Optimal control of an epidemic model of leptospirosis with time delay

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Abstract: In this paper, we consider an epidemic model of leptospirosis with nonlinear incidences by applying the optimal control technique and time delay. First, we formulate the model, with optimal control and time delay. After the formulation of the model we find the existence of the control model. We completely characterize the optimal control problem by using the Pontryagin’s maximum principle. Our aim is to minimize the infection in the host population, to do this we use three control variables. The numerical simulation of both the system solved by using the backward Runge-Kutta order four schemes for the solution of the problem. Finally, the numerical results are presented for justification purpose.

Keywords: Leptospirosis, Pontryagin’s Maximum Principle, Time Delay, Optimal control, Numerical simulations.

Subject classification: 92D25, 49J15, 93D20

1. Introduction

Mathematical modeling of infectious disease is one of the important research area now-a-days. The basic and important concern for mathematical models in epidemiology is qualitative analysis, the persistence, performance, asymptotic stability and the existence and uniqueness for the models. Many influential results related to this area have been established and can be found in many articles and books. The first epidemic model for the spread of infectious disease was introduced by [12]. They divided the population in three classes the Susceptible, infected and recovered. They assumed that the susceptible population in a fixed population become infected by contact with infected individuals, infected individuals either die or recover at a constant rate. Their models consist of three differential equations of ODE's which represent the rate of change in their respective population.

In recent years, some mathematical models incorporating delayed effects have been studied. Smith in [24] and Thieme [25] in (1990) derived a scalar delay differential equation for the population of immature and mature age classes. The maturation period is regarded as a time delay. Using the same idea, a system of delayed differential equations for mature population in a patchy environment has been proposed in So et al [26]. More recent studies consider an epidemic model with density dependence to describe disease transmission in variable population size, which can be found in Cooke et al [6,8,28]. Zaman et al [21] studied an SIR epidemic model with control strategies and using the delay. Zaman et al. [11,32] Studied the stability and optimal vaccination of a controlled SIR epidemic model without time delays and their dynamical behavior. Many mathematical models have been proposed to study the optimal control and delay such as [3,5,10,29,30,8,9,11]. For more work, [33-38].

In this paper, we consider a leptospirosis epidemic model [31] with time delay to prevent the spread of disease by using optimal treatment strategies. In order to do this, we first introduce a control variable representing the optimal treatment for infectious host and set an optimal control for our model. Moreover, we show the existence of an optimal control problem. For reducing the infection in a community, we use the control variables. We also analyzed the optimal control and optimality system using optimal control techniques. For the numerical simulation we use the real data of Thailand. Our aim is to maximize the population of susceptibles and recovered and minimize the infection in the infected individuals.

The paper is organized as follows. In Section 2 we study the basic model and applying the optimal control and time delay. Find the Jacobian and Hamiltonian to show the existence of the proposed model. Numerical Simulation of the model with the complete description of the parameters is discussed in Section 3. In the last Section 4 we wind up our work with the conclusion and references.
Optimal Control Techniques in Delay Model

To begin the optimal control procedure, it is necessary to have a model which describes the population dynamics. Youshida and Hara [27] considered an SIR model with time delay. We use an epidemic model of leptospirosis disease model to set our optimal control model. We have a population which consists of five differential equations. The system has two categories, Human and Vector. The human population consists of three sub-classes Susceptible $S_h$, Infected $I_h$ and Recovered $R_h$. The human population is denoted by $N_h$ with $N_1 = S_h + I_h + R_h$. The vector population is denoted by $N_v$ consists of two classes that is susceptible $S_v$ and infected $I_v$, and $N_2 = S_v + I_v$. The model consists of a system of non-linear differential equation is given in the following.

\[
\begin{align*}
\frac{dS_h}{dt} &= b_1 - \mu_h S_h - \frac{\beta S_h I(t)}{N_1(t)} - \frac{\beta S_h I(t-h)}{N_1(t)} + \lambda R_h, \\
\frac{dI_h}{dt} &= \frac{\beta S_h I(t)}{N_1(t)} + \frac{\beta S_h I(t-h)}{N_1(t)} - (\mu_h + \delta_h + \gamma_h) I_h(t), \\
\frac{dR_h}{dt} &= \gamma_h I_h(t) - (\mu_h + \lambda_h) R_h, \\
\frac{dS_v}{dt} &= b_2 - \gamma_v S_v - \frac{\beta S_v I(t)}{N_2(t)}, \\
\frac{dI_v}{dt} &= \frac{\beta S_v I(t)}{N_2(t)} - (\gamma_v + \delta_i) I_v,
\end{align*}
\]

