < t_p < \infty,$ the fractional derivative is provided by the Caputo derivative and

(i) $\omega \in \Omega$, the supporting set of a probability measure space $(\Omega, \mathcal{A}, P)$;

(ii) $x(t;\omega)$ is the unknown stochastic process for $t \in R_+$;

(iii) $h(t,x)$ is called the stochastic perturbing term and it is a scalar function of $t \in R_+$and scalar $x \in R$;

(iv) $k(t,\tau;\omega)$ is a stochastic kernal defined for t and τ satisfying $0 \le \tau \le t < \infty$; and

(v) $f(t,x)$ is a scalar functions of $t \in R_+$, scalar $x \in R$ and will be specified later.

The purpose of this paper is to obtain the conditions concerning the stochastic process in equation (1.1) which guarantee the existence and uniqueness of random solution $x(t;\omega)$ and to investigate the asymptotic statistical behavior of such a random solution. In addition, the usefulness of the results will be illustrated with an application to fractional stochastic differential systems. We shall utilize the spaces of functions and admissibility theory which were introduced into the study of random integral equations by Tsokos [10].The nonlocal Cauchy problem $(1.1), (1.2)$ has applications in many fields such as viscoelasticity, fluid mechanics and electromagnetic theory. See for example [11].

2. Preliminaries.

Let $(\Omega, \mathcal{A}, P)$ denote a probability measure space, that is Ω is a nonempty set known as the sample space, $\mathcal{A}$ is a sigma-algebra of subsets of Ω, and P is a complete probability measure on $\mathcal{A}$ .We let $x(t;\omega), t \in R_+ , \omega \in \Omega$, denote a stochastic process whose index set is R_+. Let $L_2(\Omega, \mathcal{A}, P)$ be the space of all random variables $x(t;\omega), t \in R_+$ which have a second moment (or square- summable) with respect to P-measure for each $t \in R_+$.That is:

$$E\{|x(t;\omega)|^2\} = \int_\Omega |x(t;\omega)|^2 \, dp(\omega) < \infty .$$

The norm of $x(t;\omega)$ in $L_2((\Omega,\mathcal{A},P))$ is defined for each $t \in R_+$by:

$$\|x(t;\omega)\| = \|x(t;\omega)\|_{L_2((\Omega,\mathcal{A},P))}$$

$$= [E\{|x(t;\omega)|^2\}]^{1/2} = \left\{\int_\Omega |x(t;\omega)|^2 dP(\omega)\right\}^{\frac{1}{2}}$$

With respect to the functions in equation (1.1), we make the following assumptions:

The random solution $x(t;\omega)$ will be considered as a function of $t \in R_+$with values in the space $L_2(\Omega,\mathcal{A},P)$. The functions $h(t,x), f(t,x)$ under convenient conditions will be functions of $t \in R_+$ with values in $L_2(\Omega,\mathcal{A},P)$.

Let $L_\infty(\Omega,\mathcal{A},P)$ be the space of all measurable and P -essentially bounded random variables of $\omega \in \Omega$. With respect to the stochastic kernel, we will assume that, for each t and τ satisfying $0 \le \tau \le t < \infty$, $k(t,\tau;\omega)$ is essentially bounded with respect to P, So that the product of $k(t,\tau;\omega)$ and $f(\tau,x(\tau;\omega))$ will always be in $L_2(\Omega,\mathcal{A},P)$ for each fixed t and τ. The norm of $k(t,\tau;\omega)$ in $L_\infty(\Omega,\mathcal{A},P)$ will be denoted and defined by

$$|||k(t,\tau;\omega)||| = P - ess\ sup_{\omega\in\Omega}|k(t,\tau;\omega)|.$$

Also the mapping $(t,\tau) \to k(t,\tau;\omega)$ from the set $\Delta = \{(t,\tau): 0 \le \tau \le t < \infty\}$ into $L_\infty(\Omega,\mathcal{A},P)$ is continuous.

And further, whenever $\tau_n \to \tau$ as $n \to \infty$, $P - ess\ sup_{\omega\in\Omega}|k(s,\tau_n;\omega) - k(s,\tau;\omega)| \to 0$ as $n \to \infty$. It will be assumed also that for each fixed t and τ,

$$P - ess\ sup_{\omega\in\Omega}|k(s,\tau;\omega)| \le M_{(t,\tau)}$$

Uniformly for $\tau \le s \le t$, Where $M_{(t,\tau)} > 0$ is some constant depending only on t and τ, $0 \le \tau \le t < \infty$.

Definition 2.1.

We define the space $C_c = C_c(R_+,L_2(\Omega,\mathcal{A},P))$ to be the space of all continuous functions from R_+ into $L_2(\Omega,\mathcal{A},P)$ and define a topology on C_c by means of the following family of seminorms

$$\|x(t;\omega)\|_n = sup_{0\le t\le n}\|x(t;\omega)\|,\ n = 1,2,\dots\dots$$

It is known that such a topology is metrizable and that the metric space C_c is complete.

Definition 2.2.

We define the space $C_g = C_g(R_+,L_2(\Omega,\mathcal{A},P))$ to be the space of all continuous functions from R_+ into $L_2(\Omega,\mathcal{A},P)$ such that there exists a constant $a > 0$ and a positive continuous function $g(t)$ on R_+ satisfying

$$\|x(t;\omega)\|_{L_2(\Omega,\mathcal{A},P)} \le a\, g(t)$$

The norm in $C_g(R_+,L_2(\Omega,\mathcal{A},P))$ will be defined by:

$$\|x(t;\omega)\|_{C_g} = sup_{t\in R_+}\left\{\frac{\|x(t;\omega)\|_{L_2((\Omega,\mathcal{A},P))}}{g(t)}\right\}$$

Definition 2.3.

