Mixed Integral Equation with Cauchy Kernel and Contact Problem

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Abstract: In this work, the existence of a unique solution of Fredholm - Volterra integral equation (**F-VIE**) of the second kind is discussed and proved in the space $L_2[-1,1] \times C(0,T)$, T ≤ 1. The Fredholm integral term (**FIT**) is considered in position with Cauchy kernel (**CK**), while the Volterra integral term (**VIT**) is considered in time with continuous kernel. A series method is used to separate the variables and obtain a **FIE** of the second kind, where its solution is obtained, using Legendre polynomials. The relation between the contact problem and the integral equation is, also investigated. Finally, numerical results are considered and the error estimate is computed.

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1. Introduction:

The singular **IEs** appear in many problems of mathematical physics and engineering. The singular **IEs** are considered to be of more interest than others cases. In addition, the singular **IEs** appear in studies involving airfoil [1], fracture mechanics [2], contact radiation and molecular conduction [3], contact problems [4] and potential theory [5]. Mkhitarian and Abdou [6, 7] discussed some different methods to solve the **FIE** of the first kind, analytically with logarithmic kernel [6] and Carleman kernel [7].

Consider the mixed integral equation (**F-VIE**)

$$\mu\,\Phi(x,t) + \lambda\int_{-1}^{1} k(y-x)\,\Phi(y,t)\,dy + \lambda\int_{0}^{t} F(t-\tau)\,\Phi(x,\tau)\,d\tau = f(x,t)$$

$$f(x,t) = \frac{\pi}{\theta_1+\theta_2}\left[\gamma(t) + \beta(t)x - h_1(x) - h_2(x)\right]$$

$$\left(|x| \le 1,\ t \in [0,T],\ T \le 1\right) \tag{1.1}$$

under the dynamic conditions

$$\int_{-1}^{1} \Phi(x,t)\,dx = N_1(t) \quad , \quad \int_{-1}^{1} x\,\Phi(x,t)\,dx = N_2(t) \tag{1.2}$$

The integral equation (1.1), under the conditions (1.2), is investigated from the contact problem, in the theory of elasticity, of two rigid surfaces $\mathbf{G_i, v_i}$, $\mathbf{i} = 1, 2$, having two elastic materials occupying the contact domain [-1, 1] where the two functions $h_i(x) \in L_2(-1,1),\ i=1,2$ represent and describe the equations of the upper and lower surfaces. The upper surface is impressed by a given variable force in time $N_1(t),\ 0 \le t \le T \le 1$, whose eccentricity of application $e(t)$, and a given moment $N_2(t)$, that case rigid displacements $\gamma(t)$ and $\beta(t)$ x respectively, through the time $t \in [0,T]$ and the position $x \in [-1,\ 1]$. The unknown function ϕ(x, t) represents the difference in the normal stresses between the two layers. Also, the kernel of position $k(|x-y|)$ depends on the properties of materials of the contact domain and it takes **CK** form. The known positive function

$F(t-\tau)$ represents the characterize function of the resistance of the material through the time t, $\tau \in [0,T]$, where F(0) = constant $\neq$ 0. Also, μ and λ are constants may be complex, that have many physical meanings.

In order to guarantee the existence of a unique solution of Eq. (1.1), under the conditions (1.2), we assume through this work the following conditions

(1) The continuous time functions $\gamma(t) < A_1$, $\beta(t) < A_2$, $F(t-\tau) < A_3$,(A_1, A_2 and A_3 are constants). τ ϵ [0,T], 0 ≤ t ≤ T ≤ 1, with its derivatives, belong to the class $C[0,T]$.

(2) The kernel of position $k(x-y)$ satisfies $\left\{\int_{-1}^{1}\int_{-1}^{1} k^2(x-y)\,dx\,dy\right\}^{1/2} = B$, B is a constant.

(3) The given function $f(x,t)$ is continuous with its derivatives with respect to the position and time in the space $L_2[-1,1]\times C[0,T]$, $t \in [0,T], T \le 1$.

(4) The unknown function $\Phi(x,t)$ satisfies Lipschitz condition for the first argument and Hölder condition for the second argument.

