# Possible application of fractional order derivative to image edges detection

Oguoma Ikechukwu Chiwueze[1] and Alain Cloot[2].

[1,2]Department of Mathematics and Applied Mathematics, Faculty of Natural and Agricultural Science, University of the Free State, Bloemfontein 9300, South Africa.
[1] Email: ikeoguoma@yahoo.com
[2]Email: ClootAH@ufs.ac.za

**Abstract:** This paper focuses on the possible application of the concept of fractional calculus to image processing. In particular, we generalized the Prewitt operator using the fractional order derivative to detect the edges in a given image. The comparison of results obtained via the modified operator gives more details than the existing operator that uses the ordinary derivative. The fractional derivative used in this work is in the Caputo sense; in addition, the numerical evaluation of the fractional operator is done using a finite difference scheme.
[Oguoma Ikechukwu Chiwueze and Cloot Alain. **Possible application of fractional order derivative to image edges detection**. *Life Sci J* 2013;10(4):171-176]. (ISSN:1097-8135). http://www.lifesciencesite.com. 22

**Keywords**: Edges detection; Fractional-Prewitt operator; Caputo derivative; numerical results.

## 1. Introduction

The edge of an image is the most basic features of the image. Edge is basically the symbol and reflection of discreteness of partial image [1]. It contains a wealth of internal information of the image. Therefore, edge detection is one of the key research works in image processing. The current image edge detection methods are mainly differential operator technique and high-pass filtration. Among these methods, the most primitive of the differential and gradient edge detection methods are complex and the effects are not satisfactory. The widely used operators such as Sobel, Prewitt, Roberts and Laplacian are sensitive to noises and their anti-noise performances are poor. The Log and Canny edge detection operators which have been proposed use Gaussian function to smooth or do convolution to the original image, but the computations are very large. Prewitt operator is a discrete differentiation operator, computing an approximation of the gradient of the image intensity function at each point in the image. It is used in image processing, particularly within edge detection algorithms. In this paper we use the fractional order derivative in the Caputo sense applied on the Prewitt operator to perform the edge detection under the gradient method.

## 2. Background of edge detection

The edge detection methods may be grouped into two categories: The gradient method and Laplacian method [10]. The gradient method detects the edges by looking for the maximum and minimum in the first derivative of the image. The Laplacian method searches for the zero crossings in the second derivative of the image to find edges. The key step is to decompose a large and complex image into small images with independent features. The primary objectives for using computers to do image processing are: Firstly, to create more suitable images for people to observe, identify and understand. Secondly, to make sure that computers can automatically recognize and understand images [8]. Mathematically, the gradient of a two-variable function or image intensity function is at each image point a two-dimensional vector with the components given by the derivatives in the horizontal and vertical directions.

### 2.1. Prewitt Operator

The Prewitt operator is one type of an edge model operator. The kernels can be applied separately to the input image, to produce separate measurements of the gradient component in each orientation say $G_x$ and $G_y$. The first derivatives in image processing are implemented using the magnitude of the gradient. For a function $f(x, y)$, the gradient of $f$ at coordinates $(x, y)$ is defined as the two-dimensional column vector

$$\nabla f = \begin{bmatrix} G_x \\ G_y \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \qquad (1)$$

The magnitude of this vector is given by

$$\nabla f = |\nabla f| = \left[ (G_x)^2 + (G_y)^2 \right]^{1/2}$$

$$= \left[ \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \right]^{1/2} \quad (2)$$

The components of the gradient vector itself are linear operators, but the magnitude of this vector obviously is not because of the squaring and square root operations. These can then be combined together to find the absolute magnitude of the gradient at each point and the orientation of that gradient [2]. The gradient magnitude is given by:

$$\nabla f = \left[ (G_x)^2 + (G_y)^2 \right]^{1/2} \approx |G_x| + |G_y| \quad (3)$$

The gradients $G_x$ and $G_y$ for the Prewitt operators are calculated using the discrete approximations:

$$G_x = (f_7 + f_8 + f_9) - (f_1 + f_2 + f_3)$$
$$G_y = (f_3 + f_6 + f_9) - (f_1 + f_2 + f_7). \quad (4)$$

Table 1 below shows the Prewitt operator consisting of a pair of 3×3 convolution kernels.

| | | |
|---|---|---|
| -1 | 0 | +1 |
| -1 | 0 | +1 |
| -1 | 0 | +1 |

$G_x$

| | | |
|---|---|---|
| +1 | +1 | +1 |
| 0 | 0 | 0 |
| -1 | -1 | -1 |

$G_y$

Table 1: A pair of 3x3 convolution kernels


Figure 1: Prewitt operator edge detection operation

Figure 1 gives a practical example of the Prewitt operator edge enhancement operation. The resulting image appears as a directional outline of the objects in the original image. The constant bright region became black and changing bright region became highlighted. This operator does not place any emphasis on pixels that are closer to the center of the masks [3].

