Solving Fractional Vibrational Problem Using Restarted Fractional Adomian's Decomposition Method

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Abstract: In this paper, the proposed Restarted Fractional Adomian's Decomposition Method (RFADM) is applied to obtain the analytical approximate solutions to the time fractional vibration equation. The fractional derivative are described in the Modified Remann-Liouville sense. The proposed method performs extremely well in terms of efficiency and simplicity. The effectiveness and good accuracy of method is verified by the numerical results. Numerical results are presented graphically.
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1. Introduction

In recent years, considerable interest in fractional differential equations and analysis of fractional differential equations, which are obtained from the classical differential equations in mathematical physics, engineering, electromagnetics, acoustics, viscoelasticity, electrochemistry and material science vibration and oscillation by replacing the second order time derivative by a fractional derivative of order α satisfying $1 < \alpha \leqslant 2$, have been a field of growing interest as evident from literature survey. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Analytical methods used to solve these equations have very restricted applications and the numerical techniques commonly used give rise to rounding of errors. Several mathematical methods including Adomian's decomposition method [1-3], Modified decomposition method [4], variational iteration method [4,5], differential transform method [6] and homotopy perturbation method [7,8] have been developed to obtain exact and approximate analytic solutions to differential equations of fractional order.

For the past three decades, a great interest has been focused on the application of Adomian's decomposition method to solve for analytic solutions of a wide variety of linear and nonlinear problems. This method was first introduced by G. Adomian's [9, 10] in the beginning of the 1980's and has led to several modifications on the method made by various researchers in an attempt to improve the accuracy and applications. Adomian's and Rach [11] introduced modified Adomian's polynomials which converge slightly faster than the original polynomials and are convenient for computer generation. Adomian's also introduced accelerated Adomian's polynomials [12], despite the various types of Adomian's polynomials available; the original Adomian's polynomials are more generally used based on the advantage of a convenient algorithm which is easily remembered. Recently, F. A. Hendi et al.[13] presented simple Mathematica program to compute Adomian's polynomials. Wazwaz [14] used padé approximants to the solution obtained using a modified decomposition method and found that not only does this improve the result, but also that the error decreases with the increase of the degree of the padé approximants. Another modification to ADM was proposed by Wazwaz [15] a reliable modification of the Adomian's decomposition method. In 2005, Wazwaz [16] presented another type of modification to the ADM. New modification was proposed by Luo [17, 18], this variation separates the ADM into two steps and therefore is termed the two-step ADM. Another recent modification is termed the restarted Adomian's method [19, 20], this method involves repeatedly updating the initial term of the series generated. Several other researchers have developed modifications to the ADM [21,22]. The modifications arise from evaluating difficulties specific for the type of problem under consideration. The modification usually involves only a slight change and is aimed at improving the convergence or accuracy of the series solution.

In this paper, we will consider fractional vibration equation by using factional restarted Adomian's decomposition method (FRADM) . This fractional vibration equation is obtained by replacing the second time derivative term in the corresponding vibration equation by a fractional derivative of order α with $1 < \alpha \leqslant 2$. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 2$, the fractional vibration equation reduces to the standard vibration equation.

2. Preliminaries

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

2.1. Definition

Assume $f: R \to R,\ x \to f(x)$ denote a continuous (but not necessarily differentiable) function and let the partition $h>0$ in the interval $[0,1]$. Jumarie's derivative is defined through the fractional difference

$$\Delta^{\alpha} = (FW-1)^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x+(\alpha-k)), \quad (1)$$

Where $FWf(x) = f(x+h)$. Then the fractional derivative is defined as the following limit.

$$f^{(\alpha)} = \lim_{h \to 0} \frac{\Delta^{\alpha}[f(x)-f(0)]}{h^{\alpha}}. \quad (2)$$

This definition is close to the standard definition of derivative, and as a direct result, the αth derivative of a constant $0<\alpha<1$; is zero.