The reader can prove that: the integral operator $K\Phi(x,t) = \int_{-1}^{1} k(|x-y|)\,\Phi(y,t)\,dy$ for all $t \in [0,T]$, $T \le 1$ is bounded and continuous in the space $L_2[-1,1]\times C[0,T]$,

In the reminder part of this work, the series method is used to give directly, two linear systems of **IEs.** The first system is **VIEs** of the second kind with continuous kernel, while, the second represents **FIE** of the second kind**.** To solve the **FIF**, we use the method of removing the singularity. Then the solution of **FIF** is expanded in term of Legendre polynomials to obtain **LAS.** Moreover, numerical applications are considered. Finally, the relation between the contact problem and the **IE** with **CK** is investigated

2. Separation of Variables: Rewrite the formula (1.1) to obtain

$$\mu\,\Phi(x,t) + \lambda\int_{-1}^{1} k(|x-y|)\,\Phi(y,t)\,dy + \lambda\int_{0}^{t} F(t-\tau)\Phi(x,\tau)\,d\tau$$
$$= \pi\,\ell\,[\gamma(t) + \beta(t)x - h_1(x) - h_2(x)],\quad \left(\ell = \frac{1}{\theta_1+\theta_2}\right) \tag{2.3}$$

Then, let t = 0 in Eq. (2.3) and (1.2), one has

$$\mu\,\Phi(x,0) + \lambda\int_{-1}^{1} k(|x-y|)\,\Phi(y,0)\,dy = \pi\,\ell\,[\gamma(0)+\beta(0)x-h_1(x)-h_2(x)] \tag{2.4}$$

and

$$\int_{-1}^{1}\Phi(y,0)\,dy = N_1(0) = M_1 \quad , \quad \int_{-1}^{1} x\,\Phi(x,0)\,dx = N_2(0) = M_2 \tag{2.5}$$

Where, M_1 **,** M_2 are absolutely constants. Rewrite (2.4) in the form

$$\mu\,\psi(x) + \lambda\int_{-1}^{1} \mathrm{k}(|\mathrm{x}-\mathrm{y}|)\,\psi(\mathrm{y})\,\mathrm{dy} = \mathrm{g}(\mathrm{x}),$$
$$g(x) = \pi\,\ell\,[\gamma(0) + \beta(0)x - h_1(x) - h_2(x)],\qquad \psi(x) = \Phi(x,0). \tag{2.6}$$

Equation (2.6) represents **FIE** of the second kind with discontinuous kernel in a Cauchy form. Again, consider the solution of Eq. (2.3) in the form

$$\Phi(x,t) = \Phi_0(x,t) + \Phi_1(x,t) \tag{2.7}$$

where $\Phi_0(x,t)$ and $\Phi_1(x,t)$ are called complementary and the particularly solution of the **FIE** (2.3), respectively.

Using (2.7) in (2.3), we get

$$\mu\,\Phi_j(x,t) + \lambda\int_{-1}^{1} k(|x-y|)\Phi_j(y,t)\,dy + \lambda\int_{0}^{t} F(t-\tau)\,\Phi_j(x,\tau)\,d\tau$$

$$= \pi \ell \delta_j \left[\gamma(t) + \beta(t) x - h_1(x) - h_2(x)\right] \quad (2.8)$$

Also, using (2.7) in (2.4) yields

$$\mu \Phi_j(x,0) + \lambda \int_{-1}^{1} k(|x-y|)\Phi_j(y,0)\,dy = \pi \ell \delta_j \left[\gamma(0) + \beta(0)x - h_1(x) - h_2(x)\right] \quad (2.9)$$

