On the $D_\infty -$ differential Banach module And spectral sequence

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Abstract: The paper is concerned with the $D_\infty -$ differential Banach module and spectral sequences. We show that On each term of the spectral sequence there is the structure of a stable $D_\infty^{(s)}$ -Banach module. We give a differential Banach module with a (1)-filtration and cohomology of spectral sequences as applications of this notion.

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1. Introduction

The conception of differential Banach module, differential A $^\infty$ - Banach algebra are defined and studied by simirnov and others In [9]. It is known that the great part of classical theory of spectral sequence depends on the concept of differential module with filtration. Lapin has studied the $D_\infty -$ differential and A_∞-structure in spectral sequences [4]. The Multiplicative A_∞-structure in terms of spectral sequences of fibrations has been studied in [6].The basic homotopy properties of D_∞ -di□erential E_∞ -(co)algebras related to a spectral sequence of a D_∞ -di□erential E_∞ -(co)algebra has been investigated in [7].The author has studied the cohomology of Banach A_∞ –module Over Admissible Banach A_∞ –algebra in [1]. In the present paper we are concerned with the $D_\infty -$ differential Banach module, stable $D_\infty^{(s)}$ -Banach module and its relation with differential Banach module with filtration and spectral sequences. We give some examples of this notion.

Firstly we recall the basic definitions and facts related to $D_\infty -$ differential Banach module and $D_\infty^{(s)}$ -differential Banach module. The main references are [9], [1],[3],[4] and [5].

Definition (1.1):

- The Pair (X,d) is called a Banach graded module, where $X=\{X_n\}$, $n\in z$ is a family of Banach module (space), equipped with a differential $d : X_\bullet \to X_{\bullet -1}$, which is a map of graded of degree (-1) such that $d^2=0$ (Z is a set of integers).
- A map of differential Banach modules $f : (X,d) \to (Y,d)$ is a map of graded Banach modules $f : X_\bullet \to Y_\bullet$ of degree 0 satisfying the condition $df = fd$.
- A homotopy $h : X \to Y$ between maps of differential Banach modules $f,g:(X,d)\to(Y,d)$ is by definition a map of graded Banach modules $h : X_\bullet \to Y_{\bullet+1}$ of degree 1 such that $dh+hd=f-g$.

Definition (1.2):

A $D_\infty -$ differential Banach module is a Banach module (space) X with a family of graded Banach module homeomorphisms $\{d^i : X_\bullet \to X_{\bullet-1}, i>0\}$ that satisfy the relation $\sum_{i+j=k} d^i d^j = 0$, for each integer $k>0$.

If $i=0$, $d^0 d^0 = 0$ and (X,d^0) is an ordinary differential module, if i=1 we have $d^1d^0+d^0d^1=0$, that is the mappings d^0 and d^1 are anticommuting. This means that the composition $d^1d^1 : X \to X$ is an endomorphism of the differential module (X,d^0). For k = 2, we obtain $d^2d^0+d^0d^2=0-d^1d^1$ .This means that the mapping $d^2:X\to X$ is a differential homotopy between the zero mapping and the mapping of differential modules $d^1d^1:(X,d^0)\to(X,d^0)$. Therefore, the mapping $d^1:X\to X$ is a differential with accuracy to a homotopy.

An example of A $D_\infty-$ differential Banach module is differential modules with filtration [2].

Definition (1.3):

Let arbitrary morphisms of $D_\infty-$ Banach modules $f=\{f^i\}:X\to Y$ and $g=\{g^i\}:Y\to S$ be given. We define the composition gf of morphisms f and g, setting $(gf)^i=\sum_{s+t=i}g^sf^t:X\to S$

An identity morphism for the $D_\infty-$ Banach module (X,d^i) is a family of mappings of modules $1_x=\{1_x^i\}:X\to X,$ where $1_x^i=0,\ i>0$ and 1_x^0 is the identity mapping of the module X. Therefore, the category of $D_\infty-$-modules is defined.