2.2. Definition

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ for a function $f \in C_{\mu},\ \mu \geq -1,$ is defined as

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi,\ \alpha>0,\ t>0. \quad (3)$$

2.3. Definition

Jumarie's fractional derivative is a mo1dified Riemann-Liouville derivative for $0<\alpha<1,$ defined as

$$D_t^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha-1}(f(t)-f(0))dt,, \\ \frac{1}{\Gamma(-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha}(f(t)-f(0))dt, \\ \left[f^{\alpha-n}(x)^n\right]^n, n \leq \alpha \leq n+1, n \geq 1 \end{cases} \quad (4)$$

Some useful formulas and results of Jumarie's modified Riemann–Liouville derivative are summarized as

$$D_x^{\alpha} c = 0, \alpha \geq 0,\ c=\text{constant}. \quad (5)$$

$$D_x^{\alpha}[cf(x)] = cD_x^{\alpha} f(x),\ \alpha \geq 0,\ c=\text{constant}. \quad (6)$$

$$D_x^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \beta \geq \alpha \geq 0. \quad (7)$$

$$D_x^{\alpha}[f(x)g(x)] = [D_x^{\alpha} f(x)g(x) + f(x)[D_x^{\alpha} g(x)]. \quad (8)$$

$$D_x^{\alpha} f(x(t)) = f_x^{'}(x).x^{\alpha}(t). \quad (9)$$

2.4. Definition

Fractional derivative of compounded functions is defined as

$$d^{\alpha} f(x) \cong \Gamma(1+\alpha) df,\ 0<\alpha<1. \quad (10)$$

2.5. Definition

The integral with respect to $(d\xi)^{\alpha}$ is defined as the solution of fractional differential equation given by Eq.

$$d\,y \cong f(x)(dx)^{\alpha}, x \geq 0, y(0)=0,\ 0<\alpha<1, \quad (11)$$

$$y \cong \int_0^x f(\xi)(d\xi)^{\alpha} = \alpha \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\zeta. \quad (12)$$

For example $f(x) = x^{\beta}$ in Eq. (12), we have

$$\int_0^x \xi^{\beta}(d\xi)^{\alpha} = \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} x^{\alpha+\beta}. \quad (13)$$

3. Analysis Fractional Adomian's Decomposition Method (M-1)

The Adomian's Decomposition Method (ADM) is a method for solving wide range of problems whose mathematical models yield equation or system of equations of algebraic, differential, integral and integro-differential equations or system of equations. In this method the solution is considered as rapidly converging, infinite series.

In order to elucidate the solution procedure of the ADM, we consider the following fractional differential equation:

$$L^{\alpha} u(x,t) = R\,u(x,t) + N\,u(x,t) + q(x,t), \quad (14)$$

$$t>0,\ x \in R,\ 1<\alpha \leq 2,$$

$$u(x,0) = f_1(x),\ u_t(x,0) = f_2(x),$$

where L^{α} is the fractional derivative, N represents general nonlinear differential operator and R is the linear differential operator in x, $f(x)$ and $q(x,t)$ are continuous functions. According to the ADM, we can construct relation for Equation (14) as follows

$$u(x,t) = u(x,0) + u_t(x,0)t + I^{\alpha}(R\,u(x,t) + N\,u(x,t) + q(x,t)),$$

$$u(x,t)=u(x,0)+u_t(x,0)t+\frac{1}{\Gamma(\alpha)}\cdot$$
$$\int_0^t (t-\xi)^{\alpha-1}\left(Ru(x,t)+Nu(x,t)+q(x,t)\right)d\xi, \tag{15}$$

Combining Eq. (12) and Eq. (15), we obtained a proposed relation

$$u(x,t)=g(x,t)+\frac{1}{\Gamma(\alpha+1)}\int_0^t \left(Ru(x,t)+Nu(x,t)\right)(d\xi)^{\alpha}. \tag{16}$$