We define the space $C = C(R_+,L_2(\Omega,\mathcal{A},P))$ to be the space of all continuous and bounded functions on $R_+$with values in $L_2(\Omega,\mathcal{A},P)$. that is C is the space of all second order stochastic processes on R_+ which are bounded and continuous in mean square. The norm in C is defined by:

$$\|x(t;\omega)\|_C = sup_{t\in R_+}\left\{\|x(t;\omega)\|_{L_2((\Omega,\mathcal{A},P))}\right\} < \infty$$

It is clear that C, C_g are Banach spaces and the following inclusion hold: $C \subset C_g \subset C_c$.

Finally, let $B, D \subset C_c(R_+,L_2(\Omega,\mathcal{A},P))$ be Banach spaces and let T be a linear operator from $C_c(R_+,L_2(\Omega,\mathcal{A},P))$ into itself. Now we give the following definitions with respect to B, D, and T.

Definition 2.4.

The pair of Banach spaces (B,D) is said to be admissible with respect to the operator T if and only if $T(B) \subset D$.

Definition 2.5.

The Banach space B is said to be stronger than the space C_c if every convergent sequence in B, with respect to its norm, will also converge in C_c.(but the converse is not true in general).

Definition 2.6.

We call $x(t;\omega)$ a random solution equation (1.1) if $x(t;\omega) \in C_c$ for each $t \in R_+$, satisfies the equation (1.1) for every $t > 0$ and satisfies the nonlocal initial condition, almost surely.

Definition 2.7.

The random solution $x(t;\omega)$ is stochastically exponentially stable if there exist constants $\gamma > 0$ and $\beta > 0$ such that for each $t \ge 0$,

$$\|x(t;\omega)\| \le \beta e^{-\gamma t}.$$

We now state the following lemma which is given by Tsokos in [4].

Lemma 2.1.

Let T be a continuous linear operator from $C_c(R_+, L_2(\Omega, \mathcal{A}, P))$ into itself. If B and D are Banach spaces stronger than C_c and if (B, D) is admissible with respect to T, then T is a continuous linear operator from B into D.

Note that:

If the operator $T: B \to D$ is continuous, then it is bounded, and there exists a constant $K_0 > 0$ such that

$$\|(Tx)(t;\omega)\|_D \leq K_0 \|x(t;\omega)\|_B$$

The infimum of such constants K_0 is called the norm of the operator T.

3. Main results

Using the definitions of the fractional derivatives and integrals, it is suitable to rewrite the considered problem in the form:

$$x(t;\omega) = x(0;\omega) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h(\tau, x(\tau;\omega))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t \int_0^s (t-s)^{\alpha-1} k(s,\tau;\omega) f(\tau, x(\tau;\omega))d\tau ds$$

Changing the order of integration

$$x(t;\omega) = x(0;\omega) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h(\tau, x(\tau;\omega))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t \boldsymbol{K}(t,\tau;\omega) f(\tau, x(\tau;\omega))d\tau \quad (3.1)$$

Where

$$\boldsymbol{K}(t,\tau;\omega) = \int_\tau^t (t-s)^{\alpha-1} k(s,\tau;\omega)ds \quad (3.2)$$

Now define the integral operators T_1 and T_2 on $C_c(R_+, L_2(\Omega, \mathcal{A}, P))$ as follows:

$$(T_1 x)(t;\omega) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} x(\tau;\omega)d\tau \quad (3.3)$$

$$(T_2 x)(t;\omega) = \frac{1}{\Gamma(\alpha)}\int_0^t \boldsymbol{K}(t,\tau;\omega) x(\tau;\omega)d\tau \quad (3.4)$$

Now we shall prove two lemmas concerning the continuity of T_1 and T_2 as mappings from $C_c(R_+, L_2(\Omega, \mathcal{A}, P))$ into itself.

Lemma 3.1.

The operator T_1 defined by the equation (3.3) is a continuous mapping from the space C_c into itself.

Proof:

Step1, we shall show that $T_1: C_c \to C_c$.

First let $x(t;\omega) \in C_c$, then

$$\|(T_1 x)(t;\omega)\| \leq \frac{1}{\Gamma(\alpha)}\|x(t;\omega)\| \int_0^t (t-\tau)^{\alpha-1}d\tau = \frac{M_t t^\alpha}{\Gamma(\alpha+1)} < \infty$$

For each $t \in R_+$, Since $x(t;\omega)$ is continuous function on the interval $[0,t]$ and hence bounded by some M_t.

Thus $(T_1 x)(t;\omega) \in L_2(\Omega, \mathcal{A}, P)$ each $t \in R_+$.

Secondly let $0 \leq t_1 < t_2 < \infty$ then

$$\|(T_1 x)(t_2;\omega) - (T_1 x)(t_1;\omega)\|$$
$$= \frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1} [(t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1}] x(\tau;\omega)d\tau + \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} x(\tau;\omega)d\tau\right\|$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\|x(t;\omega)\| [t_2{}^\alpha - t_1{}^\alpha - 2(t_2-t_1)^\alpha] \to 0 \quad as\ t_1 \to t_2.$$

Hence $T_1 x$ is a continuous function at each $t \in R_+$ with values in $L_2(\Omega, \mathcal{A}, P)$.

That is, continuous in mean square on R_+.

then $T_1: C_c \to C_c$.

Step2 we shall show that $T_1: C_c \to C_c$ is a continuous operator as follow:

Let $x_n(t;\omega) \to x(t;\omega)$ in C_c as $n \to \infty$.