Definition (1.4):

A $D_\infty-$ differential Banach module (X,d^i) is called a $D_\infty^{(S)}$-differential Banach module or $D_\infty^{(S)}-\text{Banach module}$, denoted by (X,d^{i+s}) ,if there exists such an integer s ≥ 0 such that the conditions $d^i=0$, i < s, hold. In this case, the $D_\infty-$ differential $\{d_s^i:X_\bullet\to X_{\bullet-1}\}$ satisfies the relation $\sum_{i+j=k}d^{i+s}d^{j+s}=0$.

Obviously, for s = 0, the category of $D_\infty^{(S)}-$ Banach module coincides with the above-mentioned category of $D_\infty-$ Banach module, and for every fixed number s ≥ 0, the category of $D_\infty^{(S+1)}-$ Banach module is a complete subcategory of the category of $D_\infty^{(S)}-$ Banach module.

Note that :

- for an arbitrary $D_\infty^{(S)}-$ Banach module (X,d^{i+s}), the equality $d^sd^s=0$ holds, i.e., the differential module (X,d^s) is defined.
- In particular if s=1 we get the $D_\infty^{(1)}$-Banach module (X,d^{i+1}).

Definition (1.5):

A morphism of $D_\infty^{(S)}-Banach$ modules $f:X\to Y$ is a family of module maps $F=\{f_i:X\to Y\},i>0\}$ such that the following relation is satisfied : for each integer k > 0, $\sum_{i+j=k}f^id^{j+s}=\sum_{i+j=k}d^{i+s}f^j$.

Definition (1.6):

A homotopy between morphisms $f,g:X\to Y$ of $D_\infty^{(S)}-$ Banach module is a family of homeomorphisms $h=\{h^{i-s}:X\to Y_{+1}\},i\geq 0\}$ that satisfy the following relation: for each integer k ≥ 0, $\sum_{i+j=k}d^{i+s}h^{j-s}+h^{j-s}d^{i+s}=f^k-g^k$.

For $k=o$ we have $d^sh^{-s}+h^{-s}d^s=f^0-g^0$. Hence the map of modules $h^{-s}:X\to Y$ is a differential homotopy between the maps of differential Banach modules $f^0,g^0:(X,d^s)\to(Y,d^s)$.

For given $D_\infty^{(S)}-Banach$ module (X,d^{i+s}) and (Y,d^{i+s}), we can get the SDR- case of $D_\infty^{(S)}-Banach$ modules :

$$(\eta:(X,d^s)\rightleftarrows(Y,d^s):\xi,h)$$

if the following identities are satisfied $\sum_{i+j=k}\eta^ih^{j-s}=0,\ \sum_{i+j=k}h^{i-s}\xi^j=0,$ $\sum_{i+j=k}h^{i-s}h^{j-s}=0,\ k\geq 0.$
(see [5]).

Definition (1.7):

A $D_{\infty}^{(S)} - Banach$ module (X, d^{i+s}) is said to be stable if for each x $\in$ X there exists an index k $\geq$ 0 depending on x such that $d^{i+s}(X) = 0$ for $i > k$.

Definition (1.8):

The homology module $H(X)$ of $D_{\infty}^{(S)} -$ Bancach module (X, d^{i+s}) is the homology module of the module X relative to the differential $D_s : X \to X$, where $D_s = (d^s + d^{s+1} + ... + d^{s+i} + ...)$.

§ **2. Differential Banach modules with (1)-filtrations and stable D(1)$_\infty$-differential Banach module**

In this section we consider the differential Banach modules with (1)-filtrations (analog to differential modules with (1)-filtrations in pure cases [6]) and establish a connection between differential Banach modules with (1)-filtrations and stable D(1)$_\infty$-differential Banach modules.