The Decomposition method suggests that the solution $u(x,t)$ be decomposed into an infinite series of components

$$u(x,t)=\sum_{n=0}^{\infty} u_n(x,t), \tag{17}$$

and the nonlinear function in Eq. (14) is decomposed as follows

$$Nu(x,t)=\sum_{n=0}^{\infty} A_n, \tag{18}$$

where $\sum_{n=0}^{\infty} A_n$ are the so-called Adomian's polynomials. Substituting the decomposition series Eq. (17) and Eq. (18) into the both sides of Eq. (16) gives

$$\sum_{n=0}^{\infty} u(x,t)=g(x)+\frac{1}{\Gamma(\alpha+1)}\cdot$$
$$\int_0^t \left(Ru(x,t)+\sum_{n=0}^{\infty} A_n \right)(d\xi)^{\alpha}. \tag{19}$$

From Eq. (19), the iterates are determined by the following recursive way

$$u_0=g(x),$$
$$u_{n+1}=\frac{1}{\Gamma(\alpha+1)}\int_0^t \left(Ru_n+A_n \right)(d\xi)^{\alpha}, n\geq 0, \tag{20}$$

The Adomian's polynomial can be calculated for all the types of nonlinearities and are given by

$$A_0=F(u_0),$$
$$A_1=u_1 F'(u_0),$$
$$A_2=u_2 F'(u_0)+u_1^2 F''(u_0)$$
$$A_3=u_3 F'(u_0)+u_1 u_2 F''(u_0)+\frac{u_1^3}{3}F'''(u_0),$$
$$\vdots$$

where $Nu(x,t)=F(u)$ is the nonlinear function in Eq. (19). Finally, we approximate the solution $u(x,t)$ by the series

$$\Psi_N(t)=\sum_{n=0}^{N-1} u_n(x,t) \text{ and } \lim_{N\to\infty}\Psi_N(t)=u(x,t).$$

4. Restarted Fractional Adomian's Decomposition Method (RFADM) (M-2)

The restarted ADM was used in [23] as a new method based on standard ADM for solving algebraic equations. The author in [24] applied the method to solve a system of nonlinear Fredholm integral equations of the second kind. Basically the RADM has the same structure as that of the ADM but the ADM is used more than once. In this paper, we propose the extension of RADM and test it for Fractional vibration equation. If we consider a general nonlinear equation of the form (14) and applied ADM to solve it we get the recursive relationship (20), we introduce the algorithm of restarted Adomian's method as the following.

Choose small natural numbers m,k.

Apply the Adomian's method on Equations (6) and calculate $u_0,u_1,u_2,\ldots,u_k$

Set $u_0+u_1+u_2+\ldots+u_k$

Let Z be the proper function which will be determined next for $j=2:m, Z=\varphi^{j-1}$

$$u_0=Z,$$
$$u_1=f-Z+A_0,$$
$$u_2=A_o,$$
$$\vdots$$
$$u_{k+1}=A_k,$$
$$\varphi^j=u_0+u_1+u_2+\ldots+u_k.$$

The Adomian's method usually gives sum of the some first terms as an approximation of *u*, in this algorithm we can update u_0 in each step, but we don't calculate the terms with large index, so *m* and *k* are considered small.

5. Fractional Vibration Equation

We consider the fractional calculus version of the standard vibration equation in one dimension as

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{c^2}\frac{\partial^{\alpha} u}{\partial t^{\alpha}}, \quad r\geq 0,\ t\geq 0,\ 1<\alpha\leq 2, \tag{21}$$

with the initial conditions

$$u(r,0)=r^2,\quad \frac{\partial}{\partial t}u(r,0)=cr,$$

which constitute the relation between the radial velocity of $u(r,t)$ to the fractional time derivative of order α ($1 < \alpha \leqslant 2$) of $u(r,t)$ and c is the wave velocity of free vibration. It is easily seen that the whole hierarchy of moments $M_k = < r^k(t) >$ have the same time dependence as for the fractional Brownian motion though their statistical features are quite different. Now taking the Laplace transform of eq. (21), we get

$$s^\alpha \bar{u}(r,s)=c^2\left[\frac{d^2\bar{u}}{dr^2}+\frac{1}{r}\frac{d\bar{u}}{dr}\right] \qquad (22)$$

where $\bar{u}(r,t)=L[u(r,t)]$ Eq. (22) can be written as

$$r\frac{d^2}{dr^2}\bar{u}(r,s)+\frac{d}{dr}\bar{u}(r,s)-\frac{s^\alpha}{c^2}r\bar{u}(r,s)=0, \qquad (23)$$