Then for $t \in R_+$, then

$$(T_1 x_n)(t;\omega) - (T_1 x)(t;\omega) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[x_n(\tau;\omega) - x(\tau;\omega)]d\tau$$

$$\|(T_1 x_n)(t;\omega) - (T_1 x)(t;\omega)\| \leq \frac{1}{\Gamma(\alpha)}\int_0^Q (Q-\tau)^{\alpha-1}\|x_n(\tau;\omega) - x(\tau;\omega)\| d\tau \leq \frac{Q^\alpha}{\Gamma(\alpha+1)}\|x_n(t;\omega) - x(t;\omega)\|$$

For $t \in [0,Q]$ but by definition as $n \to \infty$

$$\|x_n(t;\omega) - x(t;\omega)\| < \frac{\varepsilon\Gamma(\alpha+1)}{Q^\alpha}$$

Uniformly in $[0,Q]$, $0 < Q < \infty$. therefore, For $\varepsilon > 0$ there exists an $n_0 \in Z_+$ such that $n > n_0$ implies that $\|(T_1 x_n)(t;\omega) - (T_1 x)(t;\omega)\| < \varepsilon$

Then $T_1: C_c \to C_c$ is a continuous operator, Hence the required result.

Lemma 3.2.

The operator T_2 defined by equation (3.4) is a continuous mapping from the space C_c into itself.

Proof:

Step1, we shall show that $T_2 : C_c \to C_c$

First we must show that the function $\boldsymbol{K}(t,\tau;\omega)$ given by the equation (3.2) belongs to the space $L_\infty(\Omega,\mathcal{A},P)$ and is a continuous mapping from the set $\Delta = \{(t,\tau): 0 \le \tau \le t < \infty\}$ into $L_\infty(\Omega,\mathcal{A},P)$ as follow:

For fixed t and τ satisfying $0 \le \tau \le t < \infty$, and by using the assumptions on k, then

$$|||\boldsymbol{K}(t,\tau;\omega)||| \le \int_\tau^t (t-s)^{\alpha-1} |||k(s,\tau;\omega)||| ds$$

$$\le M_{(t,\tau)} \frac{(t-\tau)^\alpha}{\alpha}$$

Hence $\boldsymbol{K}(t,\tau;\omega) \in L_\infty(\Omega,\mathcal{A},P)$ for each t and τ satisfying $0 \le \tau \le t < \infty$.

Now let $\{(t_n,\tau_n)\}$ be a sequence in Δ such that $(t_n,\tau_n) \to (t,\tau)$ as $n \to \infty$. then

$$|||\boldsymbol{K}(t_n,\tau_n;\omega) - \boldsymbol{K}(t,\tau;\omega)|||$$

$$\le \left|\left|\left| \int_{\tau_n}^{t} (t_n - s)^{\alpha-1} k(s,\tau_n;\omega)\, ds \right|\right|\right|$$

$$+ \left|\left|\left| \int_{\tau_n}^{\tau} (t - s)^{\alpha-1} k(s,\tau;\omega)\, ds \right|\right|\right|$$

$$+ \left|\left|\left| \int_{\tau_n}^{t} [(t_n - s)^{\alpha-1} - (t-s)^{\alpha-1}][k(s,\tau_n;\omega) - k(s,\tau;\omega)]\, ds \right|\right|\right|$$

Since $\tau_n \to \tau$ as $n \to \infty$ and $|||k(s,\tau;\omega)|||$ is continuous in τ then, for $\varepsilon > 0$ there exist an $N_1 \in Z_+$ such that $n > N_1$ implies

$$\left|\left|\left| \int_{t_n}^{t} (t_n - s)^{\alpha-1} k(s,\tau_n;\omega)\, ds \right|\right|\right| \le M_{(\tau,t)} \int_{t_n}^{t} (s - t_n)^{\alpha-1} ds \le M_{(\tau,t)} \frac{(t-t_n)^\alpha}{\alpha} < \frac{\varepsilon}{3}$$

(Since $t_n \to t$ as $n \to \infty$), then $(t - t_n)^\alpha < \frac{\alpha\varepsilon}{3M_{(\tau,t)}}$

Also there exist an $N_2 \in Z_+$ such that $n > N_2$ implies that

$$|||k(s,\tau_n;\omega) - k(s,\tau;\omega)||| < \frac{\alpha\varepsilon}{6(t-\tau)^\alpha}$$

Uniformly in s for $\tau_n \le s \le t < \infty$. then

$$\left|\left|\left| \int_{\tau_n}^{t} [(t_n - s)^{\alpha-1} - (t-s)^{\alpha-1}]\, [k(s,\tau_n;\omega) - k(s,\tau;\omega)]\, ds \right|\right|\right|$$

$$\le \int_{\tau_n}^{t} [(t_n - s)^{\alpha-1} + (t-s)^{\alpha-1}] . |||k(s,\tau_n;\omega) - k(s,\tau;\omega)||| ds$$

$$< \frac{\alpha\varepsilon}{6(t-\tau)^\alpha} \int_{\tau_n}^{t} [(t_n - s)^{\alpha-1} + (t-s)^{\alpha-1}]\, ds$$

$$= \frac{\alpha\varepsilon}{6(t-\tau)^\alpha} \left[\frac{2(t-\tau)^\alpha}{\alpha}\right] = \frac{\varepsilon}{3}$$

Since $(t_n,\tau_n) \to (t,\tau)$ as $n \to \infty$. then

$$\left|\left|\left| \int_{\tau_n}^{t} [(t_n - s)^{\alpha-1} - (t-s)^{\alpha-1}]\, [k(s,\tau_n;\omega) - k(s,\tau;\omega)]\, ds \right|\right|\right| < \frac{\varepsilon}{3}$$

Similarly, there exist an $N_3 \in Z_+$ such that $n > N_3$ implies that

$$\left|\left|\left| \int_{\tau_n}^{\tau} (t - s)^{\alpha-1} k(s,\tau;\omega)\, ds \right|\right|\right| < \frac{\varepsilon}{3}$$