Definition (2.1):

Given a differential Banach module (X, d), the filtration $\{X^n\}, n \in Z$ of is a family of graded submodules $X_{\bullet}^{n} \subseteq X_{\bullet}$ satisfying the following conditions:

$$... \subseteq X_{\bullet}^{n} \subseteq X_{\bullet}^{n+1} \subseteq ..., \quad \bigcup_{n \in \mathbb{Z}} X^n = X,$$

$$\bigcap_{n \in \mathbb{Z}} X^n = 0, \quad d(X^n) \subseteq X^n, \quad n \in Z.$$

A map of differential Banach modules with filtrations $f : (X, \{X^n\}) \to (Y, \{Y^n\})$ is by definition a map of differential Banach modules $f : X \to Y$ satisfying the condition $f(X^n) \subseteq Y^n, n \in Z$. A homotopy h between maps of differential Banach modules with filtrations $f, g : (X, \{X^n\}) \to (Y, \{Y^n\})$ is by definition a homotopy $h : X \to Y$ between the maps of differential Banach modules $f, g : X \to Y$, satisfying the condition $h(X^n)) \subseteq Y^n, n \in Z$.

Definition (2.2):

A (1)-filtration of a differential Banach module (X, d) is an arbitrary filtration { X ^{n} } of this differential Banach module satisfying the condition $d(X^n)) \subseteq X^{n-1}, n \in Z$. Maps of differential Banach modules with (1)-filtrations are defined as maps of differential Banach modules with filtrations.

Note that :[2] Although the category of differential Banach modules with (1)-filtrations is a full subcategory of the category of differential Banach modules with filtrations, the embedding functor from the category of differential Banach modules with (1)-filtrations into the category of differential Banach modules with filtrations does not preserve homotopies between morphisms and, consequently, does not induce any functor.

The main result in this part is to get the connection between the differential Banach modules with (1)-filtrations and stable $D_{\infty}^{(1)}$-module.

Let (X, d) be any differential Banach module with a (1)-filtration $\{X^n\}$. We denote by Z_X^k a submodule of the graded Banach module X^k such that $X^k = Z_X^k \oplus X^{k-1}$. By using the condition $d(X_{\bullet}^{k}) \subseteq X_{\bullet}^{k-1}$) we define a stable $D_{\infty}^{(1)} - Banach$ modules (X, d^{i+1}) by setting $d^{i+1} = \bigoplus_{k \in \mathbb{Z}} d_k^{i+1} : X_{\bullet} \to X_{\bullet -1}, i \geq 0$, where the map $d_k^{i+1} : \left(Z_X^k\right)_{\bullet} \to \left(Z_X^{k-(i+1)}\right)_{\bullet -1}$ is a component of the map

$$d : \left(Z_X^k\right)_{\bullet} \to X_{\bullet -1}^{k-1} =$$

$$\left(\left(Z_X^{k-1}\right)_{\bullet -1} \oplus ... \oplus \left(Z_X^{k-(i+1)}\right)_{\bullet -1} \oplus ...\right)$$

The $D_{\infty}^{(1)}$-Banach module (X, d^{i+1}) is a stable $D_{\infty}^{(1)}$ - Banach module satisfying the condition (X,D^1) = (X, d), where D$_1$ is the summed differential of the $D_{\infty}^{(1)} - Banach$ module (X, d^{i+1}). Similarly any map of differential Banach modules over a field with (1)-filtrations and any homotopy between maps of differential Banach modules over a field with (1)-filtrations uniquely define a morphism of $D_{\infty}^{(1)} - Banach$ module and a homotopy between morphisms of $\mathrm{D}_{\infty}^{(1)}-Banach$ module, respectively. Thus, we have the following assertion.

Proposition (2.2):
Each differential Banach module (X, d) with a (1)-filtration uniquely defines on the graded module X the structure of a stable $D_{\infty}^{(1)}$ - Banach module (X,d^{i+1}) such that $(X,D_1)=(X,d)$, where D_1 is the summed differential of the $\mathrm{D}_{\infty}^{(1)}-$ Banach modules (X,d^{i+1}). Moreover, each SDR-data of differential Banach modules over a field with (1)-filtrations uniquely defines an SDR-data of stable $D_{\infty}^{(1)}$ - Banach modules for which the 'summed' SDR-data of differential modules coincides with the original SDR-data of differential Banach modules.

