Preconditioned iterative methods for linear systems

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Abstract: In this paper, the preconditioned SOR and AOR methods are established for solving systems of linear equations $Ax = b$. The convergence of the iterative methods are proved for *L*-matrices. The comparison with other preconditioned methods is given.

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1. Introduction

Many practical problems can be reduced to system of linear equations

$$Ax = b, \qquad (1.1)$$

where A, b are known matrices and x is a vector of unknowns. This type of equations play a prominent role in finance, industry, economics, engineering, physics, chemistry, computer science and other field of pure and applied sciences. System of nonlinear equations may be solved using system of linear equations.

The systems of linear equations can be solved using both direct and iterative methods. Young [8] and Frankel [1] simultaneously suggested the SOR method for solving system of linear equations. The effective preconditioners can increase the rate of convergence of stationary iterative methods by reducing the condition number of the problem. It is also possible that in some cases the original method diverges but preconditioned method rapidly converges to the solution. Hadjimos [2] proposed accelerated over-relaxation (AOR) method to improve the convergence of the relaxation methods.

In this paper, we suggest preconditioned SOR method and preconditioned AOR method for solving system of linear equations. We consider the convergence of preconditioners iterative methods, when A is an L-matrix. Several examples are given to illustrate the implementation and efficiency of the method. Comparison with other methods shows that these new methods perform better.

The basic iterative method is

$$Mx^{k+1} = Nx^k + b, \quad k = 0,1,2,\ldots, \qquad (1.2)$$

where M is nonsingular. Thus (2) can be written as

$$x^{k+1} = Tx^k + c, \; k = 0,1,2,\ldots$$

where

$$T = M^{-1}N, \quad c = M^{-1}b.$$

Assuming A has unit diagonal entries and let $A = I - L - U$, where I is the identity matrix, L and U are strictly lower and strictly upper triangular parts of A, respectively. Transform the original system (1) into the preconditioned form

$$PAx = Pb.$$

Then, we can define the basic iterative scheme:

$$M_p x^{k+1} = N_p x^k + Pb, \; k = 0,1,2,\ldots .$$

We consider the preconditioner $P = I + R + S$ introduced by Niki et al. [5].

I is the identity matrix of order n.

$$PA = (I + R + S)(I - L - U)$$
$$= I - L - U + R - RU + S - SU - SL$$

as $RL = 0$. Consider

$$RU + SL = D^* + L^* + U^*,$$

where D^* is a diagonal matrix, L^* is strictly lower triangular matrix and U^* is strictly upper triangular matrices.

$$R = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ -a_{21} & 0 & 0 & \ldots & 0 \\ -a_{31} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & 0 & 0 & \ldots & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & -a_{12} & 0 & \ldots & 0 & 0 \\ 0 & 0 & -a_{23} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -a_{n-1,n} \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

We define

$$\tilde{D} = I - D^*, \; \tilde{L} = L + L^* - R, \; \tilde{U} = U + U^* - S + SU.$$

We need the following results in the convergence of preconditioned iterative methods.

Theorem 1.1 [7]. Let $A \geq 0$ be an irreducible matrix. Then

(i) A has a positive eigenvalue equal to $\rho(A)$.

(ii) A has an eigenvector $x > 0$ corresponding to $\rho(A)$.

(iii) $\rho(A)$ is the simple eigenvalue of A.

Theorem 1.2 [7]. Let $A \geq 0$ be a matrix. Then the following hold.

(i) If $Ax \geq \beta x$ for a vector $x \geq 0$ and $x \neq 0$, then $\rho(A) \geq \beta$.

(ii) If $Ax \leq \gamma x$ for a vector $x > 0$, then $\rho(A) \leq \gamma$. Moreover, if A is irreducible and if $\beta x \leq Ax \leq \gamma x$, equality excluded, for a vector $x \geq 0$ and $x \neq 0$, then $\beta x < Ax < \gamma x$, and $x > 0$.

2. Preconditioned SOR Method

The iteration matrix of preconditioned SOR method is

$$\tilde{T}_\omega = \left(\tilde{D} - \omega\tilde{L}\right)^{-1}\left[(1-\omega)\tilde{D} + \omega\tilde{U}\right] \qquad (2.1)$$

Let $\lambda = \rho(T_\omega)$, where

$$T_\omega = (1-\omega L)^{-1}\left[(1-\omega)I + \omega U\right] \qquad (2.2)$$

In the next result, we prove the convergence of preconditioned SOR method when A is an *L*-matrix.