Taking the series solution of $\bar{u}(r,s)$ as

$$\bar{u}(r,s)=\sum_{n=0}^{\infty}a_n r^{n+\rho},\quad a_0\neq 0,\ \rho\ is\ real. \qquad (24)$$

We finally obtain

$$\bar{u}(r,s)=A(1+\ln r)+\frac{B}{c^2}s^\alpha r^2+o\left(s^{2\alpha}\right) \qquad (25)$$

where A and B are constants.
Therefore, we have

$$u(r,t)\approx t^{-\alpha-1}, \qquad (26)$$

which clearly exhibits the power law decay of $u(r,t)$ with α in contrast to the stretched exponential decay characteristic generally seen in fractional Brownian motion.

6. Application of the Method

M-1

According to Eq. (20), the components of the decomposition series are

$$u_0(r,t)=r^2+crt,$$

$$u_1(r,t)=\frac{4c^2}{\Gamma(\alpha+1)}t^\alpha+\frac{c^3}{r\Gamma(\alpha+2)}t^{1+\alpha},$$

$$u_2(r,t)=\frac{c^5}{r^3\Gamma(2\alpha+2)}t^{1+2\alpha},$$

$$u_3(r,t)=\frac{9c^7}{r^5\Gamma(3\alpha+2)}t^{1+3\alpha},$$

$$u_4(r,t)=\frac{225c^9}{r^7\Gamma(4\alpha+2)}t^{1+4\alpha},$$

$$u_5(r,t)=\frac{11025c^{11}}{r^9\Gamma(5\alpha+2)}t^{1+5\alpha},$$

and so on, in this manner the rest of components of the series solution can be obtained.

$$u=u_0+u_1+u_2+u_3+\ldots\ldots \qquad (23)$$

Thus the exact solution may be obtained by using

$$u(r,t)=\sum_{n=0}^{\infty}u_n(r,t), \qquad (24)$$

$$=r^2+crt+\frac{4c^2}{\Gamma(\alpha+1)}t^\alpha+\frac{c^3}{r\Gamma(\alpha+2)}t^{1+\alpha}+\frac{c^5}{r^3\Gamma(2\alpha+2)}t^{1+2\alpha}+\frac{9c^7}{r^5\Gamma(3\alpha+2)}t^{1+3\alpha}+\ldots \qquad (25)$$

$$=r^2+\frac{4c^2t^\alpha}{\Gamma(\alpha+1)}+crtE_{\alpha,2}\left(\frac{c^2}{r^2}kt^\alpha\right), \qquad (26)$$

where $k^n=\left[1.3.5.......(2n-3)\right]^2$

and $E_{\alpha,b}(t)=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(n\alpha+b)}$ is the generalized Mittag-Leffler function [30].

M-2

According to fractional Adomian's decomposition method, the of problem (21) upto first 6 components is

$$u(x,t)=r^2+crt+\frac{4c^2}{\Gamma(\alpha+1)}t^\alpha+\frac{c^3}{r\Gamma(\alpha+2)}t^{1+\alpha}+\frac{c^5}{r^3\Gamma(2\alpha+2)}t^{1+2\alpha}+\frac{9c^7}{r^5\Gamma(3\alpha+2)}t^{1+3\alpha}+\frac{225c^9}{r^7\Gamma(4\alpha+2)}t^{1+4\alpha}+\frac{11025c^{11}}{r^9\Gamma(5\alpha+2)}t^{1+5\alpha}. \qquad (27)$$