### 2.2. Fractional order derivatives

There are many definitions of fractional derivatives, but in this section, we present the fundamental definitions of fractional order derivatives that are mostly used, which includes Caputo [4] and Riemann-Liouville [6, 7], results relative to the Gamma function and discuss briefly the advantages and disadvantages of these fractional order definitions.

**Riemann-Liouville** gave the most popular definition of fractional derivatives of order as:

$${}_aD_{RL}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^x \frac{f(\tau)d\tau}{(x-\tau)^{\alpha-n+1}},$$
$$(n-1 \le \alpha < n) \quad (5)$$

where $\Gamma$ is the Gamma function.

**Caputo** gave the second popular definition used as:

$${}_aD_{C}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha-n)} \int_a^x \frac{f^{(n)}(\tau)d\tau}{(x-\tau)^{\alpha+1-n}}, \quad (6)$$
$$(n-1 \le \alpha < n).$$

**Advantages**

The Caputo representation has advantages over Riemann-Liouville representation. Caputo's most well known advantage is that it allows traditional initial and boundary conditions to be included in the formulation of the problem [4]. Also, its fractional derivative or Caputo derivative of a constant is zero whereas for the Riemann-Liouville, the derivative of a constant is not zero. The Laplace transform of the Riemann-Liouville derivative leads to boundary terms containing the limit values of the Riemann-Liouville fractional derivatives at the lower boundary of integration $x = a$ and inspite of the fact that mathematically, such problems can be solved, there is no physical interpretation for such type of conditions. On the other hand, the Laplace transform of Caputo derivative imposes boundary conditions involving integer-order derivatives at the lower boundary $x = a$ which usually are acceptable physical conditions. With the Riemann-Liouville fractional derivative, an arbitrary function needs not to be continuous at the origin and it needs not to be differentiable.

**Disadvantages**

Functions that have no first order derivative might have fractional derivatives of all orders less than one in the Riemann-Liouville sense.

Caputo's derivative requires higher conditions of regularity with respect to the differentiability of a function. A Caputo derivative is defined only for functions that are differentiable and in the classical sense.

**Discretization of the fractional derivatives [9]**

In this section, we describe how to discretize the fractional derivative in the Caputo sense. Firstly, we derive numerical approximations based on the Caputo derivative definition, (6):

$$D_C^{\alpha} f\left(x_j\right)=\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{j-1}\int_{x_k}^{x_k+1}\frac{d^2 f(t)}{dt^2}\left(x_j-t\right)^{1-\alpha}dt. \quad (7)$$

A usual way of approximating the Caputo derivative $D_C^{\alpha} f\left(x_j\right)$ reads:

$$D_C^{\alpha,\Delta x} f\left(x_j\right)=\frac{1}{\Gamma(2-\alpha)}\times\sum_{k=0}^{j-1}\frac{f\left(x_k+2\right)-2f\left(x_k+1\right)+f\left(x_k\right)}{\Delta x^2}\int_{x_k}^{x_k+1}\left(x_j-t\right)^{1-\alpha}dt$$

$$=\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{j-1}\frac{f\left(x_k+2\right)-2f\left(x_k+1\right)+f\left(x_k\right)\Delta x^{2-\alpha}}{\Delta x^2(2-\alpha)}d_{j,k}$$

$$=\frac{\Delta x^{-\alpha}}{\Gamma(3-\alpha)}\sum_{k=0}^{j-1}d_{j,k}\left(f\left(x_k+2\right)-2f\left(x_k+1\right)+f\left(x_k\right)\right)$$

where $d_{j,k}=(j-k)^{2-\alpha}-(j-k-1)^{2-\alpha}$.