§ 3. Differential Banach module and spectral sequence

In this part we study the relation between the spectral sequence and differential Banach module. In particular we study the spectral sequence of a differential Banach module with a (1)-filtration.

Definition (3.1).
The spectral Banach module is an any sequence of differential Banach module $\{(E_s,d_s)\}_{s\geq 1}$, where $E_{s+1}=H(E_s)=Ker\,d_s/\operatorname{Im} d_s$.

Note that, if *s=1* the spectral module (E_s,d_s) is equal to usual differential module (M,d).

Following [3], since, a D_{∞}-module over a field determines a spectral sequence, we get the following facts (related to a $\mathrm{D}_{\infty}-Banach$ modules):

1- Let (X,d^i) be a stable D_∞- Banach module. Then the spectral sequence $\{E_s,d_s\}$ of D_∞- Banach modules, where $E_S=(X_S,d^{i+s}),\ \ i\geq 0$, and $d_s^s=d_s$, determined by the D_∞-Banach module (X,d^i) converges to H(X). All terms E_s of this spectral sequence, considered as differential modules with summary differentials $D_S:E_S\to E_S$, are homotopy equivalent to each other and to the differential module $(H(X),d=0)$.

2- For any s ≥ 0, the terms E_s and E_{s+1} of the spectral sequence $\{(E_s,d_s)\}_{s\geq 1}$, considered the $\mathrm{D}_{\infty}^{(S)}-Banach$ modules, are homotopy equivalent.

The following assertion relates the spectral sequence with the stable $\mathrm{D}_{\infty}^{(S)}-Banach$ modules.

Theorem (3.1):
Let $\{(E_s,d_s)\}_{s\geq 1}$ be the spectral sequence of an arbitrary differential Banach module (X, d). Then

1- On each term (E_s,d_s) of this spectral sequence there is the structure of a stable $D_{\infty}^{(s)}$ - Banach module (X,d^{i+s}) which is connected with the differential d^s in this term by the equality $d_s^s=d_s$.

2- Let $\{(E_s,d_s)\}_{s\geq 0}$, be a spectral sequence. For any s ≥ 0, there exists a differential $d_S:E_S\to E_S$ on the term E_s such that the corresponding homology module $H(E_s,D_s)=Ker\,D_s/Im\,D_s$ is isomorphic to the limit term E_∞ of the given spectral sequence $\{E_s,d_s\}$.

An application of differential Banach module and the spectral sequence is given by the following examples;

Example(3.2):
we consider the sequence of a differential Banach module with a (1)-filtration and compare it with the spectral sequence described in Theorem 3.1 of the $D_{\infty}^{(1)}$ -module defined by the given differential module with a (1)-filtration, then we obtain the following assertion.

Theorem (3.2).
Let $\{(X_s,d_s)\}_{s\geq 1}$ be the spectral sequence of an arbitrary differential Banach module (X, d). If the (1)-filtration of the differential Banach module X is bounded below, then for each s > 1 the homology module $H(X_s)=Ker\,D_s/\operatorname{Im} D_s$ of the stable $D_{\infty}^{(S)}$ -module (X,d_s^{i+s}) is isomorphic to the limit term X_∞ of the spectral sequence $\{(X_s,d_s)\}_{s\geq 1}$ and, consequently, is isomorphic to the homology module $H(X)=Ker\,d/\operatorname{Im} d$.

Example (3.3): [6]

Let $\{(X_s, d_s)\}_{s\geq 1}$ be the (co)homology spectral sequence of an arbitrary Serre fibration $P: E \rightarrow B$ [8]. Then on each term (X_s, d_s) of this spectral sequence there is the structure of a stable $D_\infty^{(S)}-$ Banach modules (X, d_s^{i+s}) which is connected with the differential d_s by the equality $d_s^s = d_s$.

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