Intrinsic formulation for elastic line deformed on a surface by an external field in the pseudo-Galilean space $G_1^3$

Nevin Gürbüz

Eskişehir Osmangazi University, Mathematics and Computer Sciences Department
ngurbuz@ogu.edu.tr


Keywords: pseudo-Galilean space, elastic line

1. Introduction

Manning studied intrinsic formulation for elastic line deformed external field on a surface by external field $E^3$ (Manning, 1988). Intrinsic equations for a elastic line in Lorentz-Minkowski space was researched (Gürbüz and Gürbüz, 2000), (Gürbüz, 2000). In this paper we derive intrinsic formulation for elastic line deformed external field on a surface by external field in pseudo-Galilean space.

In this section we give preliminaries on pseudo-Galilean space $G_1^3$. The definitions relation to $G_1^3$ was taken (Divjak, 2008).

The pseudo-Galilean 3- space $G_1^3$ is the three dimensional real affine space with the absolute figure {$w,f,I$}, where w is a fixed plane, f a line in w and I a hyperbolic involution of the points of f. The pseudo-Galilean space length of the vector $x(x,y,z)$ is defined by

$$
\begin{cases}
x, & x \neq 0 \\
\sqrt{y^2 - z^2}, & x = 0
\end{cases}
$$

A curve parametrized by the parameter of arc length $s=x$ is given in the coordinat form by $\beta(x)=(x,y(x),z(x))$. The curvature $\kappa(x)$ and $\tau(x)$ of an curve are given by (Divjak, 2008).

$$
\kappa(x) = \sqrt{|y'^2(x) - z'^2(x)|}
$$

$$
\tau(x) = \frac{1}{\kappa^2(x)} \det(r''(x), r'''(x), r''''(x))
$$

The associated moving trihedron is given by

$$
t = r'(x) = (1, y'(x), z'(x))
$$

$$
n = \frac{1}{\kappa(x)} (0, y''(x), z''(x))
$$

$$
b = \frac{1}{\kappa(x)} (0, \varepsilon z'''(x), \varepsilon y'''(x))
$$

where $\varepsilon = 1$ or $\varepsilon = -1$ and it is called a Frenet trihedron associated to the curve. If $f$ is timelike, $n$ is a spacelike vector, $b$ is spacelike.. Frenet-Serret formulas are given as following:

$$
t'(x) = \kappa(x)n(x)
$$

$$
n'(x) = \tau(x)b(x)
$$

$$
b'(x) = \tau(x)n(x).
$$

For regular curve in $G_1^3$, $\kappa$ is defined as following

$$
\kappa = \left| \frac{\Psi' \times_{PG} \Psi''}{||\Psi||^5} \right|
$$

where $\times_{PG}$ denotes pseudo-Galilean cross product. If $e_1$ is unit spacelike vector, $e_2$ is unit spacelike vector , $e_3$ is a unit timelike vector , $a \times_{PG} b$ is given as following:

$$
a \times_{PG} b = \begin{vmatrix}
0 & e_2 & -e_3 \\
-a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix}
$$

where $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$. If $e_1$ is unit spacelike vector, $e_2$ is unit timelike vector , $e_3$
is a unit spacelike vector, \( a \times_{PG} b \) is given as following:
\[
a \times_{PG} b = \begin{bmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
0 & -e_2 & -e_3
\end{bmatrix}
\]

**Theorem 1.1.** Let F be the timelike surface in \( G_1^3 \) and \( \beta \) denote an arc on F. The analogue of the Frenet-Serret formulas in pseudo-Galilean 3-space \( G_1^3 \) is
\[
\begin{bmatrix}
T' \\
Q' \\
N'
\end{bmatrix} = \begin{bmatrix}
0 & \kappa_g & -\kappa_n \\
0 & 0 & -\tau_g \\
0 & -\tau_g & 0
\end{bmatrix} \begin{bmatrix}
T \\
Q \\
N
\end{bmatrix}
\]
where \( \kappa_g \) is the geodesic curvature, \( \kappa_n \) is the normal curvature, \( \tau_g \) is the geodesic torsion, and \( \kappa_n \) is the normal curvature.

\[
\langle T, T' \rangle = -1, \quad \langle N, N \rangle = 1, \quad \langle Q, Q \rangle = 1.
\]

**Theorem 1.2.** Let F be the spacelike surface in \( G_1^3 \) and \( \beta \) denote an spacelike arc on F. The analogue of the Frenet-Serret formulas in pseudo-Galilean 3-space \( G_1^3 \) is
\[
\begin{bmatrix}
T' \\
Q' \\
N'
\end{bmatrix} = \begin{bmatrix}
0 & \kappa_g & -\kappa_n \\
0 & 0 & -\tau_g \\
0 & -\tau_g & 0
\end{bmatrix} \begin{bmatrix}
T \\
Q \\
N
\end{bmatrix}
\]
where \( \kappa_g \) is the geodesic curvature, \( \tau_g \) is the geodesic torsion, \( \kappa_n \) is the normal curvature, and \( \kappa_n \) is the normal curvature.

\[
\langle T, T' \rangle = 1, \quad \langle Q, Q \rangle = 1, \quad \langle N, N \rangle = -1.
\]

**2. Intrinsic Method**

In this section, we study intrinsic formulation for elastic line deformed on surface by an external field in pseudo-Galilean space \( G_1^3 \).