This numerical scheme is only first order accurate and we have the following result:

**Proposition 1:** Let $f(x)$ be a function in $C^2[a,b]$ and $1<\alpha<2$. Then

$$D_C^{\alpha,\Delta x} f\left(x_j\right)=D_{C,1}^{\alpha} f\left(x_j\right)+E_{C,1}\left(x_j\right)$$

with

$$\left|E_{C,1}\left(x_j\right)\right|\leq\frac{2\left(x_j-\alpha\right)^{2-\alpha}}{\Gamma(3-\alpha)}O(\Delta x).$$

A second order accurate numerical scheme can be obtained using the concept of splines:

For $x_j, j=1,\ldots,\mathrm{N}-1$ we need to calculate

$$\frac{1}{\Gamma(2-\alpha)}\int_0^{x_j}\frac{d^2 f(t)}{dt^2}\left(x_j-t\right)^{1-\alpha}dt. \quad (8)$$

We compute these integrals by approximating the second order derivative by a linear spline $s_j(t)$, whose nodes and knots are chosen at $x_k, k=0,1,2,\ldots,j.$

The spline $s_j(t)$ is of the form

$$s_j(t)=\sum_{k=0}^{j}\frac{d^2 f\left(x_k\right)}{dt^2}s_{j,k}(t) \quad (9)$$

with $s_{j,k}(t)$, in each interval $\left[x_{k-1},x_{k+1}\right]$, $for 1\leq k\leq j-1$, given by

$$s_{j,k}(t)=\begin{cases}\frac{t-x_{k-1}}{x_k-x_{k-1}}, x_{k-1}\leq t\leq x_k\\ \frac{x_{k+1}-t}{x_{k+1}-x_{k-1}}, x_k\leq t\leq x_{k+1},\\ 0, otherwise.\end{cases}$$

For $k=0$ and $k=j, s_{j,k}(t)$ is of the form

$$s_{j,0}(t)=\begin{cases}\frac{x_1-t}{x_1-x_0}, x_0\leq t\leq x_1\\ 0, otherwise.\end{cases}$$

$$s_{j,j}(t)=\begin{cases}\frac{t-x_j-1}{x_j-x_{j-1}}, x_{j-1}\leq t\leq x_j\\ 0, otherwise.\end{cases}$$

Therefore, an approximation for (7) is of the form

$$\frac{1}{\Gamma(2-\alpha)}\int_a^{x_j}s_j(t)\left(x_j-t\right)^{1-\alpha}dt$$

$$=\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{j}\frac{d^2 f\left(x_k\right)}{dt^2}\int_a^{x_j}\left(x_j-t\right)^{1-\alpha}s_{j,k}(t)dt,$$

and after some straightforward Mathematical manipulations, we obtain

$$\frac{1}{\Gamma(2-\alpha)}\int_0^{x_j}s_j(t)\left(x_j-t\right)^{1-\alpha}dt$$

$$=\frac{\Delta x^{2-\alpha}}{\Gamma(4-\alpha)}\sum_{k=0}^{j}\frac{d^2 f\left(x_k\right)}{dt^2}a_{j,k}, \quad (10)$$

where

$$a_{j,k} = (j-1)^{3-\alpha} - j^{2-\alpha}(j-3+\alpha), k=0 \quad (11)$$

$$a_{j,k} \ (j-k+1)^{3-\alpha} - 2(j-k)^{3-\alpha} + (j-k-1)^{3-\alpha} \quad (12)$$

$1 \le k \le j-1$, $a_{j,k} = 1, k = j.$

For the mesh point $x_k, k = 1,...,N-1$ the second order derivative can be approximated by $\delta^2 f_j / \Delta x^2$ where $\delta^2$ is the central second order differential operator

$$\delta^2 f_j = f(x_{j+1}) - 2f(x_j) + f(x_{j-1}).$$

Additionally, we also need to know the value of the second order derivative at the boundary point $x_0$. If we have a physical boundary condition of the type

$$\frac{d^2 f(x_0)}{dx^2} = b_0, \quad (13)$$

we can consider the given value. If this value is not available at $x = x_0$ the second order derivative can be approximated by $\delta_0 U_0 / \Delta x^2$, where $\delta_0$ is the operator:

$$\delta_0 f_j = 2f(x_j) - 5f(x_{j+1}) + 4f(x_{j+2}) - f(x_{j+3}) \quad (14)$$

Finally, an approximation for $D_C^\alpha(x_j)$ can be written as

$$D_C^{\alpha,\Delta x} f(x_j) = \frac{\Delta x^{-\alpha}}{\Gamma(4-\alpha)}\left\{a_{j,0}\delta_0 f_0 + \sum_{k=0}^{j} a_{j,k}\delta^2 f_k\right\}.$$

