New Application of Laplace Decomposition Algorithm For Quadrtic Riccati Differential Equation by Using Adomian's Polynomials

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Abstract: In this paper, the quadratic Riccati differential equation is solved by Laplace decomposition algorithm (LDA) with considering Adomian's polynomials. This paper both describes the principle of LDA and discusses its advantages and drawbacks.
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1.Introduction

The present paper aims at offering an alternative method of solution to the existing ones concerning Riccati equation. By using Adomian decomposition method (ADM)[2], the numerical solutions of initial value problems for ordinary differential equations (ODE) were obtained in the from of infinite series.
The decomposition method has been introduced by Adomian, and it consists of splitting the given equation into linear and nonlinear terms. The linear term $y(x)$ represents an infinite sum of components $y_n(x), n = 0,1,2, \dots$ defined by

$$y(x) = \sum_{n=0}^{\infty} y_n(x).$$

The decomposition method identifies the nonlinear term $F(y(x))$ by the decomposition series

$$F(y(x)) = \sum_{n=0}^{\infty} A_n,$$

where A_{n^s} are Adomian's polynomials of $y_0, y_1, y_2, \dots y_n$. An iterative algorithm is achieved for the determination of y_{n^s} in recursive manner. By using Maple, the truncated sum $\sum_{n=0}^{k} y_n$ is calculated.

2. Numerical Laplace transform method

In this section, the Laplace transform decomposition algorithm is applied to find the solution to the following nonlinear initial value problems:

$$py' + p_1 y + p_2 y^2 = f(x)$$
$$y(0) = \alpha, y'(0) = \beta \qquad (1)$$

where p, p_1, p_2, α and β are real constants. See [5,6].
The method consists of appling Laplace transformation (denoted throughout this paper by L) to both aspects of (1), where

$$L[py'] + L[p_1 y] + L[p_2 y^2] = L[f(x). \qquad (2)$$

By using linearity of Laplace transformation, the result is

$$pL[y'] + p_1 L[y] + p_2 L[y^2] = L[f(x)]. \qquad (3)$$

Applying the formulas on Laplace transform, we obtain

$$p(sL[y] - y(0)) + p_1 L[y] + p_2 L[y^2] = L[f(x)].(4)$$

Using the initial conditions (1), we have

$$p(sL[y] - \alpha) + p_1 L[y] + p_2 L[y^2] = L[f(x)]. \quad (5)$$

or

$$L[y] = \frac{ps}{ps+p_1} - \frac{p_2}{ps+p_1} L[y^2] + \frac{1}{ps+p_1} L[f(x)]. \quad (6)$$

The Laplace transform decomposition technique consists next of representing the solution as an infinite series, In particular

$$y = \sum_{n=0}^{\infty} y_n, \qquad (7)$$

where the terms y_n are to recursively calculated. Also the nonlinear operator $h(y) = y^2$ is decomposed as

$$h(y) = y^2 = \sum_{n=0}^{\infty} A_n, \qquad (8)$$

where the A_{n^s} are Adomian polynomials of $y_0, y_1, \dots, y_n$ and are calculated by the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} h(\textstyle\sum_{i=0}^{\infty} \lambda^i y_i), n = 0,1,2,\dots \quad (9)$$

The first few polynomials are given by

$$A_0 = h(y_0), A_1 = y_1 h^{(1)}(y_0), \\ A_2 = y_2 h^{(1)}(y_0) + y_1^2, \quad (10)$$

and for $h(y) = y^2$ they are given by

$$A_0 = y_0^2, A_1 = 2y_0y_1, A_2 = 2y_0y_2 + y_1^2, \quad (11)$$

substituting (7) and (8) to (6), the result is

$$L[\textstyle\sum_{n=0}^{\infty} y_n] = \frac{p\alpha}{ps+p_1} - \frac{p_2}{ps+p_1} L[\sum_{n=0}^{\infty} A_n] \\ + \frac{1}{ps+p_1} L[f(x)]. \quad (12)$$

Matching both sides of (12), the following iterative algorithm is obtained:

$$L[y_0] = \frac{p\alpha}{ps+p_1} + \frac{1}{ps+p_1} L[f(x)], \quad (13)$$
$$L[y_1] = -\frac{p_2}{ps+p_1} L[A_0], \quad (14)$$
$$L[y_2] = -\frac{p_2}{ps+p_1} L[A_1], \quad (15)$$
$$\vdots$$
$$L[y_{n+1}] = -\frac{p_1}{ps+p_1} L[A_n]. \quad (16)$$

Applying the inverse Laplace transform to (13) we obtain the value of y_0. Substituting this value of y_0 to (11) and evaluating the Laplace transform of the quantities on the right side of $L[y^1]$, and then applying the inverse Laplace transform, we obtain the value of y_1. From (16), we obtain the other terms $y_2, y_3, \dots$ recursively.
Since the complicated excitation term $f(x)$ can cause difficult integrations and proliferation of terms, we can express $f(x)$ in Taylor series at $x_0 = 0$, which is truncated for simplification.
If we replace $f(x)$ by

$$\tilde{f}(x) = \textstyle\sum_{i=0}^{k} a_i x^i, a_i = \frac{f^{(i)}(0)}{i!}, i = 0,1,2,\dots,k. \quad (17)$$

Eq (13) becomes

$$L[y_0] = \frac{ps}{ps+p_1} + \frac{1}{ps+p_1} \textstyle\sum_{i=0}^{k} \frac{a_i i!}{s^{i+2}}. \quad (18)$$

We will obtain values $y_0, y_1, y_2, \dots$ by using LDA which is described above.