Lemma 2.1 [10]. Let $A = (a_{ij}) \in R^{n\times n}$ be an *L*-matrix and T_ω and $\tilde{T}_\omega$ be defined by (2.2) and (2.3). Suppose $0 < a_{1,\,i+1}a_{i+1,1} < 1,\ i = 1, 2, \ldots, n-1$. If $0 < \omega < 1$, then T_ω and $\overline{T}_\omega$ are nonnegative and irreducible.

Theorem 2.1. Let $A = (a_{ij}) \in R^{n\times n}$ be an L-matrix and T_ω and $\tilde{T}_\omega$ be defined by (2.2) and (2.3).
If $0 < \omega \leq 1$, and
$0 < a_{1,\,i+1}a_{i+1,1} < 1,\ i = 1, 2, \ldots, n-1$, then

(i) $\rho(\tilde{T}_\omega) < \rho(T_\omega) < 1$

(ii) $\rho(\tilde{T}_\omega) = \rho(T_\omega) = 1$

(iii) $\rho(\tilde{T}_\omega) > \rho(T_\omega) > 1$

Proof: Consider

$$\begin{aligned}
\tilde{T}_\omega x - \lambda x &= \left(\tilde{D} - \omega\tilde{L}\right)^{-1}\left[(1-\omega)\tilde{D} + \omega\tilde{U}\right]x - \lambda x \\
&= \left(\tilde{D} - \omega\tilde{L}\right)^{-1}\left[(1-\omega)\tilde{D} + \omega\tilde{U} - \lambda\left(\tilde{D} - \omega\tilde{L}\right)\right]x \\
&= \left(\tilde{D} - \omega\tilde{L}\right)^{-1}\left[\tilde{D} - \omega\tilde{D} + \omega\tilde{U} - \lambda\tilde{D} + \lambda\omega\tilde{L}\right]x \\
&= \left(\tilde{D} - \omega\tilde{L}\right)^{-1}\begin{bmatrix}(1-\lambda)\tilde{D} - \omega(I - D^*) + \omega\left(U + U^* - S + SU\right) \\ +\lambda\omega\tilde{L}\end{bmatrix}x \\
&= \left(\tilde{D} - \omega\tilde{L}\right)^{-1}\begin{bmatrix}(1-\lambda)\tilde{D} - \omega I + \omega D^* + \omega U + \omega U^* - \omega S \\ +\omega SU + \lambda\omega\tilde{L}\end{bmatrix}x \\
&= \left(\tilde{D} - \omega\tilde{L}\right)^{-1}\begin{bmatrix}(1-\lambda)\tilde{D} - \omega I + \omega\left(D^* + U^* + L^*\right) + \omega U \\ -\omega L^* - \omega S + \omega SU + \lambda\omega\tilde{L}\end{bmatrix}x \\
&= \left(\tilde{D} - \omega\tilde{L}\right)^{-1}\begin{bmatrix}(1-\lambda)\tilde{D} - \omega I + \omega U + \omega(RU + SL) \\ -\omega L^* - \omega S + \omega SU + \lambda\omega\tilde{L}\end{bmatrix}x \\
&= \left(\tilde{D} - \omega\tilde{L}\right)^{-1}\begin{bmatrix}(1-\lambda)\tilde{D} - \omega I + \omega U + \omega(RU + SU) - \omega L^* \\ -\omega S + \omega SL + \lambda\omega\tilde{L}\end{bmatrix}x \qquad (2.3)
\end{aligned}$$

From (2.2), we have

$$\lambda(I - \omega L)x = \left[(1-\omega)I + \omega U\right]x$$

$$\omega Ux = \left[\lambda(I - \omega L) - (1-\omega)I\right]x$$

$$\omega(RU + SU)x = \left[\lambda(R+S)(I - \omega L) - (R+S)(1-\omega)I\right]x$$

Put in (2.3), we get

$$\begin{aligned}
\tilde{T}_\omega x - \lambda x &= \left(\tilde{D} - \omega\tilde{L}\right)^{-1}\begin{bmatrix}(1-\lambda)\tilde{D} - \omega I + \lambda(I - \omega L) - (1-\omega)I + \lambda(R+S) - \lambda\omega(R+S)L \\ -(R+S)I + \omega(R+S) - \omega L^* - \omega S + \omega SL + \lambda\omega\left(L + L^* - R\right)\end{bmatrix}x \\
&= \left(\tilde{D} - \omega\tilde{L}\right)^{-1}\begin{bmatrix}(1-\lambda)\tilde{D} - (1-\lambda)I - (R+S)(1-\lambda) + \omega R(1-\lambda) - \omega L^*(1-\lambda) + \\ \omega SL(1-\lambda)\end{bmatrix}x \\
&= (1-\lambda)\left(\tilde{D} - \omega\tilde{L}\right)^{-1}\left[-D^* - (1-\omega)R - S - \omega L^* + \omega SL\right]x \qquad (2.4)
\end{aligned}$$

Since $\left(\tilde{D} - \omega\tilde{L}\right)^{-1}$ is a non-negative lower triangular matrix, also D^*, R, S, SL and L^* are all non-negative.