Finally, we have

$$\mu\left[\Phi_j(x,t) - \Phi_j(x,0)\right] + \lambda \int_{-1}^{1}\left[\Phi_j(y,t) - \Phi_j(y,0)\right] k(|x-y|)\,dy$$
$$+ \int_0^t F(t-\tau)\,\Phi_j(x,\tau)\,d\tau = \pi \ell \delta_j \left[\gamma(t) - \beta(t)x - \gamma(0) + \beta(0)x\right] \quad (2.10)$$

where

$$\delta_j = \begin{cases} 0 & j = 0 \\ 1 & j = 1 \end{cases} \quad (2.11)$$

Assume the solution of the two functions $\phi_0(x,t)$ and $\phi_1(x,t)$ can be expressed in the following series form

$$\Phi_j(x,t) = \sum_{n=1}^{\infty}\left[A_{2n}^{(j)}(t)\phi_{2n}(x) + A_{2n-1}^{(j)}(t)\phi_{2n-1}(x)\right]. \quad (2.12)$$

Where $\phi_{2n}(x)$ and $\phi_{2n-1}(x)$ stand from the even and odd functions respectively.

Theorem 1, *see [8]: For a symmetric and positive kernel, the integral operator*

$$K\,\Phi(x,t) = \int_{-1}^{1} k(|x-y|)\,\Phi(y,t)\,dy \quad (2.13)$$

is compact, and self-adjoint for all $t \in [0,T],\ T \le 1$. *So, we can express it as a linear combination of eigenvalues and eigenfunctions* $\alpha_n K\Phi(x,t) = \phi_n(t)$, *where* α_n *and* ϕ_n *are the eigenvalues and eigenfunctions of the integral operator, respectively.*■

In view of theorem 1 and formula (2.12), the **IE** (2.10), under the conditions (1.2), leads to the following:

$$A_n^{(1)}(t) + \beta_n \int_0^t A_n^{(1)}(\tau) F(t-\tau)\,d\tau = A_n^{(1)}(0) \qquad (n \ge 1), \quad (2.14)$$

$$A_{2n}^{(o)}(t) + \beta_{2n} \int_0^t A_{2n}^{(o)}(\tau) F(t-\tau)\,d\tau = \pi \ell \beta_{2n} b_{2n}\left[\gamma(t) - \gamma(0)\right], \quad (2.15)$$

$$A_{2n-1}^{(o)}(t) + \beta_{2n-1} \int_0^t A_{2n-1}^{(o)}(\tau) F(t-\tau)\,d\tau = \pi \ell \beta_{2n-1} b_{2n-1}\left[\beta(t) - \beta(0)\right], \quad (2.16)$$

where

$$\sum_{n=1}^{\infty} b_{2n}\phi_{2n} = 1 \quad , \quad \sum_{n=1}^{\infty} b_{2n-1}\phi_{2n-1} = x, \quad \left(n \ge 1, \quad A_n^{(o)} = 0\right) \quad (2.17)$$

and $\beta_n = \lambda \alpha_n (1 + \mu \alpha_n)^{-1}$.

The value of $\mathbf{A_n^{(1)}(0)}$ can be obtained, directly from (2.7) and (2.9) in the form

$$A_n^{(1)}(0) = \pi \beta_n \gamma(0) \quad (2.18)$$

Equations (2.14) and (2.16) represent **VIEs** of the second kind with continuous kernel $F(t-\tau) \in C([0,T] \times [0,T])$ each of it has a unique solution in $C[0,T]$.

In view of (2.14) and (2.16), the general solution (2.10) can be adapted in the form

$$\phi(x,t) = \sum_{n=1}^{\infty}\left[A_n^{(0)}(t) + A_n^{(1)}(t)\right]\phi_n(x) \quad (2.19)$$

where $A_n^{(0)}(t)$ and $A_n^{(1)}(t)$ must satisfy the inequality

$$\sum_{k=1}^{\infty}\left[A_k^{(0)}(t)+A_k^{(1)}(t)\right]^2<\varepsilon \quad , \quad \varepsilon<<1 \tag{2.20}$$

Theorem 2: *see [8]. If, for* $t\in[0,T]$, *the inequality (2.20) holds, then the series (3.19) is convergence uniformly in* $L_2[-1,1]$. *Moreover Eq. (1.1) has a unique solution in* $L_2[-1,1]\times C[0,T]$. ■