Hence there exist an $N = \max\{N_1, N_2, N_3\}$ so that for $n > N$ we have

$$|||\boldsymbol{K}(t_n,\tau_n;\omega) - \boldsymbol{K}(t,\tau;\omega)||| < \varepsilon$$

That is the mapping $\boldsymbol{K}: \Delta \to L_\infty(\Omega,\mathcal{A},P)$ is continuous. Now, since $\boldsymbol{K}(t,\tau;\omega) \in L_\infty(\Omega,\mathcal{A},P)$ for each t and τ satisfying $0 \le \tau \le t < \infty$, we have that for each $x(t;\omega) \in C_c$, the product $\boldsymbol{K}(t,\tau;\omega)\, x(t;\omega)$ is in $L_2(\Omega,\mathcal{A},P)$. Thus,

$$\|(T_2 x)(t;\omega)\|$$

$$\le \frac{1}{\Gamma(\alpha)} \int_0^t |||\boldsymbol{K}(t,\tau;\omega)|||.\, \|x(\tau;\omega)\| d\tau < \infty,$$

Since $|||\boldsymbol{K}(t,\tau;\omega)|||$ and $\|x(\tau;\omega)\|$ are continuous in τ on the interval $[0,t]$ and are therefore bounded on $[0,t]$. Then $(T_2 x)(t;\omega) \in L_2(\Omega,\mathcal{A},P)$.

Now, let $0 \le t_1 < t_2 < \infty$ then by the continuity condition on $\boldsymbol{K}(t,\tau;\omega)$ we obtain

$$\|(T_2 x)(t_2;\omega) - (T_2 x)(t_1;\omega)\|$$

$$\le \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} |||\boldsymbol{K}(t_2,\tau;\omega) - \boldsymbol{K}(t_1,\tau;\omega)|||.\, \|x(\tau;\omega)\| d\tau \right\|$$

$$+ \frac{1}{\Gamma(\alpha)} \left\| \int_{t_1}^{t_2} |||\boldsymbol{K}(t_2,\tau;\omega)|||.\, \|x(\tau;\omega)\| d\tau \right\| \to 0$$

As $n \to \infty$. thus $(T_2 x)(t;\omega)$ is continuous in the mean square for each $t \in R_+$. Then $T_2 : C_c \to C_c$.

Step2 we will show that $T_2 : C_c \to C_c$ is continuous operator as follow:

Let $x_n(t;\omega) \to x(t;\omega)$ in C_c as $n \to \infty$. then

$$\|(T_2 x_n)(t;\omega) - (T_2 x)(t;\omega)\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t |||\boldsymbol{K}(t,\tau;\omega)|||.\|x_n(\tau;\omega)-x(\tau;\omega)\|\,d\tau$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^Q |||\boldsymbol{K}(t,\tau;\omega)|||.\|x_n(\tau;\omega)-x(\tau;\omega)\|\,d\tau$$

Where $t \leq Q < \infty$

Since $|||\boldsymbol{K}(t,\tau;\omega)|||$ is continuous in (t,τ), it is bounded by some $M_Q > 0$ in the compact region $\{(t,\tau): 0 \leq t \leq Q, 0 \leq \tau \leq Q\}$.

By definition, we have for $\varepsilon > 0$ that there exists an $N \in Z_+$ so that for $n > N$,
$$\|x_n(\tau;\omega) - x(\tau;\omega)\| < \frac{\varepsilon\Gamma(\alpha)}{Q\,M_Q}$$
Uniformly in $[0,Q]$, and hence for $n > N$,
$$\|(T_2x_n)(t;\omega) - (T_2x)(t;\omega)\| < \varepsilon$$

For all $t \in [0,Q]$, where $t \leq Q < \infty$. Therefore, $(T_2x_n)(t;\omega) \to (T_2x)(t;\omega)$ in C_c as $n \to \infty$, That is $T_2: C_c \to C_c$ is continuous operator, Hence the required result.

Lemma 3.3.

Assume that $\sum_{i=1}^{p} c_i \neq -1$, then the nonlocal Cauchy problem $(1.1)(1.2)$ is equivalent to the following integral equation.

$$x(t;\omega) = \mathrm{A}x_0(\omega)$$
$$-\mathrm{A}\left(\sum_{i=1}^{p} c_i\,[(T_{1i}hx)(t_i,\omega) + (T_{2i}fx)(t_i,\omega)]\right) + (T_1hx)(t;\omega) + (T_2fx)(t;\omega)$$

Where: $\mathrm{A} = \left[1 + \sum_{i=1}^{p} c_i\right]^{-1}$, T_1 and T_2 are defined by $(3.3), (3.4)$ and

$$(T_{1i}x)(t_i;\omega) = \frac{1}{\Gamma(\alpha)}\int_0^{t_i}(t_i-\tau)^{\alpha-1}x(\tau;\omega)d\tau$$
$$(T_{2i}x)(t_i;\omega) = \frac{1}{\Gamma(\alpha)}\int_0^{t_i}\boldsymbol{K}(t_i,\tau;\omega)x(\tau;\omega)\,d\tau$$
$i = 1,2,3\ldots\ldots\ldots.p$

Proof:

Let $t = t_i$ in (3.1), multiplying both sides by c_i and taking $\sum_{i=1}^{p}$, then
$$\sum_{i=1}^{p} c_i\,x(t_i;\omega) = x(0;\omega)\sum_{i=1}^{p} c_i$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{p}\left[c_i\int_0^{t_i}(t_i-\tau)^{\alpha-1}h\big(\tau,x(\tau;\omega)\big)d\tau\right]$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{p}\left[c_i\int_0^{t_i}\boldsymbol{K}(t_i,\tau;\omega)f\big(\tau,x(\tau;\omega)\big)d\tau\right] \quad (3.5)$$