If $\lambda < 1$, then from (2.4), we have

$$\tilde{T}_\omega x \leq \lambda x$$

for irreducible $\tilde{T}_\omega$ and $x > 0$, we have

$$\rho\left(\tilde{T}_\omega\right) < \lambda$$

which is part (i). Similarly for $\lambda = 1$ and $\lambda > 1$, we get (ii) and (iii) respectively.

3 Preconditioned AOR Method

The iteration matrix of preconditioned AOR method is

$$\tilde{T}_{r,\omega} = \left(\tilde{D} - r\tilde{L}\right)^{-1}\left[(1-\omega)\tilde{D} + (\omega - r)\tilde{L} + \omega\tilde{U}\right] \qquad (3.1)$$

Let

$$\lambda = \rho(T_{r,\omega}),$$

where

$$T_{r,\omega} = (I - rL)^{-1}\left[(1-\omega)I + (\omega - r)L + \omega U\right] \qquad (3.2)$$

In the next result, we prove the convergence criteria of preconditioned AOR method (3.1).

Theorem 3.1. Let $A=(a_{ij})\in R^{n\times n}$ be an irreducible L-matrix. Suppose that $0<a_{1i}a_{i1}<1$, $i=1,2,\ldots,n$, $T_{r,\omega}$ and $\tilde{T}_{r,\omega}$ defined by (3.2) and (3.1). If $0\le r\le\omega\le 1$ $(\omega\ne 0,\ r\ne 1)$, then

(i) $\rho(\tilde{T}_{r,\omega})<\rho(T_{r,\omega})<1$

(ii) $\rho(\tilde{T}_{r,\omega})=\rho(T_{r,\omega})=1$

(iii) $\rho(\tilde{T}_{r,\omega})>\rho(T_{r,\omega})>1$

Proof. Consider

$$\tilde{T}_{r,\omega}x-\lambda x=(\tilde{D}-r\tilde{L})^{-1}\left[(1-\omega)\tilde{D}+(\omega-r)\tilde{L}+\omega\tilde{U}\right]x-\lambda x$$

$$=(\tilde{D}-r\tilde{L})^{-1}\begin{bmatrix}(1-\omega)\tilde{D}+\omega\tilde{U}+(\omega-r)\tilde{L}\\ -\lambda(\tilde{D}-r\tilde{L})\end{bmatrix}x$$

$$=(\tilde{D}-r\tilde{L})^{-1}\begin{bmatrix}\tilde{D}-\omega\tilde{D}+\omega\tilde{U}+\omega\tilde{L}-r\tilde{L}\\ -\lambda\tilde{D}+\lambda r\tilde{L}\end{bmatrix}x$$

$$\tilde{T}_{r,\omega}x-\lambda x$$

$$=(\tilde{D}-r\tilde{L})^{-1}\begin{bmatrix}\tilde{D}-\omega(\tilde{D}-\tilde{U}-\tilde{L})-r\tilde{L}\\ -\lambda\tilde{D}+\lambda r\tilde{L}\end{bmatrix}x$$

$$=(\tilde{D}-r\tilde{L})^{-1}\begin{bmatrix}(1-\lambda)\tilde{D}-\omega\begin{pmatrix}I-L-U+R\\ -RU+S-SU-SL\end{pmatrix}\\ -r(L+L^*-R)+\lambda r\tilde{L}\end{bmatrix}x$$

$$=(\tilde{D}-r\tilde{L})^{-1}\begin{bmatrix}(1-\lambda)\tilde{D}-\omega I+\omega U+\omega L\\ -\omega(R+S)+\omega SL+\omega(RU+SU)\\ -rL-rL^*+rR+\lambda r\tilde{L}\end{bmatrix}x \quad (3.3)$$