The arc \( \beta \) is called elastic line if it is extremal for the variational problem of \( (2.1) \) within the family of all arcs of length \( l \) on non-null surface \( F \) having the same initial point and initial direction as \( \beta \) in the pseudo-Galilean space \( G_1^3 \).

If elastic line is exposed to a static force field, it has a trajectory that minimizes the sum of its elastic energy and its energy of interaction with the field in \( G_1^3 \). The problem is to to minimize the energy \( E \),
\[
E = \int_0^l \left( \frac{1}{2} \kappa^2 - \theta \kappa \right) ds
\]
\[
E(t) = \frac{1}{2} b l_1(t) - \theta l_2(t)
\]

for suitable scalar functions \( p(s) \) and \( q(s) \). Define
\[
\Psi(\sigma, t) = (u(\sigma) + t\eta(\sigma), v(\sigma) + t\xi(\sigma))
\]
\[
l = \int_0^l \sqrt{\frac{\partial^2 \Psi}{\partial \sigma^2}} ds
\]
for \( 0 \leq \sigma \leq l \).

**Case I. Intrinsic formulation for elastic line deformed on a timeline surface by an external field in the pseudo-Galilean space \( G_1^3 \).**

If \( T \) is timelike, \( Q \) and \( N \) are spacelike,
\[
\frac{\partial \Psi}{\partial \sigma} \bigg|_{\sigma=0} = T, \quad 0 \leq \sigma \leq l
\]
\[
\frac{\partial^2 \Psi}{\partial \sigma^2} \bigg|_{\sigma=0} = T' = \kappa_g Q + \kappa_n N
\]
\[
\frac{\partial \Psi}{\partial t} \bigg|_{\sigma=0} = T' = \mu Q
\]
With second differentiation Equation (2.5), we obtain

$$\frac{\partial^2 \Psi}{\partial t \partial \sigma} \bigg|_{t=0} = T' = \mu' Q - \mu \tau_g N$$

(2.6)

Third differentiation Equation (2.5) gives

$$\frac{\partial^3 \Psi}{\partial t \partial \sigma^2} \bigg|_{t=0} = (\mu'' - \mu \tau_g^2)Q - (2\mu' \tau_g + \mu \tau_g')N$$

(2.7)

$$\frac{\partial \lambda}{\partial t} \bigg|_{t=0} = 0$$

Lemma 2.1. In pseudo-Galilean 3-space,

Proof.

Let $I_1(t) = \int \kappa^2 ds$ denote the total square curvature of the arc $0 \leq \sigma \leq \lambda(t)$. For $t \neq 0$, the total square curvature is

$$I_1(t) = \int_0^{2(t)} \left| \frac{\partial^2 \Psi}{\partial \sigma^2} \right|_{PG} d\sigma \left| \frac{\partial^2 \Psi}{\partial \sigma^2} \right|_{PG} d\sigma$$

Therefore

$$I_1'(0) = \int_0^2 \left| \frac{\partial^3 \Psi}{\partial t \partial \sigma^2} \bigg|_{t=0} \right|_{PG} ^{3/2} \left| \frac{\partial^3 \Psi}{\partial t \partial \sigma^2} \bigg|_{t=0} \right|_{PG} ^{3/2} d\sigma$$

From (2.9), we obtain

$$I_1'(0) = \int_0^2 \left( \mu'' \kappa_g - \mu \kappa_g \tau_g^2 - 2\mu' \kappa_n \tau_g - \mu \kappa' \tau_g' \right) ds$$

(2.10)

Using integration by parts

$$\int_0^l \mu' \kappa_n \tau_g ds = 2\mu(l)\kappa_n(l)\tau_g(l) - 2\int_0^l \mu \kappa_n \tau_g ds - 2\int_0^l \mu \kappa' \tau_g' ds$$

(2.11)

$$\int_0^l \mu'' \kappa_g ds = \mu'(l)\kappa'_g(l) - \mu(l)\kappa'_g(l) + \int_0^l \mu \kappa'' ds$$

(2.12)

Using Equations (2.5), (2.6), (2.11), (2.12), we obtain

$$I_1'(0) = \int_0^l \mu(\kappa'' + 2\kappa_n \tau_g + \kappa' \tau_g' - \kappa_g \tau_g^2) ds +$$

$$\mu'(l)\kappa'_g(l) - \mu(l)\kappa'_g(l) - 2\mu(l)\kappa_n(l)\tau_g(l)$$

(2.13)

$$I_2(t) = \int_0^t \phi ds$$

Differentiating of Equation (2.14) at $t=0$,
From (2.13) and (2.15), for all choices of the function \( \mu(s), E'(0) = 0 \), the given timelike arc \( \beta \) must satisfy two boundary conditions and differential equation in pseudo–Galilean 3-space

\[
\begin{align*}
(BC1) & \quad \kappa_g'(l) = 0 \\
(BC2) & \quad \kappa_g'(l) = -2\kappa_n(l)\tau_g(l) \\
(DE) & \quad b[\kappa_g'' + 2\kappa_n\tau_g + \kappa_n\tau_g' + \kappa_g\tau_g^2] - \theta \left[ p\left( \frac{\partial \varphi}{\partial u} \right) + q\left( \frac{\partial \varphi}{\partial v} \right) \right] = 0.
\end{align*}
\] (2.16)