Here $b_1$ is the birth rate of human population, $\beta_1, \beta_2, \beta_3$ respectively represent the transmission Coefficient between human, susceptible human and infected vector and susceptible vector and infected human. Natural mortality rate of human population is represented by $\mu_h$. $\lambda_h$ is constant of proportionality where the infected human becomes susceptible again. The disease death rate of human population is denoted by $\delta_h$, the natural mortality rate of vector population is shown by $\gamma_v$, $\delta_v$ is the disease death rate of the vector. $\alpha_1$ The parameter measure inhibitory effect on human vector population and $\alpha_2$ the parameter measure inhibitory effect of human population. $b_2$ is the birth rate for vector population.

The total dynamics of human population are represented by $N_h$ given by,

\[
N_h = b_1 - \mu_h N_1 - \delta_h I_h,
\] and the total dynamics of vector population is denoted by $N_v$, given by,

\[
N_v = b_2 - \gamma_v N_v - \delta_i I_v.
\]

Next we apply the optimal control and delay to our proposed model (1), we will derive an optimal control model to fit our control strategy. The theoretical foundation of optimal control models with underlying dynamics given by ordinary differential equations was developed by Pontryagin and his co-worker in Moscow in 1950 [9]. So by Pontryagin’s Maximum principle, its extensions and appropriate numerical methods we will set an optimal control problem in the time delayed model of leptospirosis disease. Our main goal is to investigate an effective treatment strategy to control infection diseases. We can make an epidemic model which satisfy that the number of infected individuals is not larger than the susceptible population and want to increase the recovered individuals from the infection. The definition of the control variables $u_1$ and $u_2$ is given by,

$u_1(t)$ Represents (cover all cuts, water dry, full cover boots, shoes and long sleeve shirts when handling animals ).

$u_2(t)$ Represents (wash hands thoroughly on a regular basis and shower after work).

To do this, we set an optimal control problem, with the control set defined by

\[
U = \{(u_1(t), u_2(t)) \in L^2(0,T) : 0 \leq u_1(t), u_2(t) \leq 1, 0 \leq t \leq T\},
\]

Where $u_1(t),u_2(t)$ is Lebesgue measurable and called a control variable.

\[
J(u) = \int_0^T \left( A_1 I_h + A_2 S_v + A_3 S_h + \frac{1}{2}(\xi u_1^2 + \xi u_2^2) \right) dt.
\]

Subject to the control system

\[
\begin{align*}
\frac{dS_h}{dt} &= b_1 - \mu_h S_h - \frac{\beta S_h I(t-h)}{N_1(t-h)} - \frac{\beta S_h I(t-h)}{N_1(t-h)} + \gamma_h R_h, \\
\frac{dI_h}{dt} &= \frac{\beta S_h I(t-h)}{N_1(t-h)} + \frac{\beta S_h I(t-h)}{N_1(t-h)} - (\mu_h + \delta_h + \gamma_h) I_h(t-x), \\
\frac{dR_h}{dt} &= \gamma_h I_h(t-x) - (\mu_h + \lambda_h) R_h, \\
\frac{dS_v}{dt} &= b_2 - \gamma_v S_v - \frac{\beta S_v I(t-x)}{N_2(t-x)}, \\
\frac{dI_v}{dt} &= \frac{\beta S_v I(t-x)}{N_2(t-x)} - (\gamma_v + \delta_i) I_v,
\end{align*}
\]

\[
(6)
\]

