On a Quadratic Functional Integral Equation with Deviated Arguments

Fatma M. Gaafar

Department of Mathematics, Faculty of Science, Damanhour University, Damanhour, Egypt
fatmagaafar2@yahoo.com

Abstract: In this paper, we study the existence of at least one positive solution for a quadratic functional integral equation of Fredholm type with deviated arguments by applying the technique of measure of noncompactness.
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1. Introduction

The equation

$$x(t) = g(t) + f(t,x(t)) \int_0^1 k(t,s)\, h(s,x(s))\, ds,$$

with $t \in [0,1]$ appears very often in a lot of applications to real world problems. For example, some problems considered in vehicular traffic theory, biology and queuing theory lead to the quadratic integral equations of this type (cf. [24]). Moreover, such integral equations are also applied in the theory of radiative transfer and the theory of neutron transport as well in the kinetic theory of gases (cf. [15], [21], [22], [25], [27], among others).

In this paper, we are going to prove a theorem on the existence of at least one integrable solution for the quadratic functional integral equation

$$x(t) = g(t) + f_1(t, x(\phi_1(t))) \int_0^1 k_2(t,s) f_2(s, x(\phi_2(s)))\, ds$$
$$+ f_2(t, x(\phi_2(t))) \int_0^1 k_1(t,s)\, f_1(s, x(\phi_1(s)))\, ds \qquad (1)$$

The integral equation in question has rather general form and contains as particular cases a lot of functional equations and nonlinear integral equations of Fredholm type. The main tool used in our consideration is the technique of measures of noncompactness and the fixed point theorem of Darbo [3].

Many authors have studied the existence of solutions for several classes of nonlinear quadratic integral equations (see [12]-[14], [17]-[20], [26], [28], and the references therein).

Some problems considered in the vehicular traffic theory, biology and queuing theory lead to the following nonlinear functional-integral equation (see [24]),

$$x(t) = f(t, x(t)) \int_0^1 g(t,s,x(s))\, ds,$$

Banaś et al. [8] considered quadratic integral equation,

$$x(t) = 1 + (Tx)(t) \int_0^1 k(t,s)\, \phi(s)\, x(s)\, ds\ .$$

where $t \in [0,1]$ and T is an operator which maps $C([0,1])$ continuously into itself and satisfies the Darbo condition. They proved that under certain assumptions it is solvable in the space $C([0,1])$.

In this paper, we are going to study the solvability of a class of quadratic functional integral equations of Fredholm type (1). We prove the existence of at least one integrable solution $x \in L_1[0,1]$ of the quadratic integral Equation (1) by using the technique of noncompactness which is frequently used in several branches of nonlinear analysis (see [2], [9], [10], [24]), where the functions $f_i(t,x(t)),\ i=1,2$ are L_1-Carathèodory functions.

2. Notation and auxiliary facts

Let $L_1 = L_1(I),\ I=[0,1]$ denoted the space of Lebesgue integrable functions on I and the norm in $L_1(I)$ is defined by

$$\| x \|_{L_1} = \int_0^1 |x(t)|\, dt.$$

Assume that the function $f : I \times R \to R$ satisfies Carathèodory condition i.e., it is measurable in t for any x and continuous in x for almost all t. Then to every function $x(t)$ being measurable on the interval I we may assign the function

$$(Fx)(t) = f(t, x(t)), \quad t \in I.$$

The operator F defined in such a way is called the superposition operator. This operator is one of the simplest and most important operators investigated in nonlinear functional analysis. For this operator we have the following theorem due to Krasnosel'skii [4].

Theorem 1: The superposition operator F maps continuously the space L_1 into itself if and only if

$$|f(t,x)| \le c(t) + k|x| \quad \text{for all } t \in I$$

and $x \in R$, where $c(t)$ is a function from L_1 and k is a nonnegative constant.

Now let E be a Banach space with zero element θ and let X be a nonempty bounded subset of E. Moreover denote by $B_r = B(\theta, r)$ the closed ball in E centered at θ and with radius r. For X being a nonempty subset of E we denote by $\overline{X}$, $\mathrm{Conv}\,X$ the closure and the convex closure of X, ($\mathrm{Conv}\,X$ is defined as the smallest convex closed set containing X), respectively. Further we denoted by M_E the family of nonempty and bounded subsets of E and by N_E its sub family consisting of all relatively compact and nonempty subset of E.