According to restarted fractional Adomian's decomposition method, we have

$$\varphi^1=u_0+u_1+u_2,$$

Let

$$u_0(x,t)=r^2+crt+\frac{4c^2}{\Gamma(\alpha+1)}t^\alpha+\frac{c^3}{r\Gamma(\alpha+2)}t^{1+\alpha}+\frac{c^5}{r^3\Gamma(2\alpha+2)}t^{1+2\alpha},$$

$$u_1(x,t)=\frac{4c^2}{\Gamma(\alpha+1)}t^{\alpha}+\frac{c^3}{r\,\Gamma(\alpha+2)}t^{1+\alpha}+\frac{c^5}{r^3\,\Gamma(2\alpha+2)}t^{1+2\alpha}+\frac{9c^7}{r^5\,\Gamma(3\alpha+2)}t^{1+3\alpha},$$

$$u_2(x,t)=\frac{c^5}{r^3\,\Gamma(2\alpha+2)}t^{1+2\alpha}+\frac{9c^7}{r^5\,\Gamma(3\alpha+2)}t^{1+3\alpha}+\frac{225c^9}{r^7\,\Gamma(4\alpha+2)}t^{1+4\alpha},$$

Now

$$\begin{aligned}\varphi^2&=u_0+u_1+u_2,\\&=r^2+crt+\frac{8c^2}{\Gamma(\alpha+1)}t^{\alpha}+\frac{2c^3}{r\,\Gamma(\alpha+2)}t^{1+\alpha}+\frac{3c^5}{r^3\,\Gamma(2\alpha+2)}t^{1+2\alpha}+\frac{18c^7}{r^5\,\Gamma(3\alpha+2)}t^{1+3\alpha}+\frac{225c^9}{r^7\,\Gamma(4\alpha+2)}t^{1+4\alpha}\end{aligned}\tag{28}$$

Let

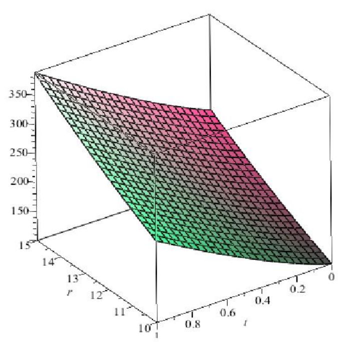
$$u_0(x,t)=r^2+crt+\frac{8c^2}{\Gamma(\alpha+1)}t^{\alpha}+\frac{2c^3}{r\,\Gamma(\alpha+2)}t^{1+\alpha}+\frac{3c^5}{r^3\,\Gamma(2\alpha+2)}t^{1+2\alpha}+\frac{18c^7}{r^5\,\Gamma(3\alpha+2)}t^{1+3\alpha}+\frac{225c^9}{r^7\,\Gamma(4\alpha+2)}t^{1+4\alpha},$$

$$u_1(x,t)=\frac{4c^2}{\Gamma(\alpha+1)}t^{\alpha}+\frac{c^3}{r\,\Gamma(\alpha+2)}t^{1+\alpha}+\frac{2c^5}{r^3\,\Gamma(2\alpha+2)}\cdot t^{1+2\alpha}+\frac{25c^7}{r^5\,\Gamma(3\alpha+2)}t^{1+3\alpha}+\frac{450c^9}{r^7\,\Gamma(4\alpha+2)}t^{1+4\alpha}+\frac{11025c^{11}}{r^9\,\Gamma(5\alpha+2)}t^{1+5\alpha},$$

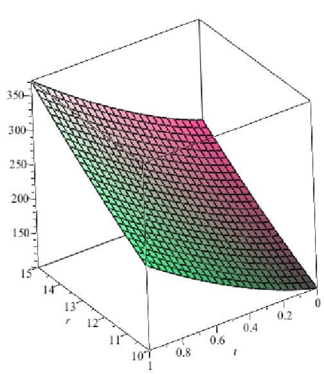
$$u_2(x,t)=\frac{c^5}{r^3\,\Gamma(2\alpha+2)}t^{1+2\alpha}+\frac{18c^7}{r^5\,\Gamma(3\alpha+2)}t^{1+3\alpha}+\frac{625c^9}{r^7\,\Gamma(4\alpha+2)}t^{1+4\alpha}+\frac{22050c^{11}}{r^9\,\Gamma(5\alpha+2)}t^{1+5\alpha}+\frac{893025c^{13}}{r^{11}\,\Gamma(6\alpha+2)}t^{1+6\alpha},$$