3. Volterra Integral Equation: Consider the integral equation

$$A(t)+\lambda\int_0^t F(t-\tau)\,A(\tau)\,d\tau=f(t) \tag{3.21}$$

where $\mathbf{F}(t-\tau)$ is continuous kernel. Different methods are used, in [9], to obtain the solution of (4.21). Here, we use Laplace transform method to solve (3.21), see [10].
For this, assume

$$L\{f(t)\}=\int_0^{\infty}e^{-st}f(t)\,dt \tag{3.22}$$

and use the convolution theorem, to get

$$L\{A(t)\}+\mu\,L\{F(t)\}L\{A(t)\}=L\{f(t)\} \tag{3.23}$$

Hence, the solution of (3.21) can be written in the following form

$$A(t)=L^{-1}\left\{\frac{L\{f(t)\}}{1+\mu\,L\{F(t)\}}\right\} \tag{3.24}$$

where L^{-1} is the inverse of Laplace transformation. The formula (3.24) represents a general solution of the **VIE** (3.21) for any values of $f(t)$ and $F(t-\tau)$.

4. *Fredholm equation*:

Consider the **FIE** of the second kind with **CK**

$$\phi(x)+\lambda\int_{-1}^{1}\frac{\phi(y)}{x-y}\,dy=f(x) \quad , \quad \lambda \text{ is a constant} \tag{4.25}$$

under the static conditions

$$\phi(\pm 1)=0 \tag{4.26}$$

The sign $\int$ denotes integration with Cauchy principal value sense. For this aim, the singularity of the integral term of Eq. (4.25) will be weakened as follows

$$\int_{-1}^{1}\frac{\phi(y)}{y-x}\,dy=\int_{-1}^{1}\frac{\phi(y)-\phi(x)}{y-x}\,dy-\phi(x)\cdot\ln\frac{1+x}{1-x} \tag{4.27}$$

The integral term in the right hand side of (4.27) is regular and will be evaluated. In this aim, assume that, the unknown function of the following integral equation

$$\phi(x)+\lambda\int_{-1}^{1}\frac{\phi(y)-\phi(x)}{y-x}\,dy-\lambda\,\phi(x)\log\frac{1+x}{1-x}=f(x) \qquad -1\le x\le 1 \tag{4.28}$$

can be expanded in term of Legendre polynomials form, i.e.

$$\phi(x)=\sum_{j=0}^{\infty}a_j P_j(x) \tag{4.29}$$

where a_j are constants and $P_j(x)$ are the Legendre polynomials. Substituting from (4.29) in (4.28), we get

$$\left[1-\lambda\,\log\left(\frac{1+x}{1-x}\right)\right]\sum_{j=0}^{\infty} a_j\,P_j(x)+\lambda\sum_{j=0}^{\infty} a_j\int_{-1}^{1}\frac{P_j(y)-P_j(x)}{y-x}\,dy=f(x) \tag{4.30}$$

The value of $\mathbf{a_0}$ in (4.29) can be obtained, after using the following orthogonal relation,

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx=\begin{cases}\frac{2}{2n+1} & n=m\\ 0 & n\neq m\end{cases}. \tag{4.31}$$

When n = m = 0, we obtain a_0 = c, where **c** is a constant equivalent to the value $c=\frac{1}{2}\int_{-1}^{1}\phi(y)\,dy$ .To determine the rest of the coefficients a_j, j = 1, 2,…, we use the Rodriguez formula of the Legendre polynomial $P_j(x)$ of degree j, where , see [12],

$$P_j(x)=\sum_{k=0}^{\left[\frac{j}{2}\right]} b_k\,x^{j-2k}\quad ,\quad b_k=\frac{(-1)^k\,(2j-2k)!}{2^k\,k!(j-k)!(j-2k)!} \tag{4.32}$$

to obtain

$$\int_{-1}^{1}\frac{P_j(y)-P_j(x)}{y-x}\,dy=\sum_{k=0}^{\left[\frac{j-1}{2}\right]} b_k\sum_{\ell=0}^{j-2k-1}\gamma_{j,k,\ell}\,x^{\ell}\quad ,\quad \gamma_{j,k,\ell}=\frac{b_k\left[\ell-(-1)^{j-1}\right]}{j-2k-1} \tag{4.33}$$