Definition 1: (See [3]) A function $\mu : M_E \to [0,\infty)$ is said to be a measure of noncompactness in the space E if it satisfies the following conditions,

1. the family $\ker\mu = X \in M_E : \mu(X) = 0$ is nonempty and $\ker\mu \subset N_E$;
2. $X \subset Y \Rightarrow \mu(X) \le \mu(Y);$
3. $\mu(\overline{X}) = \mu(\mathrm{Conv}\,X) = \mu(X);$
4. $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y),$ for $\lambda \in [0,1]$;
5. If X_n is a sequence of closed sets from M_E such that $X_{n+1} \subset X_n$, for $n = 1, 2, 3, \ldots$, and if $\lim_{n\to\infty} \mu(X_n) = 0,$ then $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is not empty.

The family $\ker\mu$ described above is called the kernel of the measure of noncompactness μ.

For further details concerning measures of noncompactness and their properties may be found in [3]. In the sequel we shall need some criteria for compactness in measure; the complete description of compactness in measure was given by Fre'chet [4], but the following sufficient condition will be more convenient for our purposes (see[4]).

Theorem 2: Let X be a bounded subset of L_1. Assume that there is a family of subsets $(\Omega_c)_{0 \le c \le b-a}$ of the interval (a,b) such that $\mathrm{meas}\,\Omega_c = c$ for every $c \in [0, b-a]$, and for every $x \in X$, $x(t_1) \le x(t_2)$, $(t_1 \in \Omega_c,\ t_2 \notin \Omega_c)$, then the set X is compact in measure.

The measure of weak noncompactness defined by De Blasi ([1] and [23]) is given by,

$$\beta(X) = \inf(r > 0 : \text{there exists a weakly compact subset } Y \text{ of } E \text{ such that } X \subset Y + K_r).$$

The function $\beta(X)$ possesses several useful properties which may be found in [23].

The convenient formula for the function $\beta(X)$ in L_1 was given by Appel and De Pascale (see [1], [3])

$$\beta(X) = \lim_{\varepsilon \to 0} (\sup_{x \in X} (\sup[\int_D |x(t)|\,dt : D \subset [a,b], \ \mathrm{meas}\, D \le \varepsilon])),$$

where the symbol meas D stands for Lebesgue measure of the set D.

Next, we shall also use the notion of the Hausdorff measure of noncompactness χ (see [4]) defined by

$$\chi(X) = \inf(r > 0 : \text{there exists a finite subset } Y \text{ of } E \text{ such that } X \subset Y + K_r).$$

In the case when the set X is compact in measure, the Hausdoff and De Blasi measures of noncompactness will be identical. Namely we have (see [1] and [23]).

Theorem 3: Let X be an arbitrary nonempty bounded subset of L_1. If X is compact in measure then $\beta(X) = \chi(X)$.

Finally, we will recall the fixed point theorem due to

Darbo [3].

Theorem 4: Let Q be a nonempty, bounded, closed and convex subset of E and let $H : Q \to Q$ be a continuous transformation which is a contraction with respect to the Hausdorff measure of noncompactness χ , i.e., there exists a constant $\alpha \in [0,1)$ such that $\chi(H\,X) \le \alpha\,\chi(X)$ for any nonempty subset X of Q. Then H has at least one fixed point in the set Q.

3. Existence of solutions

Let the integral operator H_i be defined as

$$(H_i x)(t) = \int_0^1 k_i(t,s)\, f_i(s, x(\phi_i(s)))\, ds, \quad i = 1, 2.$$

Then equation (1) may be written in operator form as:

$$(Ax)(t)=g+(F_1x)(t).(H_2x)(t)+(F_2x)(t).(H_1x)(t),$$

where $(F_ix)(t) = f_i(s, x(\phi_i(s)))\, ds, \quad i = 1, 2.$

Consider the quadratic equation (1), under the following assumptions:

(i) $g \in L_1$ and is a.e. nondecreasing and positive on the interval I;

(ii) $f_i : I \times R_+ \to R_+$ satisfy Carathèodory condition (i.e. measurable in t for all $x \in R_+$ and continuous in x for all $t \in [0,1]$) and there exist two functions $a_i \in L_1$ and two constants $b_i \geq 0, \quad i=1,2$ such that

$$f_i(t,x) \leq a_i(t) + b_i\,|x| \quad \forall (t,x) \in I \times R_+.$$

Moreover, $f_i(t,x),\ i=1,2$ are a.e. nondecreasing with respect to each of both variables;

(iii) $\phi_i : I \to I$ is increasing, absolutely continuous on I and there exists a constant $M_i > 0$ such that $\phi_i'(t) > M_i$ on I;

(iv) $k_i : I \times I \to R_+,\ i = 1,2$ are continuous and $k_i(t,x)$ is nondecreasing with respect to each variables t and x, separately. And there exist positive constants N_i, such that $k_i(t,x) \leq N_i$.