3. Numerical Example

The Laplace transform decomposition algorithm is illustrated by following example.
The following example clarifies the effectiveness of LDA. We should remind that there is a similar example in [1,3].

Example 1. 1 *Consider the quadratic Riccati differential equation*

$$y' - 2y + y^2 = 1. \quad (19)$$

with initial conditions

$$y(0) = 0, y'(0) = 1. \quad (20)$$

The analytic solution of this equation is

$$y(x) = 1 + \sqrt{2}\tanh\left(\sqrt{2}x + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \quad (21)$$

Taylor Expanding $y(x)$ using expansion about $x = 0$ gives

$$y(x) = x + x^2 + \frac{1}{3}x^3 - \frac{1}{3}x^4 - \frac{7}{15}x^5 - \frac{7}{45}x^6 \\ + \frac{53}{315}x^7 + \frac{71}{315}x^8 + \cdots. \quad (22)$$

If we reapply the given above algorithm, considering $p = 1, p_1 = -2, p_2 = 1, \alpha = 0, \beta = 1$ to (1), we obtain following iterative algorithm:

$$L[y_0] = \frac{1}{(s-2)} L[1], \\ L[y_1] = \frac{1}{(s-2)}(-L[A_0]), \quad (23) \\ L[y_2] = \frac{1}{(s-2)}(-L[A_1]).$$

In general,

$$L[y_{n+1}] = \frac{1}{(s-2)}(-L[A_n]). \quad (24)$$

Substituting the inverse Laplace transform to (24) we obtain

$$y_0 = \frac{1}{2}e^{2x} - \frac{1}{2}. \quad (25)$$

Substituting this value of y_0 and $A_0 = y_0^2$ given in (25) to (14), then the result is

$$L[y_1] = -\frac{2}{s(s-4)(s-2)^2}. \quad (26)$$

Operating with Laplace inverse on both side of (26) we get

$$y_1 = \frac{1}{8} + \frac{1}{2}e^{2x}x - \frac{1}{8}e^{4x}. \qquad (27)$$

Substituting (27) to (23) and using the value A_1 given in (11), we obtain

$$L[y_2] = \frac{16(-3+s)}{s(s-6)(s-4)^2(s-2)^3}. \qquad (28)$$

The inverse Laplace transform applied to (28) yields

$$y_2 = -\frac{1}{16} + \frac{1}{32}e^{6x} + \frac{1}{32}e^{2x}(-4x + 8x^2 - 1) - \frac{1}{16}e^{4x}(4x - 1). \qquad (29)$$

Higher iterates can be easily obtained by using the computer algebra system Maple. For example,

$$\begin{aligned} y_3 &= \frac{5}{128} - \frac{1}{128}e^{8x} \\ &+ \frac{1}{96}e^{2x}(3 - 12x^2 + 3x + 8x^3) \\ &- \frac{1}{32}e^{4x}(4x - 1)(2x - 1) \\ &+ \frac{1}{32}e^{6x}(-1 + 3x), \\ y_4 &= \;.... \end{aligned}$$

The partial sum $\widetilde{\phi}_n(x) = \sum_{m=0}^{n} y_m$ is determinated, and in particularly $\widetilde{\phi}_3$ is calculated.

$$\begin{aligned} \widetilde{\phi}_3(x) = \; & \frac{1}{2}e^{2x} - \frac{51}{128} + \frac{13}{32}e^{2x}x \\ & - \frac{3}{32}e^{4x} + \frac{1}{8}e^{2x}x^2 - \frac{1}{16}e^{4x}x \\ & - \frac{1}{128}e^{8x} + \frac{1}{12}e^{2x}x^3 - \frac{1}{4}e^{4x}x^2 + \frac{3}{32}e^{6x}x. \end{aligned}$$

Expanding $\widetilde{\phi}_3(x)$ using Taylor expansion $x = 0$ gives

$$\widetilde{\phi}_3(x) = x + x^2 + \frac{1}{3}x^3 - \frac{1}{3}x^4 - \frac{7}{15}x^5 - \frac{7}{45}x^6 + \frac{53}{315}x^7 + \frac{71}{315}x^8 + \frac{1}{21}x^9 + \cdots.$$

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4. Conclusion

In this work, we present the Laplace decomposition algorithm of quadratic Riccati deferential equation. It gives a simple and a powerful mathematical tool for nonlinear problems. In our work we use the Maple Package to calculate the series obtained from the Laplace decomposition algorithm.

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