$S_h(0) \geq 0, I_h(0) \geq 0, R_h(0) \geq 0, S_v(0) \geq 0, I_v(0) \geq 0.$
Here $\xi_1, \xi_2$ are the positive constants to keep balanced of the sized of infected human individuals, infected vector individuals, susceptible vector individuals, $I_h, I_v, S_v$ at time $t$ and $w \in [0,1]$ and $\epsilon_1$ and $\epsilon_2$ are positive constants. In epidemic dynamics, stability, existence and optimal control theory are important research area. At first we will show the existence of solutions for the control system (6). In this control problem , we assume the restriction on the control variables such that $0 \leq u_1, u_2 \leq 1$, where $(u_1, u_2) \geq 0$ for all $t \in [0,T]$. The total population of host individuals is shown by $N_1$, and $N_2$ the populations of vector. Susceptible individuals acquire infection at a per capita $\beta_1 I_h(t-h)N_1(t-h), \beta_2 I_v(t-h)N_1(t-h)$. In our model the incidence rate is $\beta_1 S_h I_h(t-h)N_1(t-h)$ and $\beta_2 S_v I_v(t-h)N_1(t-h)$ and $\frac{\beta_2 S_v I_v(t-h)}{N_2(t-h)}$. This incidence rate seems more reasonable than $\beta I_h(t)N_1(t), \beta I_v(t)N_1(t)$ because the force of infection is proportional to $\frac{I_h(t-h)}{N_1(t-h)}$ with time delay. Note that in some epidemic models, bilinear incidence rate $\beta S_h(t)I_h(t)$ and standard incidence rate $\beta S_h(t)/N$ are frequently used. Actually the infection probability per contact is likely influenced by the number of infected individuals because more infected individuals can increase infection risk. For instance, during SARS outbreak in 2003. The Chinese government did a lot of protection measures and control policies: closing schools, closing restaurants, postponing conferences, isolating infection etc. These actions greatly reduced the contact number per unit time. Then we write the control system (6) in the following form:

$$\frac{dW(t)}{dt} = AW + F(W(t)) \quad (8)$$

Where

$$W(t) = \begin{bmatrix} S_h(t) \\ I_h(t) \\ R_h(t) \\ S_v(t) \\ I_v(t) \end{bmatrix},$$

$$A = \begin{bmatrix} -\mu_h & 0 & \lambda_h & 0 & 0 \\ 0 & -P_2 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_h & 0 & 0 \\ 0 & 0 & 0 & -P_1 - \epsilon_1 u_1 \\ 0 & 0 & 0 & 0 & -P_1 - \epsilon_2 u_2 \end{bmatrix}.$$
\[ H = L(I_h, S^v, I_v, u_t, u_2) + \lambda_1 \frac{dS^h}{dt} + \lambda_2 \frac{dI_h}{dt} + \lambda_3 \frac{dR^h}{dt} + \lambda_4 \frac{dS^v}{dt} + \lambda_5 \frac{dI_v}{dt}. \]  

In order to find an optimal control pair, we consider the optimal control problem (6-7). First we have to find the Lagrangian and Hamiltonian for the optimal control problem (6-7).

The conditions of the methods of optimal control with delay

The state equations,

\[ x_1(t) = H(x, x_h, u, \lambda)(t), \]

The optimality conditions

\[ 0 = H_u(x, x_h, u, \lambda)(t), \]  

and the adjoint equations,

\[ -\dot{\lambda}_i(t) = H(x, x_h, u, \lambda) \]  

Actually, the Lagrangian of the optimal control problem is given by

**Theorem:** Let \( S^*_h(t), I^*_h(t), R^*_h(t), S^*_v(t) \) and \( I^*_v(t) \) be the state variables with associated optimal solutions for the corresponding optimal control variables \( u^*_1(t), u^*_2(t) \) for the optimal control problem (4-6). Then there exist adjoint variables \( \lambda_i, i = 1, 2, 3, 4, 5 \), satisfying

\[ \frac{d\lambda_i}{dt} = \lambda_i(t) \mu_h \frac{\beta S^*_h I^*_h(t-h)}{N^*_1(t-h)} + \frac{\beta S^*_h I^*_h(t-h)}{N^*_1(t-h)} - \frac{u^*_1(t)I^*_h(t)}{(N^*_1(t))} + \frac{u^*_2(t)I^*_h(t)}{(N^*_1(t))}, \]  

\[ + \lambda_2(t)(1-w) \frac{u^*_1(t)I^*_h(t)}{(N^*_1(t))} - \lambda_3(t) - \lambda_4(t) - \lambda_5(t). \]  

**Proof:** To prove the above result, i.e the adjoint equation and the transversality conditions, we use the Hamiltonian (9). The adjoint system was obtained by using the adjoint equation (10).
\[
 \dot{\lambda}(t) = H_S(t) + \lambda(t) H_{SV}(t), \\
 \dot{\lambda}(t) = H_I(t) + \lambda(t) H_{IV}(t),
\]