(v) Let $d > 2\sqrt{c\,(\|g\| + \|a_1\|\,\|a_2\|\,(N_1+N_2))}$, where

$$d=1-\frac{(b_2 M_1\|a_1\|+b_1 M_2\|a_2\|)\,(N_1+N_2)}{M_1 M_2},$$

and $c = \dfrac{b_1 b_2 (N_1+N_2)}{M_1 M_2}$.

Let r be a positive root of the equation

$$c\,r^2 - d\,r + (\|g\| + \|a_1\|\,\|a_2\|(N_1+N_2))\, M_1 M_2 = 0,$$

and define the set

$$B_r = \{x \in L_1 : \|x\| \leq r\}.$$

For the existence of at least one L_1- positive solution of the quadratic integral equation (1) we have the following theorem.

Theorem 5: Let the assumptions (i)-(v) are satisfied. If $r c < 1$; then the quadratic integral equation (1) has at least one solution $x \in L_1$ which is positive and a.e. nondecreasing on I.

Proof. Take an arbitrary $x \in L_1$ then, we get

$$\begin{aligned}|(Ax)(t)| \leq |g(t)| &+ (a_1(t) + b_1|x(\phi_1(t))|)\int_0^1 k_2(t,s)\,(a_2(s) \\ &+ b_2|x(\phi_2(s))|)\, ds \\ &+ (a_2(t) + b_2|x(\phi_2(t))|)\int_0^1 k_1(t,s)\,(a_1(s) \\ &+ b_1|x(\phi_1(s))|)\, ds,\end{aligned}$$

which implies that

$$\begin{aligned}\|(Ax)(t)\| &= \int_0^1 |(Ax)(t)|\, dt \\
&\leq \int_0^1 |g(t)|\,dt + \int_0^1 a_1(t) \int_0^1 |k_2(t,s)|\, a_2(s)\, ds\, dt \\
&\quad + b_2 \int_0^1 a_1(t) \left(\int_0^1 |k_2(t,s)|\,|x(\phi_2(s))|\, ds \right) dt \\
&\quad + b_1 \int_0^1 |x(\phi_1(t))| \int_0^1 |k_2(t,s)|\, a_2(s)\, ds\, dt \\
&\quad + b_1 b_2 \int_0^1 |x(\phi_1(t))| \left(\int_0^1 |k_2(t,s)|\,|x(\phi_2(s)|\, ds \right) dt \\
&\quad + \int_0^1 a_2(t) \int_0^1 |k_1(t,s)|\, a_1(s)\, ds\, dt \\
&\quad + b_1 \int_0^1 a_2(t) \left(\int_0^1 |k_1(t,s)|\,|x(\phi_1(s))|\, ds \right) dt \\
&\quad + b_2 \int_0^1 |x(\phi_2(t))| \int_0^1 |k_1(t,s)|\, a_1(s)\, ds\, dt \\
&\quad + b_1 b_2 \int_0^1 |x(\phi_2(t))| \left(\int_0^1 |k_1(t,s)|\,|x(\phi_1(s)|\, ds \right) dt, \\
&\leq \|g\| + N_2 \int_0^1 a_1(t) \int_0^1 a_2(s)\, ds\, dt \\
&\quad + b_2\, N_2 \int_0^1 a_1(t) \int_0^1 |x(\phi_2(s))|\, ds\, dt \\
&\quad + b_1 N_2 \int_0^1 |x(\phi_1(t))| \int_0^1 a_2(s)\, ds\, dt \\
&\quad + b_1 b_2\, N_2 \int_0^1 |x(\phi_1(t))| \int_0^1 |x(\phi_2(s)|\, ds\, dt \\
&\quad + N_1 \int_0^1 a_2(t) \int_0^1 a_1(s)\, ds\, dt \\
&\quad + b_1\, N_1 \int_0^1 a_2(t) \int_0^1 |x(\phi_1(s))|\, ds\, dt \\
&\quad + b_2 N_1 \int_0^1 |x(\phi_2(t))| \int_0^1 a_1(s)\, ds\, dt \\
&\quad + b_1 b_2\, N_1 \int_0^1 |x(\phi_2(t))| \int_0^1 |x(\phi_1(s)|\, ds\, dt,\end{aligned}$$