For which the following proposition holds:

**Proposition 2:** Let $f(x)$ be a function in $C^3[a,b]$ and $1 < \alpha < 2$. Then

$$D_C^{\alpha,\Delta x} f(x_j) = D_C^\alpha f(x_j) + E_C(x_j)$$

with

$$|E_C(x_j)| \le \frac{2(x_j - \alpha)^{2-\alpha}}{\Gamma(3-\alpha)} O(\Delta x^2).$$

Note that

$$D_C^\alpha f(x) = D_{RL}^\alpha - \sum_{k=0}^{1} \frac{d^k f(a)}{dx^k} \frac{(x-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)},$$

that is

$$D_{RL}^\alpha f(x) = D_C^\alpha f(x) + f(a)\frac{(x-a)^{-\alpha}}{\Gamma(-\alpha+1)} + f'(a)\frac{(x-a)^{-\alpha+k}}{\Gamma(-\alpha+2)}.$$

**Justification of the Algorithm**

In this section, the results obtained by applying Caputo derivative numerical scheme to the Error function are presented numerically and analytically with different values of alpha. The analytical results are in agreement with the numerical results which indicated that the numerical code is efficient and accurate. The red dots in Figure 2 to Figure 5 below shows the numerical results while the green line indicates the analytical results.

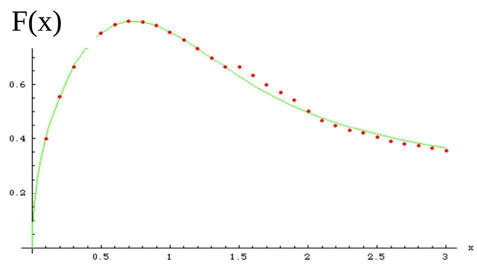



Figure 2: The analytical and numerical result with alpha = 0.5

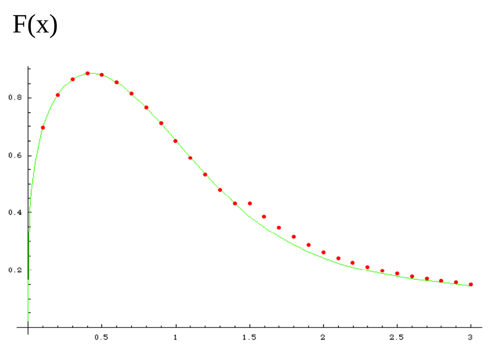



Figure 3: The analytical and numerical result with alpha = 0.75

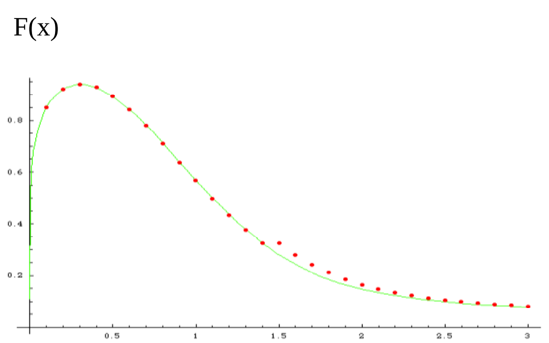
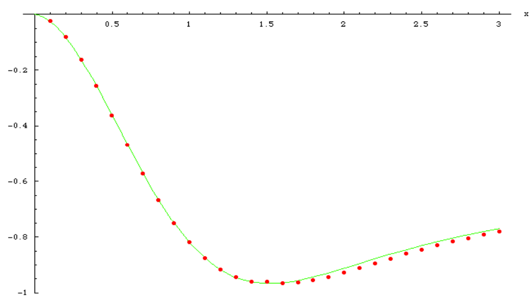



Figure 4: The analytical and numerical result with alpha = 0.85




F(x)

Figure 5: The analytical and numerical result with alpha = 1.25

## 3. Numerical Simulation

In order to assess the possible effect of the order of the fractional derivative in detecting edges in an image or enhancing, we make use of the fractional-Prewitt operator method. The matrix (also called in image processing language “mask”) obtained are represented in $x$ -direction and $y$ -direction. The approximation of the addition of both $x$ - direction and $y$ -direction in this work is called the fractional – Prewitt operator. The fractional-Prewitt operator (mask) is used to convolute with the original image. The aim of this convolution is to detect the edges of an image.