**Case II. Intrinsic formulation for elastic line deformed on a spacelike surface by an external field in the pseudo-Galilean space \( G^3_1 \).**

N is imelike, T and Q are spacelike:

For \( \left| \kappa_g^2 - \kappa_n^2 \right| = \kappa_g^2 - \kappa_n^2 \), we have

\[
I_1'(0) = \int_0^l \mu(\kappa_g'' + 2\kappa_n\tau_g + \kappa_n\tau_g' + \kappa_g\tau_g^2) ds + \\
\mu'(l)\kappa_g(l) - \mu(l)\kappa_n'(l) + 2\mu(l)\kappa_n(l)\tau_g(l)
\] (2.17)

For all choices of the function \( \mu(s), E'(0) = 0 \), the given spacelike arc \( \beta \) must satisfy two boundary conditions and differential equation in pseudo–Galilean 3-space

\[
\begin{align*}
(BC1) & \quad \kappa_g(l) = 0 \\
(BC2) & \quad \kappa_g'(l) = -2\kappa_n(l)\tau_g(l) \\
(DE) & \quad b[\kappa_g'' + 2\kappa_n\tau_g + \kappa_n\tau_g' + \kappa_g\tau_g^2] - \theta \left[ p\left( \frac{\partial \varphi}{\partial u} \right) + q\left( \frac{\partial \varphi}{\partial v} \right) \right] = 0
\end{align*}
\] (2.18)

Given spacelike arc \( \beta \) must satisfy two boundary conditions and differential equation in pseudo–Galilean 3-space

\[
\begin{align*}
(BC1) & \quad \kappa_g(l) = 0 \\
(BC2) & \quad \kappa_g'(l) = 2\kappa_n(l)\tau_g(l) \\
(DE) & \quad b[\kappa_g'' + 2\kappa_n\tau_g + \kappa_n\tau_g' + \kappa_g\tau_g^2] + \theta \left[ p\left( \frac{\partial \varphi}{\partial u} \right) + q\left( \frac{\partial \varphi}{\partial v} \right) \right] = 0
\end{align*}
\] (2.19)

**3. Results**

**Theorem 3.1.** On the timelike surface in in pseudo-Galilean space for the case \( \Theta = 0 \), an timelike geodesic arc is elastic line if and only if it satisfies

\[
\kappa_n^2\tau_g = 0.
\] (2.20)

Since \( \kappa_g = 0 \), from the third equation of (2.16),

\[-2\kappa_n\tau_g - \kappa_n\tau_g' = 0\] (2.21)

From (2.21), first integral is obtained
\( \kappa_n^2 \tau_g = \text{const} \tan t \) . The constant must vanish, from the second equation of (2.16).

**Theorem 3.2.** An timelike geodesic arc on the timelike surface in pseudo-Galilean space for the case \( \Theta \neq 0 \), is elastic line if and only if it satisfies

\[
b \kappa_n^2 \tau_g - \Theta [ p \left( \frac{\partial \varphi}{\partial u} \right) + q \left( \frac{\partial \varphi}{\partial v} \right) ] = 0
\]

(2.22)

**Proof.** From (2.16), we get (2.22).

**Theorem 3.3.** An spacelike geodesic arc on the spacelike surface in pseudo-Galilean space for the case \( \Theta \neq 0 \), is elastic line if and only if it satisfies

\[
b \kappa_n^2 \tau_g - \Theta [ p \left( \frac{\partial \varphi}{\partial u} \right) + q \left( \frac{\partial \varphi}{\partial v} \right) ] = 0
\]

(2.23)

**Proof.** From (2.18), we have (2.23).

**Example 3.1.** An timelike arc on timelike plane for \( \Theta = 0 \) in \( G^3_i \), is elastic line if and only if it lies on a geodesic.

**Proof.** On timelike plane, \( \tau_g = (k_2 - k_1) \cos \theta \sinh \theta \) and \( \kappa_n = k_1 \cosh^2 \theta - \sinh^2 \theta = 0 \). From the third equation of (2.16),

\[
\kappa_n'' = 0
\]

The first integral is \( \kappa_n' = \text{const} \).

The boundary conditions of (2.16), \( \kappa_n' (l) = 0 \). Thus \( \kappa_n = 0 \).

**Example 3.2.** An arc on first kind helicoid for \( \Theta = 0 \) in \( G^3_i \), is elastic line.

**Proof.** On first kind helicoid, \( \kappa_n = 0 \) and \( \tau_g = 0 \). Thus (2.16) is satisfied.

**References**


10/21/2013