Substitute from (1.2) into (3.5) then
$$x(0;\omega) = \mathrm{A}x_0(\omega)$$
$$-\frac{\mathrm{A}}{\Gamma(\alpha)}\sum_{i=1}^{p}\left[c_i\int_0^{t_i}(t_i-\tau)^{\alpha-1}h\big(\tau,x(\tau;\omega)\big)d\tau\right]$$
$$-\frac{\mathrm{A}}{\Gamma(\alpha)}\sum_{i=1}^{p}\left[c_i\int_0^{t_i}\boldsymbol{K}(t_i,\tau;\omega)f\big(\tau,x(\tau;\omega)\big)d\tau\right] \quad (3.6)$$
Substitute from (3.6) into (3.1), then
$$x(t;\omega) = \mathrm{A}x_0(\omega)$$
$$-\mathrm{A}\sum_{i=1}^{p} c_i\,[(T_{1i}hx)(t_i,\omega) + (T_{2i}fx)(t_i,\omega)]$$
$$+(T_1hx)(t;\omega) + (T_2fx)(t;\omega) \quad (3.7)$$
Hence the required result.

We now prove the following existence theorem.

Theorem 3.1.

Suppose the random equation (1.1) satisfies the following conditions:

(i) B and D are Banach spaces stronger than C_c and the pair (B,D) is admissible with respect to each of the operators, T_1 and T_2 defined by $(3.3), (3.4)$;

(ii) $x(t;\omega) \to h\big(t,x(t;\omega)\big)$ is an operator on $S = \{x(t;\omega) \in D: \|x(t;\omega)\|_D \leq \rho\}$, With values in B satisfying:
$$\big\|h\big(t,x(t;\omega)\big) - h\big(t,y(t;\omega)\big)\big\|_B \leq \lambda_1\|x(t;\omega) - y(t;\omega)\|_D$$
For $x(t;\omega), y(t;\omega) \in S$, $\rho > 0$ and $\lambda_1 > 0$ are constants;

(iii) $x(t;\omega) \to f\big(t,x(t;\omega)\big)$ is an operator on S With values in B satisfying:
$$\big\|f\big(t,x(t;\omega)\big) - f\big(t,y(t;\omega)\big)\big\|_B \leq \lambda_2\|x(t;\omega) - y(t;\omega)\|_D$$
For $x(t;\omega), y(t;\omega) \in S$ and $\lambda_2 > 0$ constant; $x_0(\omega) \in D$.

Then there exists a unique random solution $x(t;\omega) \in S$ of equation (1.1), provided that
$$\left[(K_1\lambda_1 + K_2\lambda_2)\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right)\right] < 1$$
$$|\mathrm{A}|\,\|x_0(\omega)\|_D +$$
$$(K_1\|h(t,0)\|_B + K_2\|f(t,0)\|_B)\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right)$$
$$\leq \rho\left(1 - (K_1\lambda_1 + K_2\lambda_2)\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right)\right)$$
Where K_1 and K_2 are the norms of T_1 and T_2, respectively

Proof:

By condition (i), lemmas 2.1, 3.1, and 3.2 T_1 and T_2 are continuous from B into D. Hence, their norms K_1 and K_2 exist.

Define the operator $U: S \to D$ by

$$(Ux)(t;\omega) = \mathrm{A}x_0(\omega) - \mathrm{A}\sum_{i=1}^{p} c_i\,[(T_{1i}hx)(t,\omega) + (T_{2i}fx)(t,\omega)] + (T_1hx)(t;\omega) + (T_2fx)(t;\omega) \quad (3.8)$$

We must show that $U(S) \subset S$ and that the operator U is a contraction operator on S .Then we may apply Banach's fixed-point theorem to obtain the existence of a unique random solution. Let $x(\tau;\omega) \in S$. Taking the norm in D in (3.8), we get

$$\|(Ux)(t;\omega)\|_D \le |\mathrm{A}|\,\|x_0(\omega)\|_D$$
$$|\mathrm{A}|\sum_{i=1}^{p}|c_i|\,[\|(T_{1i}hx)(t_i;\omega)\|_D + \|(T_{2i}fx)(t_i;\omega)\|_D]$$
$$+\|(T_1hx)(t;\omega)\|_D + \|(T_2fx)(t;\omega)\|_D$$

$$\le |\mathrm{A}|\|x_0(\omega)\|_D + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\,\left[K_1\|h(t_i,x(t_i;\omega))\|_B + K_2\|f(t_i,x(t_i;\omega))\|_B\right] + K_1\|h(t,x(t;\omega))\|_B + K_2\|f(t,x(t;\omega))\|_B$$

$$\le |\mathrm{A}|\|x_0(\omega)\|_D + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\,\{K_1[\lambda_1\rho + \|h(t_i,0)\|_B] + K_2[\lambda_2\rho + \|f(t_i,0)\|_B]\} + K_1[\lambda_1\rho + \|h(t,0)\|_B] + K_2[\lambda_2\rho + \|f(t,0)\|_B]$$

$$\le |\mathrm{A}|\|x_0(\omega)\|_D + |\mathrm{A}|\rho(K_1\lambda_1 + K_2\lambda_2)\sum_{i=1}^{p}|c_i| + |\mathrm{A}|(K_1\|h(t,0)\|_B + K_2\|f(t,0)\|_B)\sum_{i=1}^{p}|c_i| + \rho(K_1\lambda_1 + K_2\lambda_2) + (K_1\|h(t,0)\|_B + K_2\|f(t,0)\|_B)$$

$$\le |\mathrm{A}|\|x_0(\omega)\|_D + \rho(K_1\lambda_1 + K_2\lambda_2)\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right) + (K_1\|h(t,0)\|_B + K_2\|f(t,0)\|_B)\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right) \le \rho$$

By the last condition of the theorem.
Thus $U(S) \subset S$.
Let $y(t;\omega)$ be another element of S.