With \( \lambda(T) = 0 \). To obtain the required characterization of the optimal control given by (12-13), solving the equations,

\[
\frac{\partial H}{\partial u_1} = 0, \quad \frac{\partial H}{\partial u_2} = 0
\]

In the interior of the control set and by the control space \( U \), we derive the equation (10-13). Substituting the corresponding derivatives in the above equations and after the arrangement we get the adjoint equations (10-13). In addition, the second derivative of the Lagrangian with respect to \( u_1^* \) is positive, which shows that the optimal problem is minimum at control \( u_1^*, u_2^* \). By substituting the value of \( u_1^*, u_2^* \) in the control system (6) we get the followingsystem.

\[
\begin{align*}
\frac{dS}{dt} &= b_1 - \mu_S - \beta S(t) I_{(t-h)} N_{(t-h)} + \beta S(t) I_{(t-h)} N_{(t-h)} + \lambda_1 (1-w) I(t), \\
\frac{dI}{dt} &= \beta S(t) I_{(t-h)} N_{(t-h)} - \mu_I I(t) - \gamma I(t) - \delta I(t) - \lambda_2 (1-w) I(t), \\
\frac{dR}{dt} &= \gamma I(t) - (\mu + \lambda_3) R(t) - \lambda_4 I(t), \\
\frac{dA}{dt} &= \mu I(t) - \delta A(t) - \lambda_5 A(t),
\end{align*}
\]

\[
\text{Numerical Simulation and Summary}
\]

In this section, we present the numerical simulations of the proposed model (1) and the delay control model (6) by using Runge-Kutta method. We solve first the model (1) and then solving the proposed model (6).
Then we solve the adjoint equation (10) with the boundary conditions (11) numerically by Runge-kutta order four backward scheme. The constants used in the objective functional with their numerical values we assumed in the numerical simulation is $A_2 = 0.001, A_3 = 0.002, \xi_1 = 0.7, \xi_2 = 0.3$ and $\varepsilon_2 = 0.0031$. The values of parameters used in the numerical simulations are presented in Table 1.

In this simulation the bold line shows the system with no control and the dashed line shows the system with control throughout Figure 1 to Figure 5. Figure 6 and Figure 7 represents the control variable $u_1$ and $u_2$ respectively. The aim of this paper is to control the infection in the host population by using the control variables in the form of treatment or prevention or educational campaign. The control shows in the Figure 1, the population of susceptible human increases and Figure 2 the infection in the host decreases. Figure 3 shows the recovered individuals of human population which increases. Also the population of susceptible vector and infected vector and susceptible vector also decreases in Figure 4 and Figure 5.

![Figure 1](image1.png)

Figure 1. Represents the comparison of susceptible human in both the system without control and with control.

![Figure 2](image2.png)

Figure 2. Represents the comparison of infected human in both the system without control and with control.

In this paper, we have presented an epidemic model by applying the optical control and time delay. First we have obtained the formulatation of the model and then we applied the time delay and optimal control with control variables $u_1, u_2$. Then we have proved the existence of the control system and obtained the numerical solution of the both the system without control and with control. Finally, we conclude our work by references.

![Figure 6](image6.png)

Figure 6. Represents the the contro variable $u_1$

![Figure 7](image7.png)

Figure 7. Represents the the control variable $u_2$

Table 1. Parameters value used in the numerical simulation of the optimal control problem.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Parameters Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_1$</td>
<td>Recruitment rate for human population</td>
<td>13</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>Transmission rate for human population</td>
<td>0.01</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>Transmission rate for vector population</td>
<td>0.95</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>Transmission rate for $S_v$ and $I_h$</td>
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</tr>
<tr>
<td>$\mu_h$</td>
<td>Natural mortality rate of human population</td>
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</tr>
<tr>
<td>$\lambda_h$</td>
<td>Proportionality constant</td>
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<tr>
<td>$\delta_h$</td>
<td>Disease death rate for human population</td>
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<tr>
<td>$\gamma_v$</td>
<td>Natural mortality rate of vector population</td>
<td>0.051</td>
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<tr>
<td>$\delta_v$</td>
<td>Disease death rate for vector population</td>
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<tr>
<td>$b_2$</td>
<td>Recruitment rate for vector population</td>
<td>3</td>
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</table>

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References