From (3.2), we have

$$\lambda(1-rL)x=\left[(1-\omega)I+(\omega-r)L+\omega U\right]x$$

$$\omega Ux=\left[\lambda(1-rL)-(1-\omega)I-(\omega-r)L\right]x$$

$$\omega(RU+SU)x=\left[\lambda(R+S)(1-rL)-(1-\omega)(R+S)I-(\omega-r)(R+S)L\right]x$$

Put in (3.3), we have

$$\tilde{T}_{r,\omega}x-\lambda x=(\tilde{D}-r\tilde{L})^{-1}\begin{bmatrix}(1-\lambda)\tilde{D}-\omega I+\omega U+\omega L-\omega(R+S)\\ +\omega SL+\lambda(R+S)(1-rL)-(1-\omega)(R+S)I\\ -(\omega-r)(R+S)L-rL-rL^*+rR+\lambda r\tilde{L}\end{bmatrix}x$$

$$=(\tilde{D}-r\tilde{L})^{-1}\begin{bmatrix}(1-\lambda)\tilde{D}-\omega I+\omega U+\omega L+\lambda(R+S)-\lambda rLS\\ -(R+S)I-rL-rL^*+rSL+rR+\lambda r(L+L^*-R)\end{bmatrix}x \quad (3.4)$$

Now putting the value of ωUx from (3.2) in (3.4), we have

$$\tilde{T}_{r,\omega}x-\lambda x=(\tilde{D}-r\tilde{L})^{-1}\begin{bmatrix}(1-\lambda)\tilde{D}-\omega I+\omega L+\lambda(1-rL)-(1-\omega)I\\ -(\omega-r)L+\lambda(R+S)-(R+S)I-rL-rL^*\\ -\lambda rSL+rR+\lambda rL+\lambda rL^*-\lambda rR\end{bmatrix}x$$

$$=(\tilde{D}-r\tilde{L})^{-1}\begin{bmatrix}(1-\lambda)\tilde{D}+\lambda(R+S)+\lambda I-I-(R+S)I\\ -rL^*+rR+\lambda rL^*-\lambda rSL+rSL-\lambda rR\end{bmatrix}x$$

$$=(\tilde{D}-r\tilde{L})^{-1}\begin{bmatrix}(1-\lambda)\tilde{D}-(1-\lambda)I-(1-\lambda)(R+S)I\\ -(1-\lambda)rL^*+(1-\lambda)rSL+(1-\lambda)rR\end{bmatrix}x$$

$$=(1-\lambda)(\tilde{D}-r\tilde{L})^{-1}\left[\tilde{D}-I-(R+S)I-rL^*+rSL+rR\right]x$$

$$=(1-\lambda)(\tilde{D}-r\tilde{L})^{-1}\left[-D^*-(1-r)R-S-rL^*+rSL\right]x \quad (3.5)$$

Since $(\tilde{D}-\omega\tilde{L})^{-1}$ is a non-negative lower triangular matrix, also D^*, R, S, L^* and SL are all non-negative.

If $\lambda<1$, then from (3.4), we have

$$\tilde{T}_{r,\omega}x\le\lambda x,$$

for irreducible $\tilde{T}_{\omega}$ and $x>0$, we have

$$\rho(\tilde{T}_{r,\omega})<\lambda,$$

which is part (i). Similarly for $\lambda=1$ and $\lambda>1$, we get (ii) and (iii) respectively.

In the next section, we compare our methods numerically with different methods.

4. Numerical results

In this section, we consider several examples to show the implementation of the proposed method. All the experiments are performed with Intel(R) Core (TM) 2 × 2.1GHz, 1GB RAM, and the codes are written in Matlab 7.

Example 4.1[3]. Let

$$A=\begin{bmatrix}1 & q & r & s & q & \cdots\\ s & 1 & q & r & \ddots & q\\ q & s & 1 & q & \ddots & s\\ r & \ddots & \ddots & \ddots & \ddots & r\\ s & \ddots & q & s & \ddots & q\\ \cdots & s & r & q & s & 1\end{bmatrix},$$

where $q=-p/n$, $r=-p/(n+1)$ and $s=-p/(n+2)$. For $n=7$ and $p=1$ the comparison among SOR, modified SOR method [3] and preconditioned SOR method is given in Table 4.1.

Table 4.1.

ω	$\rho(T_\omega)$	$\rho(\hat{T}_\omega)$	$\rho(\tilde{T}_\omega)$
1	0.5913	0.5403	0.4521
0.8	0.7173	0.6811	0.6332
0.4	0.8859	0.8703	0.8559
0.1	0.9749	0.9713	0.9669

In Table 4.1 $\rho(T_\omega)$, $\rho(\hat{T}_\omega)$ and $\rho(\tilde{T}_\omega)$ denote spectral radius of SOR, modified SOR method [3] and preconditioned SOR method (2.1) respectively. The comparison shows that preconditioned SOR method perform better.