From the assumptions, it is clear that $[(Ux)(t;\omega) - (Uy)(t;\omega)] \in D$ since the difference of two elements of a Banach space is in the Banach space,

$$\|(Ux)(t;\omega) - (Uy)(t;\omega)\|_D$$
$$\le |\mathrm{A}|\sum_{i=1}^{p}|c_i|\,K_1\,\|h(t_i,x(t_i;\omega)) - h(t_i,y(t_i;\omega))\|_B$$
$$+|\mathrm{A}|\sum_{i=1}^{p}|c_i|\,K_2\|f(t_i,x(t_i;\omega)) - f(t_i,y(t_i;\omega))\|_B$$
$$+K_1\|h(t,x(t;\omega)) - h(t,y(t;\omega))\|_B + K_2\|f(t,x(t;\omega)) - f(t,y(t;\omega))\|_B$$
$$\le \left(|\mathrm{A}|(K_1\lambda_1 + K_2\lambda_2)\sum_{i=1}^{p}|c_i|\right)\|x(t;\omega) - y(t;\omega)\|_D + (K_1\lambda_1 + K_2\lambda_2)\|x(t;\omega) - y(t;\omega)\|_D$$
$$\le \left[(K_1\lambda_1 + K_2\lambda_2)\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right)\right]\|x(t;\omega) - y(t;\omega)\|_D$$

Since by hypothesis

$$\left[(K_1\lambda_1 + K_2\lambda_2)\left(1 + |\mathrm{A}|\textstyle\sum_{i=1}^{p}|c_i|\right)\right] < 1,$$

then U is a contraction operator on S. Applying Banach's fixed-point theorem, there exists a unique element of S so that $(Ux)(t;\omega) = x(t;\omega)$, completing the proof.

Corollary 3.1.

If the stochastic fractional integro-differential equation

$$\frac{\partial^\alpha x(t;\omega)}{\partial t^\alpha} = \int_0^t k(t,\tau;\omega)\,f(\tau,x(\tau;\omega))\,d\tau \quad (3.9)$$

With the nonlocal condition

$$x(0;\omega) + \sum_{i=1}^{p} c_i x(t_i,\omega) = x_0(\omega) \quad (3.10)$$

Satisfies the following conditions:

(i) B and D are Banach spaces stronger than C_c and the pair (B,D) is admissible with respect the operator T_2 defined by (3.4);

(ii) $x(t;\omega) \to f(t,x(t;\omega))$ is an operator on $S = \{x(t;\omega) \in D : \|x(t;\omega)\|_D \le \rho\}$ With values in B satisfying:

$$\|f(t,x(t;\omega)) - f(t,y(t;\omega))\|_B \le \lambda\|x(t;\omega) - y(t;\omega)\|_D$$

For $x(t;\omega), y(t;\omega) \in S$ and $\rho > 0, \lambda > 0$ are constants;

(iii) $x_0(\omega) \in D$.

Then there exists a unique random solution $x(t;\omega) \in S$ of equation (3.9) provided that:

$$\left[\lambda K\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right)\right] < 1$$

$$|\mathrm{A}|\|x_0(\omega)\|_D + K\|f(t,0)\|_B\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right) \le \rho\left(1 - \lambda K\left(1 + |\mathrm{A}|\sum_{i=1}^{p}|c_i|\right)\right)$$

Where K is the norm of T_2.

Proof:

Since (3.9) is the equivalent of (3.7) with $h(t,x)$ equal to zero, the proof follows from that theorem 3.1 with T_1 being the null operator.

4. Boundedness and asymptotic behavior of random solution.

Using the spaces $C_g(R_+, L_2(\Omega, \mathcal{A}, P))$ and $C(R_+, L_2(\Omega, \mathcal{A}, P))$ we now give some results concerning the asymptotic behavior of the random solution of (1.1). We first consider the unperturbed case (3.9).

Theorem 4.1.

Suppose the equations (3.9). satisfy the following conditions:

(i) $|||k(s,\tau;\omega)||| \le \Lambda_1\, e^{-\gamma(t-\tau)}$ for some constants $\Lambda_1 > 0$ and $\gamma > 0,\ 0 \le \tau \le s \le t;$

(ii) $x(t;\omega) \to f(t, x(t;\omega))$ satisfies, for some $\Lambda_2 > 0$ and $\gamma > \beta > 0,$

$\|f(t,x(t;\omega))\| \le \Lambda_2 e^{-\beta t}, \quad t \ge 0,$

$\|f(t,x(t;\omega)) - f(t,y(t;\omega))\| \le \lambda \|x(t;\omega) - y(t;\omega)\|$

For $\|x(t;\omega)\|$ and $\|y(t;\omega)\| \le \rho e^{-\beta t}$ at each $t \ge 0$ and λ constant;

(iii) $x_0(\omega) = 0.\ P - a.e.$

Then there exists a unique random of solution of (3.9) which is stochastically exponentially stable, provided that λ is small enough.