$$\le \|g\| + N_2 \int_0^1 a_1(t)\,dt \int_0^1 a_2(s)\,ds$$
$$+ b_2\ N_2 \int_0^1 a_1(t)\,dt \int_0^1 |x(\phi_2(s))|\ ds$$
$$+ b_1 N_2 \int_0^1 |x(\phi_1(t))|\,dt \int_0^1 a_2(s)\ ds$$
$$+ b_1 b_2\ N_2 \int_0^1 |x(\phi_1(t))|\,dt \int_0^1 |x(\phi_2(s)|\ ds$$
$$+ N_1 \int_0^1 a_2(t)\,dt \int_0^1 a_1(s)\,ds$$
$$+ N_1 \int_0^1 a_2(t)\,dt \int_0^1 a_1(s)\,ds$$
$$+ b_1\ N_1 \int_0^1 a_2(t)\,dt \int_0^1 |x(\phi_1(s))|\,ds$$
$$+ b_2 N_1 \int_0^1 |x(\phi_2(t))|\,dt \int_0^1 a_1(s)\,ds$$
$$+ b_1 b_2\ N_1 \int_0^1 |x(\phi_2(t))|\,dt \int_0^1 |x(\phi_1(s)|\,ds,$$
$$\le \|g\| + N_2\ \|a_1\|\,\|a_2\|$$
$$+ \frac{b_2\ N_2 \|a_1\|}{M_2} \int_0^1 |x(\phi_2(s))|\ \phi_2'(s)\ ds$$
$$+ \frac{b_1\ N_2 \|a_2\|}{M_1} \int_0^1 |x(\phi_1(t))|\ \phi_1'(t)\ dt$$
$$+ \frac{b_1 b_2 N_2}{M_1 M_2} \int_0^1 |x(\phi_1(t))|\phi_1'(t)\,dt \int_0^1 |x(\phi_2(s)|\phi_2'(s)\,ds$$
$$+ N_1\ \|a_1\|\,\|a_2\|$$
$$+ \frac{b_1\ N_1 \|a_2\|}{M_1} \int_0^1 |x(\phi_1(s))|\ \phi_1'(s)\ ds$$
$$+ \frac{b_2\ N_1 \|a_1\|}{M_2} \int_0^1 |x(\phi_2(t))|\ \phi_2'(t)\ dt$$
$$+ \frac{b_1 b_2 N_1}{M_1 M_2} \int_0^1 |x(\phi_2(t))|\phi_2'(t)\,dt \int_0^1 |x(\phi_1(s)|\phi_1'(s)\,ds,$$
$$\le \|g\| + (N_1 + N_2)\ \|a_1\|\,\|a_2\|$$
$$+ \frac{b_2\ N_2 \|a_1\|}{M_2} \int_{\phi_2(0)}^{\phi_2(1)} |x(u)|\ du$$
$$+ \frac{b_1\ N_2 \|a_2\|}{M_1} \int_{\phi_1(0)}^{\phi_1(1)} |x(u)|\ du$$
$$+ \frac{b_1 b_2 N_2}{M_1 M_2} \int_{\phi_1(0)}^{\phi_1(1)} |x(u)|\ du \int_{\phi_2(0)}^{\phi_2(1)} |x(u)|\ du$$
$$+ \frac{b_1\ N_1 \|a_2\|}{M_1} \int_{\phi_1(0)}^{\phi_1(1)} |x(u)|\ du$$
$$+ \frac{b_2\ N_1 \|a_1\|}{M_2} \int_{\phi_2(0)}^{\phi_2(1)} |x(u)|\ du$$
$$+ \frac{b_1 b_2 N_1}{M_1 M_2} \int_{\phi_2(0)}^{\phi_2(1)} |x(u)|\ du \int_{\phi_1(0)}^{\phi_1(1)} |x(u)|\ du,$$
$$\le \|g\| + (N_1 + N_2)\ \|a_1\|\,\|a_2\|$$
$$+ \frac{b_2\ N_2 \|a_1\|}{M_2} \int_0^1 |x(u)|\ du$$
$$+ \frac{b_1\ N_2 \|a_2\|}{M_1} \int_0^1 |x(u)|\ du$$
$$+ \frac{b_1 b_2 N_2}{M_1 M_2} \int_0^1 |x(u)|\ du \int_0^1 |x(u)|\ du$$
$$+ \frac{b_1\ N_1 \|a_2\|}{M_1} \int_0^1 |x(u)|\ du$$
$$+ \frac{b_2\ N_1 \|a_1\|}{M_2} \int_0^1 |x(u)|\ du$$
$$+ \frac{b_1 b_2 N_1}{M_1 M_2} \int_0^1 |x(u)|\ du \int_0^1 |x(u)|\ du$$
$$\le \|g\| + (N_1 + N_2)\ \|a_1\|\,\|a_2\|$$
$$+ \left(\frac{b_2\ N_2 \|a_1\|}{M_2} + \frac{b_1\ N_2 \|a_2\|}{M_1} + \frac{b_1\ N_1 \|a_2\|}{M_1} \right.$$
$$\left. + \frac{b_2\ N_1 \|a_1\|}{M_2} \right) \|x\| \ + c\,\|x\|^2$$
$$\le \|g\| + (N_1 + N_2)\ \|a_1\|\,\|a_2\|$$
$$+ \frac{(b_2\ M_1 \|a_1\| + b_1\ M_2 \|a_2\|)(N_1 + N_2)}{M_1 M_2} \|x\|$$
$$+ c\,\|x\|^2 \le r.$$