Using equations (4.29) and (4.33) in (4.30), yields

$$\left[1-\lambda\log\frac{1+x}{1-x}\right]\sum_{j=0}^{\infty} a_j\,P_j(x)+\lambda\sum_{j=1}^{\infty}\sum_{k=0}^{\left[\frac{j-1}{2}\right]}\sum_{\ell=0}^{j-2k-1} a_j\,b_k\,\gamma_{j,k,\ell}\,x^{\ell}=f(x) \tag{4.34}$$

Multiply both sides of (4.34) by x^{i-1} for i= 1, 2,…,N and integrate the result over the interval [-1, 1], to obtain

$$\sum_{j=0}^{\infty}a_j\{\int_{-1}^{1}(1+\lambda\sum_{\beta=1}^{M}\frac{x^{2\beta}}{\beta})x^{i-1}P_j(x)\,dx+\lambda\sum_{k=0}^{\left[\frac{j-1}{2}\right]}b_k\sum_{\ell=0}^{j-2k-1}\frac{2[1-(-1)^{j-\ell}]}{(j-2k-\ell)(\ell+i)}\}$$

$$=\int_{-1}^{1}x^{i-1}f(x)\,dx-2c\,[\frac{\delta_{i-1}}{i}-\frac{2\lambda\delta_i}{i+1}]\quad ,\quad \delta_c=\begin{cases}1 & c\ \ even\\ 0 & c\ \ odd\end{cases}\quad , \tag{4.35}$$

here, we used the approximation $$\log\frac{1+x}{1-x}\cong 2\sum_{\beta=1,3,5}^{M}\frac{x^{\beta}}{\beta}\ ,\quad (M\ \ odd)\ ,\quad |x|<1 .$$

Using the Rodriguez formula, see [12]

$$\int_{-1}^{1}x^i\,P_i(x)\,dx=2\sum_{k=0}^{\left[\frac{j}{2}\right]}\frac{b_k\delta_{i,j}}{j+i-2k+1},\quad \delta_{c,d}=\begin{cases}1 & c+d\quad even\quad c\geq d\\ 0 & otherwise,\end{cases} \tag{4.36}$$

the formula (4.35), becomes

$$\sum_{j=1}^{\infty}a_j\left\{\sum_{k=0}^{\left[\frac{j}{2}\right]}[\frac{2b_k\,\delta_{i-1,j}}{j+i-2k}+2\sum_{\beta=1}^{M}\frac{\lambda\,b_k\,\delta_{2\beta+i-1,j}}{\beta(j+i-2k+2\beta)}]\right.$$

$$\left.+2\sum_{k=0}^{\left[\frac{j-1}{2}\right]}\sum_{\ell=0}^{j-2k-1}\frac{\lambda b_k\left[1-(-1)^{j-\ell}\right]}{(j-2k-\ell)(\ell+i)}\,\delta_{\ell+i-1}\right\}=$$

$$\int_{-1}^{1}x^{i-1}f(x)dx-2c\left[\frac{\delta_{i-1}}{i}+\frac{2\lambda\delta_i}{i+1}\right]\qquad(i=1,2,...,N) \tag{4.37}$$

If we truncate the infinite series of the left hand side of Eq. (4.16) to the first N terms, thus we have

$$\sum_{j=1}^{N} D_{ij}\, a_j = d_j \qquad (1 \le i \le N) \tag{4.38}$$

where

$$D_{ij} = \sum_{k=0}^{\left[\frac{j}{2}\right]} \Big[\frac{2\, b_k\, \delta_{i-1,\, j}}{j+i-2k} + 2\sum_{\beta=1}^{M} \frac{\lambda\, b_k\, \delta_{2\beta+i-1,\, j}}{\beta\,(j+i-2k+2\beta)} \Big] + 2\sum_{k=0}^{\left[\frac{j-1}{2}\right]} \sum_{\ell=0}^{j-2k-1} \frac{\lambda\, b_k \left[1-(-1)^{j-\ell}\right]}{(j-2k-\ell)(\ell+i)}\, \delta_{\ell+i-1} \;,$$

$$b_i = \int_{-1}^{1} x^{i-1} f(x)\, dx - 2c \left[\frac{\delta_{i-1}}{i} + \frac{2\lambda\, \delta_i}{i+1} \right] \tag{4.39}$$

