Intrinsic formulation for elastic line deformed on a surface by an external field in the pseudo-Galilean space G_1^3

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Abstract: In this paper, we derive intrinsic formulation for elastic line deformed on a surface by an external field in the pseudo-Galilean space G_1^3

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1.Introduction

Manning studied intrinsic formulation for elastic line deformed external field on a surface by external field E^3 (Manning, 1988). Intrinsic equations for a elastic line in Lorentz-Minkowski space was researched (Gürbüz and Görgülü, 2000), (Gürbüz, 2000). In this paper we derive intrinsic formulation for elastic line deformed external field on a surface by external field in pseudo-Galilean space.

In this section we give preliminaries on pseudo-Galilean space G_1^3. The definitions relation to G_1^3 was taken (Divjak, 2008).

The pseudo-Galilean 3- space G_1^3 is the three dimensional real affine space with the absolute figure {w,f,I} , where w is a fixed plane, f a line in w and I a hyperbolic involution of the points of f. The pseudo-Galilean space length of the vector x(x,y,z) is defined by

$$\begin{cases} x, & x \neq 0 \\ \sqrt{\left|y^2 - z^2\right|}, & x = 0 \end{cases}$$

A curve parametrized by the parameter of arc length s=x is given in the coordinat form by β(x)=(x,y(x),z(x)). The curvature $\kappa(x)$ and $\tau(x)$ of an curve are given by (Divjak, 2008).

$$\kappa(x) = \sqrt{\left|y'^2(x) - z'^2(x)\right|},$$

$$\tau(x) = \frac{1}{\kappa^2(x)} \det(r'(x), r''(x), r'''(x)).$$

The associated moving trihedron is given by

$$t = r'(x) = (1, y'(x), z'(x))$$

$$n = \frac{1}{\kappa(x)}(0, y''(x), z''(x)),$$

$$b = \frac{1}{\kappa(x)}(0, \varepsilon z''(x), \varepsilon y''(x))$$

where $\varepsilon = 1$ or $\varepsilon = -1$ and it is called a Frenet trihedron associated to the curve. If t is timelike, n is a spacelike vector, b is spacelike,. Frenet-Serret formulas are given as following:

$$t'(x) = \kappa(x) n(x)$$

$$n'(x) = \tau(x) b(x)$$

$$b'(x) = \tau(x) n(x).$$

For regular curve in G_1^3, κ is defined as following

$$\kappa = \frac{\left\|\Psi' \times_{PG} \Psi''\right\|}{\left\|\Psi'\right\|^3}$$

where $\times_{PG}$ denotes pseudo-Galilean cross product. If e_1 is unit spacelike vector, e_2 is unit spacelike vector , e_3 is a unit timelike vector , $a \times_{PG} b$ is given as following:

$$a \times_{PG} b = -\begin{vmatrix} 0 & e_2 & -e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$. If e_1 is unit spacelike vector, e_2 is unit timelike vector , e_3

is a unit spacelike vector, $a \times_{PG} b$ is given as following:

$$a \times_{PG} b = \begin{vmatrix} 0 & -e_2 & -e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Theorem 1.1. Let F be the timelike surface in G_1^3 and β denote an arc on F. The analogue of the Frenet-Serret formulas in pseudo-Galilean 3-space G_1^3 is

$$\begin{bmatrix} T' \\ Q' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ 0 & 0 & -\tau_g \\ 0 & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ Q \\ N \end{bmatrix} \quad (1.1)$$

where κ_g is the geodesic curvature, τ_g is the geodesic torsion, κ_n is the normal curvature.

$$\langle T,T \rangle = -1, \quad \langle N,N \rangle = 1, \quad \langle Q,Q \rangle = 1 .$$

Theorem 1.2. Let F be the spacelike surface in G_1^3 and β denote an spacelike arc on F. The analogue of the Frenet-Serret formulas in pseudo-Galilean 3-space G_1^3 is

$$\begin{bmatrix} T' \\ Q' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & -\kappa_n \\ 0 & 0 & -\tau_g \\ 0 & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ Q \\ N \end{bmatrix} \quad (1.2)$$

where κ_g is the geodesic curvature, τ_g is the geodesic torsion, κ_n is the normal curvature. Also, $\langle T,T \rangle = 1, \quad \langle Q,Q \rangle = 1, \quad \langle N,N \rangle = -1$.