From this estimate we show that the operator A maps the ball B_r into itself with

$$r = \frac{d - \sqrt{d^2 - 4c\,(\,\|g\| + (N_1 + N_2)\ \|a_1\|\,\|a_2\|)}}{2c}.$$

From assumption (v) we have

$$0 < d^2 - 4c\,(\,\|g\| + (N_1 + N_2)\ \|a_1\|\,\|a_2\|) < d^2,$$

which implies that

$$0 < \sqrt{d^2 - 4c\,(\,\|g\| + (N_1 + N_2)\ \|a_1\|\,\|a_2\|)} < d.$$

Then d is positive which implies that r is a positive constant.

Now, let Q_r denote the subset of $B_r \in L_1$ consisting of all functions which are a.e. nondecreasing on I.

The set Q_r is nonempty, bounded, convex and closed (see Banaś [4] pp. 780). Moreover this set is compact in measure (see Lemma 2 in [7] pp. 63).

From assumption (ii) we deduce that the operator A maps Q_r into itself. Since the operator $(F_i x)(t) = f_i(t, x(t))$ is continuous (Theorem 1 in section 2), then the operator H_i is continuous and hence the product $F_i H_i$ is continuous. Thus the operator A is continuous on Q_r.

Let X be a nonempty subset of Q_r. Fix $\varepsilon>0$ and take a measurable subset $D \subset I$ such that $\text{meas } D \le \varepsilon$. Then, for any $x \in X$ using the same reasoning as in [4] and [7], we get

$$\|(Ax)(t)\|_{L_1(D)} = \int_D |(Ax)(t)|\, dt$$

$$\le \int_D |g(t)|dt + \int_D a_1(t) \int_0^1 |k_2(t,s)|\, a_2(s) ds\, dt$$
$$+ b_2 \int_D a_1(t) \int_0^1 |k_2(t,s)|\,|x(\phi_2(s))|\, ds\, dt$$
$$+ b_1 \int_D |x(\phi_1(t))| \int_0^1 |k_2(t,s)|\, a_2(s)\, ds\, dt$$
$$+ b_1 b_2 \int_D |x(\phi_1(t))| \left(\int_0^1 |k_2(t,s)|\,|x(\phi_2(s)|\, ds\right) dt$$
$$+ \int_D a_2(t) \int_0^1 |k_1(t,s)|\, a_1(s) ds\, dt$$
$$+ b_1 \int_D a_2(t) \left(\int_0^1 |k_1(t,s)|\,|x(\phi_1(s))|\, ds\right) dt$$
$$+ b_2 \int_D |x(\phi_2(t))| \int_0^1 |k_1(t,s)|\, a_1(s)\, ds\, dt$$
$$+ b_1 b_2 \int_D |x(\phi_2(t))| \left(\int_0^1 |k_1(t,s)|\,|x(\phi_1(s)|\, ds\right) dt,$$