5. *Numerical Result*:

The solution of the **IE** (4.25) depends on the **CK**, the value of λ and the given function $f(x)$. Here, we assume the following: $f(x) = x,\ c = 0.8,\ \lambda = 0.25$ and N = 20 and we use the general assumption of the solution $\phi(x) = \sum_{j=0}^{20} a_j\, P_j(x),$ where $P_j(x)$ is Legendre polynomial and the coefficients $\mathbf{a_j}$ are the solution of the linear system (4.38), we used Maple 10 to solve such system. These coefficients are tabulated in table (1).

Table 1: The coefficients $\mathbf{a}_j$ **,** $\boldsymbol{j}$ **= 1, 2,…., 20** (ordered in

-.796513983	1.282960283	-1.39797130	1.456975757	-1.549072869
1.479876614	-1.567379746	1.417647537	-1.559090422	1.178337053
-1.392267336	2.0371206564	-1.596255523	-2.924756126	-5.248330647
8.38415880	6.33866610	1.704701286	-.2528251584	-1.196720603

6. Physical Meaning of FIE with CK:

The solution of the **IE** (4.25) under the static condition (4.26) is equivalent to the solution of the following contact problem: Consider the semi-symmetric contact problem of stamp, of equation $f_1(x)$, is impressed on the strip of equation $f_2(x)$, then consider the tangent force $t(x)$ is related with the normal pressure $\mathbf{p(x)}$ in the contact region, $\Omega = \left\{ (\mathbf{x}, \mathbf{y}) \in \Omega : -1 \le \mathbf{x}, \mathbf{y} \le 1 \right\}$, of the two surfaces by the relation [13]

$$t(x) = k\, p(x). \tag{6.40}$$

Also, let the tangent stress σ and the normal stress t_{xy} satisfy the relation

$$t_{xy} = k\, \sigma_y \tag{6.41}$$

where k is the fraction coefficient. Consider $v_i^*\ (i = 1, 2)$ represents the displacement components in **y** direction, satisfying the following [14]

$$\frac{d\, v_1^*}{dx} = \frac{t(x)}{G_1} \quad , \quad \frac{d\, v_2^*}{dx} = \frac{t(x)}{G_2} \tag{6.42}$$

Where, G_1 and G_2 are the displacement compressible materials of the two surfaces $f_1(x)$ and $\mathbf{f}_2(\mathbf{x})$ respectively. It is known that, see [14], such problem reduces to the following integral equation

$$k_1 \frac{G_1 + G_2}{G_1 G_2} \int_0^x \phi(t)\, dt + (v_1 + v_2) \int_{-1}^{1} G\left(\frac{x-y}{\lambda} \right) \phi(t)\, dt = \delta - f_1(x) - f_2(x), \quad \lambda \in [0, \infty],$$

$$G(z) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh u}{u}\, e^{iuz}\, du \quad , \quad z = \left(\frac{x-y}{\lambda} \right) \tag{6.43}$$

under the condition $\int_{-1}^{1} \phi(y)\, dy = P < \infty$, where $\phi(t)$ is the unknown potential function which is continuous through the interval of integration $[-1,1]$, the contact domain between the two surfaces $f_i(x)(i=1,2)$, δ is the rigid displacement under the action of a constant force P. Also, k_1 is a physical constant, $G(t)$ is the discontinues kernel of the problem with singularity at the point **x** = **y**, and $v_i = \frac{1-\mu_i^2}{\pi E_i}(i=1,2)$. Here, μ_i are the Poisson's coefficients and E_i are the coefficients of Young. As in [15], the kernel can be written in the form $G(t) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh u}{u} e^{iut}\, du = -\ln\left| \tanh\frac{\pi t}{4} \right|$. If $\lambda \longrightarrow \infty$ and the term $\frac{x-y}{\lambda}$ is very small, so that it satisfies the condition $\tanh z \cong z$, we have $\ln\left|\tanh\frac{\pi t}{4}\right| = \ln t - d\ , \ (d = \ln\frac{4\lambda}{\pi})$. Multiplying Eq. (6.43) by the term $\frac{G_1 G_2}{k_1(G_1 + G_2)}$, and then using (6.45), we get

$$\int_0^x \phi(t)dt + v\int_{-1}^{1} [-\ln(y-x) + d]\phi(y)dy = f^*(x), \tag{6.46}$$

where,

$$v = \frac{G_1 G_2 (v_1 + v_2)}{k_1(G_1 + G_2)}, \qquad f^* = \frac{G_1 G_2 [\delta - f_1(x) - f_2(x)]}{k_1(G_1 + G_2)},$$