Example 4.2[7]. Consider a 4×4 matrix A of the form

$$A=\begin{bmatrix} 1 & 0 & 0 & -0.3 \\ -0.3 & 1 & -0.3 & -0.3 \\ 0 & -0.3 & 1 & -0.3 \\ -0.3 & 0 & -0.3 & 1 \end{bmatrix}$$

Spectral radii of the iteration matrices $\rho(T_\omega)$, $\rho(\hat{T}_\omega)$ and $\rho(\bar{T}_\omega)$ of SOR method, preconditioned SOR method by [7] and preconditioned SOR method given by (2.1), with various values of ω are given in the below table 4.2.

Table 4.2.

ω	$\rho(T_\omega)$	$\rho(\hat{T}_\omega)$	$\rho(\tilde{T}_\omega)$
0.95	0.3456	0.2391	0.1674
0.8	0.5081	0.4275	0.3669
0.6	0.6695	0.6141	0.5699
0.4	0.7984	0.7638	0.7349

Table 4.2 shows that our method is faster than both SOR method and preconditioned SOR method [7].
In the next Table, we compare our method with AOR method and precondition AOR method by Yun et al. [9] considering example 4.2.
Spectral radii of the iteration matrices $\rho(T_\omega)$, $\rho(\hat{T}_\omega)$ and $\rho(\bar{T}_\omega)$ of AOR method, preconditioned AOR method by [9] and preconditioned AOR method given by (3.1), with various values of ω are given in the below table 4.3.

5 Conclusion

In the paper, we proposed preconditioned iterative methods for solving system of linear equations. The convergence of preconditioned SOR method and preconditioned AOR method was considered under the condition that A is L-matrix. We compared our methods numerically with other preconditioned methods. From comparison we found that preconditioned iterative method defined by (2.1) and (3.1) performed better.

Table 4.3

r	ω	$\rho(T_{r,\omega})$	$\rho(\hat{T}_{r,\omega})$	$\rho(\bar{T}_{r,\omega})$
0.9	1	0.3406	0.2321	0.1577
0.8	1	0.3852	0.2331	0.2083
0.7	1	0.4202	0.3240	0.2480
0.7	0.8	0.5361	0.4592	0.3986
0.6	0.8	0.5593	0.4854	0.4232
0.5	0.8	0.5793	0.5079	0.4440
0.5	0.6	0.6845	0.6309	0.5842
0.4	0.6	0.6976	0.6458	0.5978
0.3	0.6	0.7093	0.6590	0.6159
0.3	0.4	0.8062	0.7727	0.7419
0.2	0.4	0.8133	0.7807	0.7514
0.1	0.4	0.8197	0.7879	0.7551

References

[1] S. Frankel, Convergence rates of iterative treatments of partial differential equations. Math. Tab. Other Aids Comput., 4 (1950) 65-75.

[2] A. Hadjimos, Accelerated overrelaxation method. Math. Comp., 32 (1978)149-157.

[3] T. Z. Huang, G. H. Cheng and X. Y. Cheng, Modified SOR-type method iterative method for Z-matrices, Appl. Math. Comput. 175 (2006) 258-268.

[4] W. Li, A note on the preconditioned Gauss-Seidel method for linear systems, J.comput. Appl. Math. 182 (2005) 81-90.

[5] H. Niki, K. Harada, M. Morimoto and M. Sakakihara, A survey of preconditioners used for accelerating the rate of convergence of Gauss-Seidel method, J. Comput. Appl. Math. 165 (2004) 587-600.

[6] M. A. Noor, J. Iqbal, K. I. Noor and E. Al-Said, On an iterative method for solving absolute value equations, Optim. Lett. 6(5) (2012) 1027-1033.

[7] S. Varga, Matrix iterative analysis, Springer Berlin, 2000.

[8] D. M.Young, Iterative methods for solving partial differential equations of elliptic type, Ph D. Thesis. Harvard university Cambridge Mass (1950).

[9] J. H. Yun and S. W. Kim, Convergence of the preconditioned AOR method for irreducible *L*-matrices, Appl. Math. Comput. 201 (2008) 54-64.

[10] J. H. Yun, A note on the modified SOR method for Z-matrices, Appl. Math. Comput. 201 (2008) 54-64.

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