2. Intrinsic Method

In this section, we study intrinsic formulation for elastic line deformed on surface by an external field in pseudo-Galilean space G_1^3.

The arc β is called elastic line if it is extremal for the variational problem of (2.1) within the family of all arcs of length l on non-null surface F having the same initial point and initial direction as β in the pseudo-Galilean space G_1^3.

If elastic line is exposed to a static force field, it has a trajectory that minimizes the sum of its elastic energy and its energy of interaction with the field in G_1^3. The problem is to to minimize the energy E,

$$E = \int_0^l \left(\frac{1}{2} b\kappa^2 - \theta\varphi\right) ds$$
$$E(t) = \frac{1}{2} b I_1(t) - \theta I_2(t) \quad (2.1)$$

,

among elastic lines with trajectories $\phi(u(s),v(s))$ of fixed length l and arc length, $0 \le s \le l$, contained pseudo-Galilean surface $\phi(u,v)$ in pseudo-Galilean space G_1^3.

$-\theta$ is constant measuring the strength of the external field, $\phi(u,v)$ gives its shape and κ denotes elastic bending energy in the pseudo-Galilean 3-space .

The equilibrium trajectory are the extrema of the sum of stress and potentiel energies in G_1^3. The path of the elastic line have to satisfy a differential equation, which is derived by variational methods on the pseudo-Galilean 3-space.

Assume β lies in a coordinat patch $\phi(u,v)$ of F. Thus β is given as $\beta(s) = \phi(u(s),v(s))$. Also,

$$T(s) = \beta'(s),$$
$$Q(s) = p(s)\phi_u + q(s)\phi_v \quad (2.2)$$

for suitable scalar functions p(s) and q(s). Define

$$\Psi(\sigma;t) = \phi(u(\sigma) + t\eta(\sigma), v(\sigma) + t\xi(\sigma))$$

for $0 \le \sigma \le l$. $\quad l = \int_0^{\lambda(t)} \sqrt{\left| \left\langle \frac{\partial \psi}{\partial \sigma}, \frac{\partial \psi}{\partial \sigma} \right\rangle_{PG} \right|} d\sigma$

Case I. Intrinsic formulation for elastic line deformed on a timelike surface by an external field in the pseudo-Galilean space G_1^3.

i. If T is timelike , Q and N are spacelike ,

$$\left. \frac{\partial \Psi}{\partial \sigma} \right|_{t=0} = T, \quad 0 \le \sigma \le l \quad (2.3)$$

$$\left. \frac{\partial^2 \Psi}{\partial \sigma^2} \right|_{t=0} = T' = \kappa_g Q + \kappa_g N \quad (2.4)$$

$$\left. \frac{\partial \Psi}{\partial t} \right|_{t=0} = T' = \mu Q \quad (2.5)$$

With second differentiation Equation (2.5) , we obtain

$$\left.\frac{\partial^2\Psi}{\partial t\partial\sigma}\right|_{t=0} = T' = \mu' Q - \mu\tau_g N \quad (2.6)$$

Third differentiation Equation (2.5) gives

$$\left.\frac{\partial^3\Psi}{\partial t\partial\sigma^2}\right|_{t=0} = (\mu'' - \mu\tau_g^2)Q - (2\mu'\tau_g + \mu\tau_g')N$$

(2.7)

Lemma 2.1. In pseudo-Galilean 3-space, $\left.\frac{\partial\lambda}{\partial t}\right|_{t=0} = 0$

Proof.

$$\left.\frac{d\lambda}{dt}\right|_{t=0} \int_0^{\lambda(t)} \sqrt{\left|\left\langle \left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0}\right\rangle_{PG}\right|} +$$

$$\int_0^l \frac{\left\langle \left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial\psi}{\partial\sigma\partial t}\right|_{t=0}\right\rangle_{PG}}{\left|\left\langle \left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0}\right\rangle_{PG}\right|^{-1/2}} \left\langle \left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial\psi}{\partial\sigma}\right|_{t=0}\right\rangle_{PG}^{-1} = 0$$