$$\le \|g\|_{L_1(D)} + N_2 \int_D a_1(t) \int_0^1 a_2(s) ds\, dt$$
$$+ b_2 N_2 \int_D a_1(t) \int_0^1 |x(\phi_2(s))| ds\, dt$$
$$+ b_1 N_2 \int_D |x(\phi_1(t))| \int_0^1 a_2(s) ds\, dt$$
$$+ b_1 b_2 N_2 \int_D |x(\phi_1(t))| \int_0^1 |x(\phi_2(s)|\, ds\, dt$$
$$+ N_1 \int_D a_2(t) \int_0^1 a_1(s)\, ds\, dt$$
$$+ b_1 N_1 \int_D a_2(t) \int_0^1 |x(\phi_1(s))|\, ds\, dt$$
$$+ b_2 N_1 \int_D |x(\phi_2(t))| \int_0^1 a_1(s)\, ds\, dt$$
$$+ b_1 b_2 N_1 \int_D |x(\phi_2(t))| \int_0^1 |x(\phi_1(s)|\, ds\, dt,$$

$$\le \|g\|_{L_1(D)} + N_2 \int_D a_1(t)\, dt \int_D a_2(s) ds$$
$$+ b_2 N_2 \int_D a_1(t)\, dt \int_D |x(\phi_2(s))| ds$$
$$+ b_1 N_2 \int_D |x(\phi_1(t))|\, dt \int_D a_2(s)\, ds$$
$$+ b_1 b_2 N_2 \int_D |x(\phi_1(t))|\, dt \int_D |x(\phi_2(s)|\, ds$$
$$+ N_1 \int_D a_2(t) dt \int_D a_1(s)\, ds$$
$$+ b_1 N_1 \int_D a_2(t) dt \int_D |x(\phi_1(s))| ds$$
$$+ b_2 N_1 \int_D |x(\phi_2(t))| dt \int_D a_1(s)\, ds$$
$$+ b_1 b_2 N_1 \int_D |x(\phi_2(t))|\, dt \int_D |x(\phi_1(s)|\, ds,$$

$$\le \|g\|_{L_1(D)} + N_2 \|a_1\|_{L_1(D)} \|a_2\|_{L_1(D)}$$
$$+ \frac{b_2 N_2 \|a_1\|_{L_1(D)}}{M_2} \int_D |x(\phi_2(s))|\, \phi_2'(s) ds$$
$$+ \frac{b_1 N_2 \|a_2\|_{L_1(D)}}{M_1} \int_D |x(\phi_1(t))|\, \phi_1'(t)\, dt$$
$$+ \frac{b_1 b_2 N_2}{M_1 M_2} \int_D |x(\phi_1(t))| \phi_1'(t) dt \int_D |x(\phi_2(s)| \phi_2'(s) ds$$
$$+ N_1 \|a_1\|_{L_1(D)} \|a_2\|_{L_1(D)}$$
$$+ \frac{b_1 N_1 \|a_2\|_{L_1(D)}}{M_1} \int_D |x(\phi_1(s))|\, \phi_1'(s)\, ds$$
$$+ \frac{b_2 N_1 \|a_1\|_{L_1(D)}}{M_2} \int_D |x(\phi_2(t))|\, \phi_2'(t)\, dt$$
$$+ \frac{b_1 b_2 N_1}{M_1 M_2} \int_D |x(\phi_2(t))| \phi_2'(t) dt \int_D |x(\phi_1(s)| \phi_1'(s)\, ds,$$

$$\le \|g\|_{L_1(D)} + (N_1 + N_2) \|a_1\|_{L_1(D)} \|a_2\|_{L_1(D)}$$
$$+ \frac{b_2 N_2 \|a_1\|_{L_1(D)}}{M_2} \int_D |x(u)|\, du$$
$$+ \frac{b_1 N_2 \|a_2\|_{L_1(D)}}{M_1} \int_D |x(u)|\, du$$
$$+ \frac{b_1 b_2 N_2}{M_1 M_2} \int_D |x(u)|\, du \int_D |x(u)|\, du$$
$$+ \frac{b_1 N_1 \|a_2\|_{L_1(D)}}{M_1} \int_D |x(u)|\, du$$
$$+ \frac{b_2 N_1 \|a_1\|_{L_1(D)}}{M_2} \int_D |x(u)|\, du$$
$$+ \frac{b_1 b_2 N_1}{M_1 M_2} \int_D |x(u)|\, du \int_D |x(u)|\, du$$