Now we make use of the Caputo derivative for x-direction, y-direction and the fractional- Prewitt operator (both directions) to access the edges contained in the x-ray picture below. This is done for different values of alpha (0.25, 0.75, 0.85, and 0.95) as indicated in the Figure 6 to Figure 10.

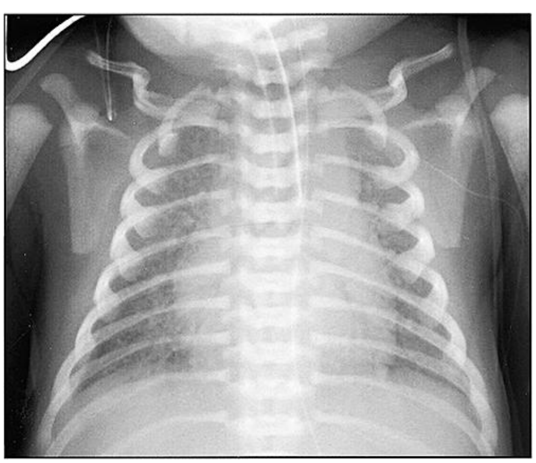


Figure 6: Original image

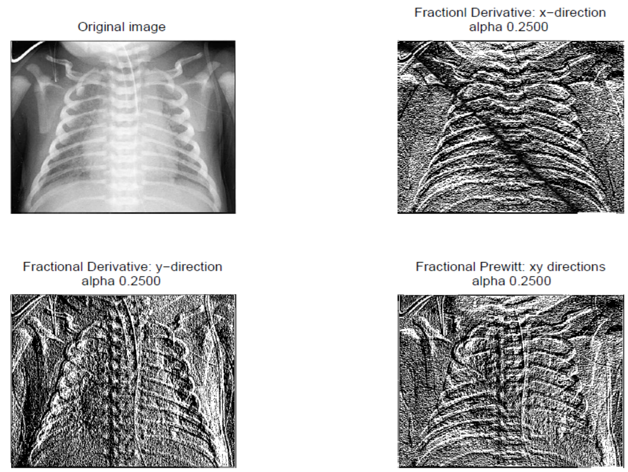



Figure 7: Derivative with alpha = 0.25

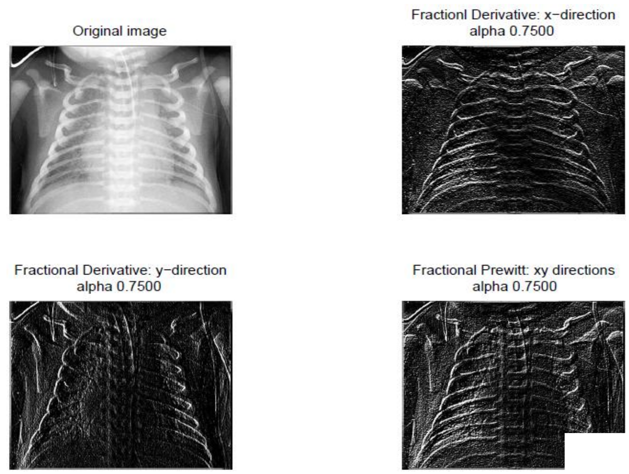



Figure 8: Derivative with alpha = 0.75

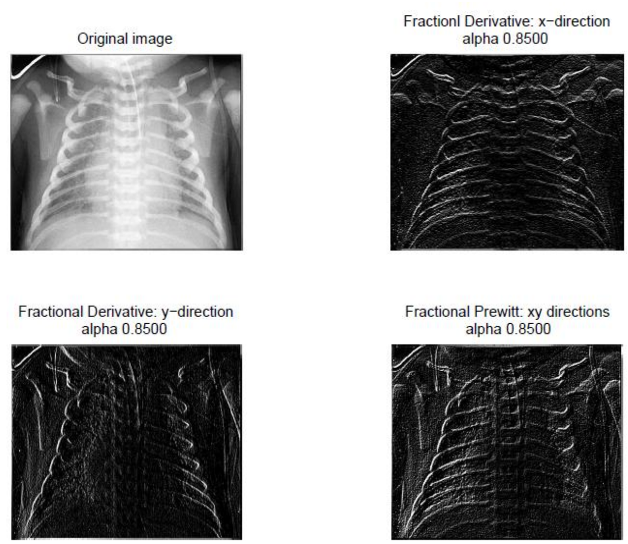



Figure 9: Derivative with alpha = 0.85

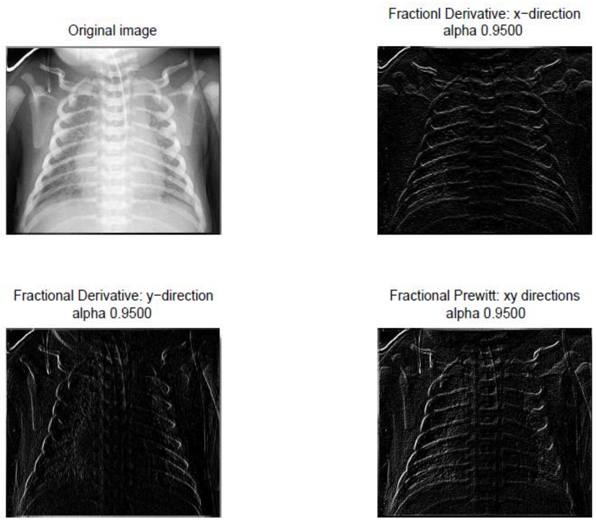



Figure 10: Derivative with alpha = 0.95

It is observed from the above results that, applying Caputo derivative to this original image, we are able to see the effect of fractional order derivative on an image in all directions.

We observed from these images that, there are significant effects of different alpha values on the same original image. The smoothing or edge detecting power of alpha is noticeable as alpha increases for 0 to 1. More image intensity details and sharp edges are detected and seen on both directions with the Fractional – Prewitt operator.

## 4. Conclusion

Given that the principal objective of enhancement is to process an image so that the result is more suitable than the original image for a specific application and obscured details are detected and enhanced or certain features of interest in an image are highlighted or sharpening of image features such as edges, boundaries, or contrast to make an image more useful for display and analysis, knowing that there is no particular way to determine a perfect, or ideal or good enhanced image, but whenever an image look good, we say it has been enhanced.