From (2.3) and (2.6), we obtain

$$\left.\frac{\partial\lambda}{\partial t}\right|_{t=0} = 0 \quad (2.8)$$

Let $I_1(t) = \int \kappa^2 ds$ denote the total square curvature of the arc . $0 \le \sigma \le \lambda(t)$. For $t \ne 0$, the total square curvature is

$$I_1(t) = \int_0^{\lambda(t)} \left|\left\langle \frac{\partial\Psi}{\partial\sigma} \times_{PG} \frac{\partial^2\Psi}{\partial\sigma^2}, \frac{\partial\Psi}{\partial\sigma} \times_{PG} \frac{\partial^2\Psi}{\partial\sigma^2}\right\rangle\right| \left|\left\langle \frac{\partial\Psi}{\partial\sigma}, \frac{\partial\Psi}{\partial\sigma}\right\rangle_{PG}\right|^{-5/2} d\sigma$$

Therefore

$$I_1'(0) = \int_0^{\lambda(0)} \frac{\left\langle \left.\frac{\partial^3\Psi}{\partial t\partial\sigma}\right|_{t=0}, \left.\frac{\partial^2\Psi}{\partial\sigma^2}\right|_{t=0}\right\rangle_{PG}}{\left|\left\langle \left.\frac{\partial\Psi}{\partial\sigma}\right|_{t=0}, \left.\frac{\partial\Psi}{\partial\sigma}\right|_{t=0}\right\rangle_{PG}\right|^{3/2}} \frac{\left|\left\langle \left.\frac{\partial^2\Psi}{\partial\sigma^2}\right|_{t=0}, \left.\frac{\partial^2\Psi}{\partial\sigma^2}\right|_{t=0}\right\rangle\right|_{PG}}{\left\langle \left.\frac{\partial^2\Psi}{\partial\sigma^2}\right|_{t=0}, \left.\frac{\partial^2\Psi}{\partial\sigma^2}\right|_{t=0}\right\rangle_{PG}} d\sigma \quad (2.9)$$

From (2.9), we obtain

$$I'_1(0) = \int_0^l (\mu''\kappa_g - \mu\kappa_g\tau_g^2 - 2\mu'\kappa_n\tau_g - \mu\kappa_n\tau_g')ds \quad (2.10)$$

Using integration by parts

$$\int_0^l \mu'\kappa_n\tau_g ds = 2\mu(l)\kappa_n(l)\tau_g(l) - 2\int_0^l \mu\kappa_n'\tau_g ds - 2\int_0^l \mu\kappa_n\tau_g' ds \quad , \quad (2.11)$$

$$\int_0^l \mu''\kappa_g ds = \mu'(l)\kappa_g(l) - \mu(l)\kappa_g'(l) + \int_0^l \mu\kappa_g'' ds \quad (2.12)$$

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Using Equations (2.5) ,(2.6), (2.11),(2.12) ,we obtain

$$I'_1(0) = \int_0^l \mu(\kappa_g'' + 2\kappa_n'\tau_g + \kappa_n\tau_g' - \kappa_g\tau_g^2)ds +$$

$$\mu'(l)\kappa_g(l) - \mu(l)\kappa_g'(l) - 2\mu(l)\kappa_n(l)\tau_g(l) \quad (2.13)$$

$$I_2(t) = \int_0^l \varphi ds \quad (2.14)$$

Differentiating of Equation (2.14) at t=0,

$$I_2'(0)=\int_0^l (\frac{\partial\varphi}{\partial t})\bigg|_{t=0} = \int_0^l \mu[p(\frac{\partial\varphi}{\partial u})+q(\frac{\partial\varphi}{\partial v})]ds$$

(2.15)

From (2.13) and (2.15), for all choices of the function $\mu(s)$, $E'(0)=0$, the given timelike arc β must satisfy two boundary conditions and differential equation in pseudo –Galilean 3-space

(BC1) $\kappa_g(l)=0$

(BC2) $\kappa_g'(l)=-2\kappa_n(l)\tau_g(l)$ (2.16)

(DE) $b[\kappa_g''+2\kappa_n'\tau_g+\kappa_n\tau_g'-\kappa_g\tau_g^2]-\theta[p(\frac{\partial\varphi}{\partial u})+q(\frac{\partial\varphi}{\partial v})]=0.$

Case II. Intrinsic formulation for elastic line deformed on a spacelike surface by an external field in the pseudo-Galilean space G_1^3.