$$\leq \|g\|_{L_1(D)} + (N_1+N_2)\|a_1\|_{L_1(D)}\ \|a_2\|_{L_1(D)}$$
$$+\left(\frac{b_2\ N_2\|a_1\|_{L_1(D)}}{M_2}+\frac{b_1\ N_2\|a_2\|_{L_1(D)}}{M_1}\right.$$
$$\left.+\frac{b_1\ N_1\|a_2\|_{L_1(D)}}{M_1}+\frac{b_2\ N_1\|a_1\|_{L_1(D)}}{M_2}\right)\|x\|_{L_1(D)}$$
$$+c\,\|x\|_{L_1(D)}\ \|x\|_{L_1(D)}$$
$$\leq \|g\|_{L_1(D)} + (N_1+N_2)\|a_1\|_{L_1(D)}\ \|a_2\|_{L_1(D)}$$
$$+\frac{(b_2\ M_1\|a_1\|_{L_1(D)}+b_1\ M_2\|a_2\|_{L_1(D)})(N_1+N_2)}{M_1 M_2}\|x\|_{L_1(D)}$$
$$+r\,c\,\|x\|_{L_1(D)}.$$

Since,
$$\lim_{\varepsilon\to 0}\{\sup\{\int_D |a_i(t)|\,dt:\ D\subset I,\ \text{meas}\,D\leq\varepsilon\}\}=0,$$
and
$$\lim_{\varepsilon\to 0}\{\sup\{\int_D |g(t)|\,dt:\ D\subset I,\ \text{meas}\,D\leq\varepsilon\}\}=0,$$
We obtain
$$\beta(Ax(t))\leq r\,c\,\beta(x(t)),$$
this implies
$$\beta(A\,X)\leq r\,c\,\beta(X), \tag{2}$$
where β is the De Blasi measure of week noncompactness, and by Theorem 2 we can write (2) in the form
$$\chi(A\,X)\leq r\,c\,\chi(X),$$
where χ is the Hausedorff measure of noncompactness. Since $r\,c<1$, then from Theorem 4 it follows that A is contraction with respect to the measure of noncompactness χ. Thus A has at least one fixed point in Q_r which is a solution of the quadratic integral equation (1).

4. Quadratic equations with deviated argument

Let the assumptions of Theorem 5 are satisfied:

(i) Let $\phi_i(t)=\beta_i\,t,\ \beta_i\in(0,1)$, then equation (1) can be written as:
$$x(t)=g(t)+f_1(t,x(\beta_1\,t))\int_0^1 k_2(t,s)f_2(s,x(\beta_2\,t))\,ds$$
$$+f_2(t,x(\beta_2\,t))\int_0^1 k_1(t,s)\,f_1(s,x(\beta_1\,t))\,ds$$
where $\phi_i(t)=\beta_i\,t:I\to I$ is increasing, absolutely continuous on I with $M_i=1$. Then the quadratic integral equation has at least one solution $x\in L_1$ which is positive and a.e. nondecreasing on I.

(ii) Let $\phi_i(t)=t^{\beta_i},\ \beta_i\in(0,1)$, then equation (1) can be written as:
$$x(t)=g(t)+f_1(t,x(t^{\beta_1}))\int_0^1 k_2(t,s)f_2(s,x(t^{\beta_2}))\,ds$$
$$+f_2(t,x(t^{\beta_2}))\int_0^1 k_1(t,s)\,f_1(s,x(t^{\beta_1}))\,ds$$
where $\phi_i(t)=t^{\beta_i}:I\to I$, is increasing, absolutely continuous on I with $M_i=1$. Then the quadratic integral equation has at least one solution $x\in L_1$ which is positive and a.e. nondecreasing on I.

4. Discussions

Necessary conditions for the existence of at least one solution $x\in L_1$ for the quadratic functional integral equation Eq. (1) which is positive and a.e. nondecreasing are proposed in this paper by using the method of measure of noncompactness. As a special cases, the existence results for quadratic functional integral equation with deviated argument can be obtained by taking $\phi_i(t)=\beta_i\,t,\ \ \phi_i(t)=t^{\beta_i},\ \beta_i\in(0,1)$.

Corresponding Author:
Dr. Fatma M. Gaafar
Department of Mathematics
Faculty of Science, Damanhour University
Damanhour, Egypt
E-mail: fatmagaafar2@yahoo.com

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