The purpose of this study is to show that fractional order derivatives can be used as a tool for edge detection during image enhancement. Therefore, using Caputo derivative to detect image edges as seen above shows that fractional order derivative can be used as one of the tools in image processing enhancement methods.

**Reference**

[1] L.P. Han and W.B. Yin. An Effective Adaptive Filter Scale Adjustment Edge Detection Method (China, Tsinghua university, 1997).
[2] Gonzalez R.C. and Woods R.E.; Digital Image Processing; Second Edition; Prentice; Hall; 2002. University of Tennessee and MedData Interavtive.
[3] A. Seif, et.al.; “A hardware architecture of Prewitt edge detection”, Sustainable Utilization and Development in Engineering and Technology (STUDENT), 2010 IEEE Conference, Malaysia, pp. 99 – 101, 20-21 Nov. 2010.
[4] M. Caputo, [1967] “Linear model of dissipation whose W is almost frequency independent II” Geophysical Journal International. Astr. Soc. 13:529-539.
[5] S. G. Samko, A.A. Kilbas and O.I. Maritchev [1987]. Integrals and Derivatives of the Fractional Order and some of their Applications. Nauka: Tekhnika, Minsk, [in Russian].
[6] I. Podlubny; Geometric and Physical interpretation of fractional integration and fractional differentiation, Fract. Calculus Appl. Anal. 5(2002), 357- 386.
[8] G. Wenshuo; Y. Lei; Z. Xiaoguang and L. Huizhong; An improved Sobel Edge detection; Digital media department. Communication University of China; ICICT 2010; Beijing, China. 978-14244-5540-9.
[9] E. Sousa, Fractional Differentiation and its Applications; CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal.
[10] D. Marr and E. Hildreth, Theory of Edge Detection (London, 1980).

10/6/2013