N is imelikee , T and Q are spacelike:

For $\left|\kappa_g^2-\kappa_n^2\right|=\kappa_g^2-\kappa_n^2$, we have

$$I'_1(0)=\int_0^l \mu(\kappa_g''+2\kappa_n'\tau_g+\kappa_n\tau_g'+\kappa_g\tau_g^2)ds+$$

$$\mu'(l)\kappa_g(l)-\mu(l)\kappa_g'(l)+2\mu(l)\kappa_n(l)\tau_g(l)$$

(2.17)

For all choices of the function $\mu(s)$, $E'(0)=0$, the given spacelike β arc must satisfy two boundary conditions and differential equation in pseudo –Galilean 3-space .

(BC1) $\kappa_g(l)=0$

(BC2) $\kappa_g'(l)=-2\kappa_n(l)\tau_g(l)$ (2.18)

(DE) $b[\kappa_g''+2\kappa_n'\tau_g+\kappa_n\tau_g'+\kappa_g\tau_g^2]-\theta[p(\frac{\partial\varphi}{\partial u})+q(\frac{\partial\varphi}{\partial v})]=$

Given spacelike β arc must satisfy two boundary conditions and differential equation in pseudo –Galilean 3-space .

(BC1) $\kappa_g(l)=0$

(BC2) $\kappa_g'(l)=2\kappa_n(l)\tau_g(l)$ (2.19)

(DE) $-b[\kappa_g''+2\kappa_n'\tau_g+\kappa_n\tau_g'+\kappa_g\tau_g^2]+\theta[p(\frac{\partial\varphi}{\partial u})+q(\frac{\partial\varphi}{\partial v})]=0$

3. Results

Theorem 3.1. On the timelike surface in in pseudo-Galilean space for the case $\Theta=0$, an timelike geodesic arc is elastic line if and only if it satisfies

$$\kappa_n^2\tau_g=0. \qquad (2.20)$$

Since $\kappa_g=0$, from the third equation of (2.16),

$$-2\kappa_n'\tau_g-\kappa_n\tau_g'=0. \qquad (2.21)$$

From (2.21), first integral is obtained

$\kappa_n^2 \tau_g = cons\tan t$ .The constant must vanish, from the second equation of (2.16).

Theorem 3.2. An timelike geodesic arc on the timelike surface in pseudo-Galilean space for the case $\Theta \neq 0$, is elastic line if and only if it satisfies

$$b\kappa_n^2 \tau_g - \Theta[p(\frac{\partial \varphi}{\partial u}) + q(\frac{\partial \varphi}{\partial v})] = 0 \tag{2.22}$$

Proof. From (2.16) , we get (2.22).

Theorem 3.3. . An spacelike geodesic arc on the spacelike surface in pseudo-Galilean space for the case $\Theta \neq 0$, is elastic line if and only if it satisfies

$$b\kappa_n^2 \tau_g - \Theta[p(\frac{\partial \varphi}{\partial u}) + q(\frac{\partial \varphi}{\partial v})] = 0 \tag{2.23}$$

Proof. From (2.18) , we have (2.23).

Example 3.1. An timelike arc on timelike plane for $\Theta = 0$ in G_1^3, is elastic line if and only if it lies on a geodesic.

Proof. On timelike plane , $\tau_g = (k_2 - k_1)\cosh\theta \sinh\theta$ =0 and $\kappa_n = k_1 \cosh^2\theta - \sinh^2\theta = 0$. From the third equation of (2.16),

$$\kappa_g'' = 0$$

The first integral is $\kappa_g' = const.$

The boundary coinditions of (2.16), $\kappa_g'(l) = 0$. Thus $\kappa_g = 0$.

Example 3.2. An arc on first kind helicoid for $\Theta = 0$ in G_1^3 , is elastic line .

Proof. On first kind helicoid, $\kappa_n = 0$ and $\tau_g = 0$. Thus (2.16) is satisfied..

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