Proof:

It is sufficient to show that condition (i) implies the admissibility of the pair of spaces (C_g, C_g) with respect to the operator T_2 defined by (3.4), and that condition (ii) is equivalent to condition (ii) of Corollary 3.1 with $B = D = C_g(R_+, L_2(\Omega, \mathcal{A}, P)), g(t) = e^{-\beta t}, \beta > 0.$

Let $x(t;\omega) \in C_g(R_+, L_2(\Omega, \mathcal{A}, P))$, taking the norm in $L_2(\Omega, \mathcal{A}, P)$ of (3.4), we obtain

$$\|(T_2\, x)(t;\omega)\| \le \frac{\Lambda_1 e^{-\gamma t}}{\Gamma(\alpha+1)} \|x(t;\omega)\|_{C_g} \int_0^t (t-\tau)^\alpha\, e^{(\gamma-\beta)\tau} d\tau$$

Let $t - \tau = \theta$, then.

$$\|(T_2\, x)(t;\omega)\| \le \frac{\Lambda_1 e^{-\beta t}}{\Gamma(\alpha+1)} \|x(t;\omega)\|_{C_g} \int_0^t \theta^\alpha\, e^{-(\gamma-\beta)\theta} d\theta$$

Let $(\gamma-\beta)\theta = y$, then

$$\|(T_2\, x)(t;\omega)\| \le \frac{\Lambda_1 e^{-\beta t}}{(\gamma-\beta)^{\alpha+1}\Gamma(\alpha+1)} \|x(t;\omega)\|_{C_g} \int_0^{(\gamma-\beta)t} y^\alpha\, e^{-y} dy$$

Since $\gamma > \beta$, then

$$\int_0^{(\gamma-\beta)t} y^\alpha\, e^{-y} dy \le \Gamma(\alpha+1).$$

Now

$$\|(T_2 x)(t;\omega)\| \le \frac{\Lambda_1 e^{-\beta t}}{(\gamma-\beta)^{\alpha+1}\Gamma(\alpha+1)} \|x(t;\omega)\|_{C_g} .\Gamma(\alpha+1)$$

$$= \frac{\Lambda_1}{(\gamma-\beta)^{\alpha+1}} \|x(t;\omega)\|_{C_g} . e^{-\beta t} = \rho e^{-\beta t}$$

Since $\gamma > \beta > 0,\ \Lambda_1 > 0$.

Hence for $x(t;\omega) \in C_g, \rho$ is a positive constant we have $(T_2 x)(t;\omega) \in C_g$; that is (C_g, C_g) is admissible with respect to T_2.

Now let $f(t, y(t;\omega)),\ f(t, x(t;\omega)) \in C_g$, then

$$\|f(t,x(t;\omega)) - f(t,y(t;\omega))\|_{C_g}$$

$$= sup_{t\ge 0}\left\{\frac{\|f(t,x(t;\omega)) - f(t,y(t;\omega))\|}{e^{-\beta t}}\right\}$$

$$\le sup_{t\ge 0}\left\{\frac{\|x(t;\omega) - y(t;\omega)\|}{e^{-\beta t}}\right\}$$

$$= \|x(t;\omega) - y(t;\omega)\|_{C_g}$$

Then condition (ii) implies that condition (ii) of Corollary 3.1 holds.Therefore, by Corollary3.1 there exists a unique random solution and

$$\|(Tx)(t;\omega)\| = \|x(t;\omega)\| \le \rho e^{-\beta t}, \quad t \ge 0$$

Then the solution is stochastically exponentially stable, Hence the required result.

Now, if $h(t,x)$ is not identically equal to zero, then we can still obtain the result that there is a unique random solution of (1.1) which is bounded in the mean square for all $t \in R_+$.

Theorem 4.2.

Assume that equations (1.1) satisfies the following conditions:

(i) $|||k(s,\tau;\omega)||| \le \Lambda_1$ for some constants $\Lambda_1 > 0, 0 \le \tau \le s \le t;$

(ii) $x(t;\omega) \to h(t, x(t;\omega))$ satisfies, for some $\Lambda_2 > 0,\ \|h(t,x(t;\omega))\| \le \Lambda_2,\ \ t \ge 0,$ and

$\|h(t,x(t;\omega)) - h(t,y(t;\omega))\| \le \lambda_1 \|x(t;\omega) - y(t;\omega)\|$

For $\|x(t;\omega)\|$ and $\|y(t;\omega)\| \le \rho,\ t \ge 0$ and $\lambda_1,\ \Lambda_2$ are constant;

(iii) $x(t;\omega) \to f(t, x(t;\omega))$ satisfies, for some $\Lambda_3 > 0,\ \|f(t,x(t;\omega))\| \le \Lambda_3,\ \ t \ge 0,$ and

$\|f(t,x(t;\omega)) - f(t,y(t;\omega))\| \le \lambda_2 \|x(t;\omega) - y(t;\omega)\|$

For $\|x(t;\omega)\|$ and $\|y(t;\omega)\| \le \rho$ at each $t \ge 0$ and λ_2, Λ_3 are constant;

(iv) $x_0(\omega) \in C$

Then there exists a unique random of solution of (1.1) which is bounded in the mean square on R_+, provided that λ_1, λ_2,

$\|x_0(\omega)\|_C$, $\| h(t,0)\|_C$ and $\| f(t,0)\|_C$ are sufficiently small.

Proof:

It will suffice to show that the pair of spaces (C,C) is admissible with respect to the integral operators T_1 and T_2 defined by (3.3), (3.4), respectively under condition (i).

Let $x(t;\omega) \in C$. then from (3.3) we have that

$$\|(T_1 x)(t;\omega)\| \leq \frac{t^{\alpha}}{\Gamma(\alpha+1)} \|x(t;\omega)\|_C < \infty$$

Thus $(T_1 x)(t;\omega) \in C$ and the pair (C,C) is admissible with respect to T_1.

By the same way we have

$$\|(T_2 x)(t;\omega)\| \leq \frac{\Lambda_1 t^{\alpha+1}}{\Gamma(\alpha+2)} \|x(t;\omega)\|_C < \infty$$

Thus $(T_2 x)(t;\omega) \in C$ and the pair (C,C) is admissible with respect to T_2. Clearly conditions (ii), (iii) of theorem 4.2 implies conditions (ii), (iii) of theorem 3.1. Therefore, the conditions of theorem 3.1 hold with $B = C$, $g(t) = 1$, and $D = C$, and then there exists a unique random solution of (1.1), $x(t;\omega)$, bounded in the mean square by ρ for all $t \in R_+$.