Now

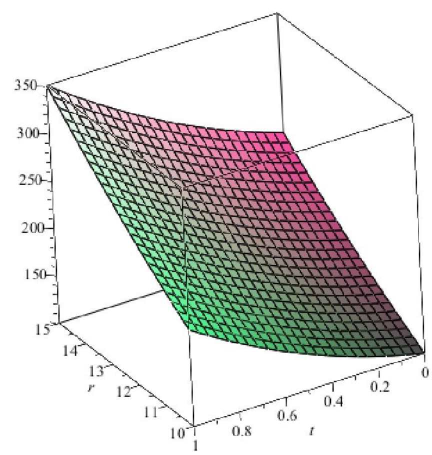
$$\begin{aligned}\varphi^3&=u_0+u_1+u_2,\\&=r^2+crt+\frac{12c^2}{\Gamma(\alpha+1)}t^{\alpha}+\frac{3c^3}{r\,\Gamma(\alpha+2)}t^{1+\alpha}+\frac{6c^5}{r^3\,\Gamma(2\alpha+2)}t^{1+2\alpha}+\frac{61c^7}{r^5\,\Gamma(3\alpha+2)}t^{1+3\alpha}+\frac{1300c^9}{r^7\,\Gamma(4\alpha+2)}\cdot t^{1+4\alpha}+\frac{33075c^{11}}{r^9\,\Gamma(5\alpha+2)}t^{1+5\alpha}+\frac{893025c^{13}}{r^{11}\,\Gamma(6\alpha+2)}t^{1+6\alpha}.\end{aligned}\tag{29}$$



(a)

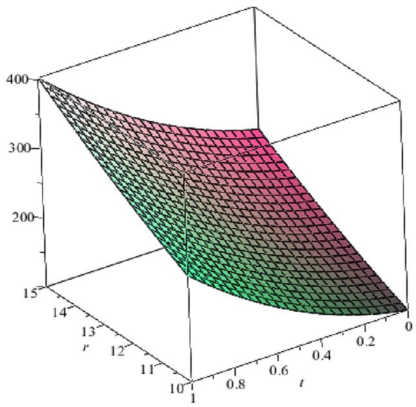


(b)

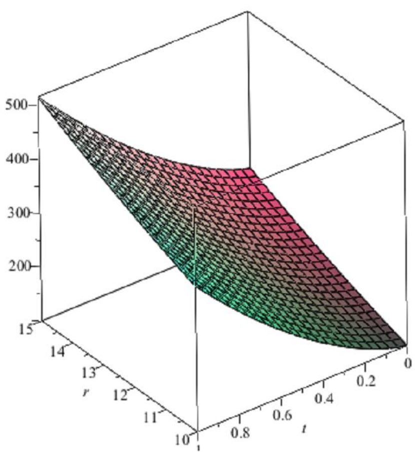


(c)

Fig. 1. Plot of *u(r, t)* with respect to *r* and *t* at *c = 5* $\alpha=1.3, 1.6, 2$.



(a)



(b)

Fig. 2. Plot of $u(r, t)$ with respect to r and t at $c = 5$ (a) Graph of eq. (28) for $\alpha = 2$, (b) Graph of eq. (29) for $\alpha = 2$

7. Numerical results and discussion

It is observed from the Fig. 3 and Fig. 4 and Fig. 5 that the displacement increases with the increase of both r and t with wave velocities. Numerical results coupled with graphical representation explicitly reveal the complete reliability of the proposed algorithm. It is found that the proposed modification is valid and give results in the form of fast convergent series solution as to FADM.

8. Conclusıons

In this work, we applied the proposed modification of Fractional Adomian's method (FADM) called restarted fractional Adomian's method (RFADM) fractional order vibration equation and showed that the new algorithm gives better approximate solutions than the standard Adomian's method. Restarted fractional Adomian's method is very powerful in finding the solutions for various physical, vibration and oscillation problems. The main interest is to construct a competitive study of finding numerical solutions of vibration equation. It is seen that our method is efficient for finding the solutions in higher degree of accuracy. Our method is direct and straightforward and it avoids the volume of calculations. The present study of solving fractional vibration equation for very large membrane constitutes a significant change from the usual approach and thus will considerably benefitial for the engineers and scientist working in this field.

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