Differentiating equation (6.46) with respect to x, we have

$$\phi(x) + v\int_{-1}^{1} \frac{\phi(y)}{y-x} dy = f(x) \qquad \left(f(x) = \frac{df^*(x)}{dx} \right). \tag{6.47}$$

Equation (6.50) represents a **FIE** of the second kind with **CK**. As an important special case, if we let in Eq. (6.46) $G_1 + G_2 = 0,\ f_2(x) = 0$, we have a **FIE** of the first kind with logarithmic kernel. Abdou and Hassan [14] used potential theory method to obtain the eigenvalues and eigenfunctions such problem. Also, Abdou and Ezz-Eldin [16] used Krein's method to obtain the solution of the **FIE** of the first kind.

References

[1] T. Diogo, P. Lima, Super Convergence of Collocation Methods for Class of Weakly Singular Integral Equations, J. Cam. Appl. Math. 218(2008) 307-316.

[2] M. Hori, S. N. Nasser, Asymptotic Solution of a Class of Strongly Singular Integral Equations", SIAM. J. Appl. Math. Vol. 50 No. 3 June (1990) 716 - 727.

[3] J. I. Frankel, A. Gherkin Solution to a Regularized Cauchy Singular Integro-Differential Equation", Quart. Appl. Math. Vol. LIII June (1995), 245 - 259.

[4] R. Kangro, P. Oja, Convergence of Spline Collection for Volterra Integral Equation, Appl. Num. Math. 58(2008)1434-1447.

[5] M. A. Abdou, "Fredholm - Volterra Integral Equation and Generalized Potential Kernel", Appl. Math. Comput. 131 (2002), 81 - 94.

[6] S. M. Mkhitarian,, M.A. Abdou, "On Different Method of Solution of the Integral Equation for the Planer Contact Problem of Elasticity", Dakl. Acad. Nauk Arm. SSR 89(2) (1990), 59 - 74.

[7] S. M. Mkhitarian, M.A. Abdou, On Various Methods for the Solution of the Carleman

Integral Equation, Dakl. Acad. Nauk Arm. SSR 89(3) (1990), 125 - 129.
[8] M. A. Abdou, Integral Equation of Mixed Type and Integrals of Orthogonal Polynomials, J. Comp. Appl. Math. 138 (2002), 273 - 285.
[9] T. A. Burton, Volterra Integral and Differential Equations", London, New York, 1983.
[10] G. Bateman and A. Ergelyi, "Higher Transcendental Function", Vol. II, Reprinted Mc Graw-Hill, New York, 1989.
[11] N. N. Lenedev, R. A. Silverman, Special Functions and Their Applications, Dover Publications, Inc., New York, 1972.
[12] I.S. Gradstein, I. M. Ryzhik, Tables of Integrals, Series and Products", Academic Press, New York, 1980.
[13] V. M. Aleksandrov, E. V. Kovalenko, Problems in the Mechanics of Continuous Media with Mixed Boundary Conditions", Nauka Moscow, 1986.
[14] M. A. Abdou, S. A. Hassan, Boundary value of a Contact Problem, PUMA, Vol. 5, No. 3 (1994), 311 - 316.
[15] G. Ya. Popov, Contact Problems for a Linearly Deformable Function,, Visha Shkola, Kiev, Odessa, 1982.
[16] M. A. Abdou, N. Y. Ezz-Eldin, Krein's Method with Certain Singular Kernel for Solving the Integral Equation of the first kind, Period, Math. Hung. Vol. 28, No. 2 (1994), 143 - 149.

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