5. Application to stochastic differential systems.

Consider the following nonlinear fractional differential system with random parameters:

$$\frac{dy(t;\omega)}{dt} = \Pi(\omega)\, y(t;\omega) + \mathrm{b}(\omega)\,\Phi\big(t,\sigma(t;\omega)\big) \quad (5.1)$$

$$\frac{\partial^{\alpha}\sigma(t;\omega)}{\partial t^{\alpha}} = c^T(t;\omega)\, y(t;\omega), \quad (5.2)$$

With the following initial conditions

$$y(0;\omega) = y_0(\omega)$$

$$\sigma(0;\omega) + \sum_{i=1}^{p} c_i \sigma(t_i,\omega) = \sigma_0(\omega),$$

Where $0 < \alpha \leq 1$, $t \in R_+ = [0,\infty)$, $0 < t_1 < \ldots < t_p < \infty$, the fractional derivative is provided by the Caputo derivative, $\Pi(\omega)$ is an $n \times n$ matrix of measurable functions, $x(t;\omega)$ and $c(t;\omega)$ are $n \times 1$ vectors of random variables for each $t \in R_+$, $\mathrm{b}(\omega)$ is an $n \times 1$ vector of measurable functions, $\sigma(t;\omega)$ is a scalar random variable for each $t \in R_+$, $\Phi(t,\sigma)$ is a scalar functions of $t \in R_+$, $\sigma \in R$, and T denotes the transpose of a matrix.

The system $(5.1)-(5.2)$ may be reduced to a stochastic fractional integro-differential equation of the form (1.1).Now integrating (5.1), we have

$$y(t;\omega) = e^{\Pi(\omega)t} y(0;\omega)$$

$$+ \int_0^t e^{\Pi(\omega)(t-\tau)} \mathrm{b}(\omega)\,\Phi\big(\tau,\sigma(\tau;\omega)\big)d\tau \quad (5.3)$$

Substituting from (5.3) into (5.2), we obtain

$$\frac{\partial^{\alpha}\sigma(t;\omega)}{\partial t^{\alpha}} = c^T(t;\omega)\, e^{\Pi(\omega)t}\, y_0(\omega)$$

$$+ \int_0^t e^{\Pi(\omega)(t-\tau)} c^T(t;\omega)\,\mathrm{b}(\omega)\,\Phi\big(\tau,\sigma(\tau;\omega)\big)d\tau \ (5.4)$$

Now assume that $|||c^T(t;\omega)||| \leq K_1$ for all $t \geq 0$ and $K_1 > 0$ a constant. Also, let $y_0(\omega) \in C$, and $\mathrm{b}(\omega) \in L_\infty(\Omega,\mathcal{A},P)$. if we assume that the matrix $\Pi(\omega)$ is stochastically stable, that is there exist an $\propto > 0$ such that

$$P\{\omega: Re\ \psi_k(\omega) < -\propto,\ \ k = 1,2,\ldots\ldots\ldots,n\} = 1,$$

Where $\psi_k(\omega)$, $k = 1,2,\ldots\ldots\ldots,n$, are the characteristic roots of the matrix, then it has been shown by Morozan [12] that

$$|||e^{\Pi(\omega)t}||| \leq K_2 e^{-\propto t} \leq K_2$$

For some constant K_2. we also let $\Phi\big(t,\sigma(t;\omega)\big) \in C\big(R_+, L_2(\Omega,\mathcal{A},P)\big)$ for each $t \in R_+$, and

$$\|\Phi\big(t,\sigma_1(t;\omega)\big) - \Phi\big(t,\sigma_2(t;\omega)\big)\| \leq \lambda\|\sigma_1(t;\omega) - \sigma_2(t;\omega)\|.$$

Let

$$\mathrm{h}\big(t,\sigma(t;\omega)\big) = c^T(t;\omega)e^{\Pi(\omega)t}\, y_0(\omega)$$

Then

$$\|\mathrm{h}\big(t,\sigma(t;\omega)\big)\|$$
$$\leq |||c^T(t;\omega)|||.\,|||e^{\Pi(\omega)t}|||.\,\|y_0(\omega)\|$$
$$\leq aK_1K_2e^{-\propto t} \leq aK_1K_2$$

Where $a > 0$ is a constant, since $y_0(\omega) \in C$.

Thus by definition $\mathrm{h}\big(t,\sigma(t;\omega)\big) \in C$,

Also,

$$\|\mathrm{h}\big(t,\sigma_1(t;\omega)\big) - \mathrm{h}\big(t,\sigma_2(t;\omega)\big)\| = 0$$

So that it satisfies a Lipschitz condition.

Now, by the assumptions on $c^T(t;\omega)$, $\mathrm{b}(\omega)$, and $\Pi(\omega)$, we have

$$k(s,\tau;\omega) = e^{\Pi(\omega)(s-\tau)} c^T(s;\omega)\,\mathrm{b}(\omega)$$

Satisfying

$$|||k(s,\tau;\omega)|||$$
$$\leq |||e^{\Pi(\omega)(s-\tau)}|||\,|||c^T(s;\omega)|||\,|||\mathrm{b}(\omega)|||$$
$$\leq K_1K_2e^{-\propto(s-\tau)}\,|||\mathrm{b}(\omega)||| \leq K_1K_2|||\mathrm{b}(\omega)|||$$

Therefore, all conditions of theorem 4.2 are satisfied and there exists a unique random solution of the system (5.1)(5.2) which is bounded in the mean square on R_+.

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