An inverse fractional abstract Cauchy problem with nonlocal conditions

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Abotract: This note is devoted to the study of an inverse Cauchy problem in a Hilbert space H for the abstract fractional differential equation of the form: $\frac{d^\alpha u(t)}{dt^\alpha} = A\,u(t) + f(t)\,g(t),$ with the nonlocal initial condition: $u(0) = u_0 + \sum_{k=1}^{p} c_k u(t_k),$ and the overdetermination condition: $(u(t), v) = w(t),$ where $(.,.)$ is the inner product in H, f is a real unknown function w is a given real function, u_0, v are given elements in H, g is a given abstract function with values in H, $0 < \alpha \le 1$, u is unknown, and A is a linear closed operator defined on a dense subset of H.It is supposed that A generates a bounded semigroup. An application is given to study a nonlocal inverse problem in a suitable Sobolev space for general fractional parabolic partial differential equations with unknown source functions.

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1. Introduction

Several authors [1-7]considered the identification of an unknown state independent source term in the heat equation and some parabolic equations. In other words some inverse problems must be solved to find, on basis of the observations, the coefficients, free term and sometimes initial and boundary conditions. To solve some problems related to the theory of viscoelasticity, we need to study inverse and nonlocal Cauchy problem for suitable fractional partial differential equations, see[8-11]. The purpose of this paper is to study an inverse problem in a real Hilbert space H for the abstract fractional differential equation of the form:

$$\frac{d^\alpha u(t)}{dt^\alpha} = A\,u(t) + f(t) g(t), (1.1)$$

with the nonlocal initial condition

$$u(0) = u_0 + \sum_{k=1}^{p} c_k u(t_k), (1.2)$$

and the overdetermination condition:

$$(u(t), v) = w(t), (1.3)$$

where $(.,.)$ is the inner product in H, f is a real unknown function, w is a given real function, u_0, v are given elements in H, g is a given abstract function with values in H,

$0 \le t_1 < \ldots < t_p \le a,$ $c_1, \ldots, c_p$ are real numbers, and A is a linear closed operator defined on a dense set S in H to H.

It is supposed that A generates an analytic semigroup $Q(t)$ such that

$\|Q(t)\| \le K$ for all $t \in J = [0, a],$ Q(t) $h \in$ S ,$\|AQ(t)h\| \le \frac{K}{t}\|h\|$ for every element h in H, and all $t \in (0, a],$ where $\|.\|$ is the norm in H, (see [12]).

We shall consider the integral of operator - valued functions;

$$\psi(t) = \int_0^\infty \zeta_\alpha(\theta)\, Q(t^\alpha \theta) d\theta,$$

and

$$\psi^*(t) = \alpha \int_0^\infty \theta t^{\alpha-1} \zeta_\alpha(\theta) Q(t^\alpha \theta)\, d\theta, t > 0,$$

where ζ_α is a probability density function defined on $(0, \infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\theta x} \zeta_\alpha(\theta) d\theta = \sum_{j=0}^{\infty} \frac{(-x)^j}{\Gamma(1+\alpha j)},$$

where $0 < \alpha \le 1,$
$x > 0,$ *and* Γ *is the gamma function*,[12,13].

It is clear that ψ represents a uniformly continuous function of t in the uniform topology (i.e., in the set of all linear bounded operators $B(H)$ defined on H).

We shall assume the following conditions;

$A_1: u_0, v \in S, and\ g(t) \in S$ for all $t \in J$,

$A_2: |g_1(t)| \geq g_0, t \in J$, where $g_1(t) = (g(t), v)$

and g_0 is a positive constant,

A_3: The abstract functions g and Ag are continuous on J with respect to the norm in H,

$A_4: \frac{d^\alpha w}{dt^\alpha} \in C(J)$, where $C(J)$ is the set of all continuous functions on J.

It is suitable to rewrite the problem (1.1), (1.2) in the form:

$$u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1}[Au(\theta) + f(\theta)g(\theta)]d\theta. (1.4)$$

In section 2, the nonlocal inverse Cauchy problem (1.4)is studied under the overdetermination condition (1.3). In section 3 an application is given to the nonlocal inverse Cauchy problem (1.4) for partial differential equations of the form:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \sum_{|q| \leq 2m} a_q(x) D^q u(x,t) = f(t) g(x,t), (1.5)$$

with the initial condition

$$u(x,0) = u_0(x) + \sum_{k=1}^{p} c_k u(x, t_k), (1.6)$$

and the integral overdetermination condition

$$\int_G u(x,t) v(x) dx = w(t), (1.7)$$

where $q = (q_1, ..., q_n)$ is an n-dimensional multi-index, $x \in G \subset R^n, R^n$ is the n-dimensional Euclidean space, G is a bounded region with smooth boundary $\partial G, D^q = D_1^{q_1} ... D_n^{q_n}$,

$$D_j = \frac{\partial}{\partial x_j},\ j = 1, ..., n,\ |q| = q_1 + ... q_n.$$

It is assumed that equation (1.5) is fractional uniformly parabolic. In other words

$$(-1)^m \sum_{|q|=2m} a_q(x) y^q \geq c|y|^{2m},$$

for all $x \in \bar{G} = GU\partial G, y \in R^n$, where $|y|^2 = y_1^2 + ... + y_n^2$, $y^q = y_1^q ... y_n^{q_n}$ and c is a positive constant.

We suppose that $a_q \in C^{2m}(G)$, for all $|q| \leq 2m$, where $C^j(G)$ is the set of all continuous real-valued functions defined on G, which have continuous partial derivatives of order less than or equal to j.

The functions u_0, v and w are given. The unknown functions u and f are determined in a suitable space. There are many applications of the theory of fractional calculus and non local Cauchy problem (see [4], [5], [6]).

2. Representation of solutions

A pair of functions $\{u, f\}$ is said to be a strict solution of the nonlocal inverse problem (1.3)-(1.4) if

$$u \in S, and\ \frac{d^\alpha u(t)}{dt^\alpha} \in H$$

for each $t \in (0, T], f \in C(J)$ and the relations (1.3)-(1.4) are satisfied. In this case we say that the nonlocal inverse problem (1.3)-(1.4) is solvable.

Let us consider the following equation:

$$f = h + Pf, (2.1)$$

where

$$h(t) = \frac{1}{g_1(t)} \frac{d^\alpha w(t)}{dt^\alpha}$$

and P is a linear operator defined on $C(J)$ with values:

$$(Pf)(t) = -\frac{1}{g_1(t)}(Au(t), v)(2.2)$$

We shall prove now the equivalence between the inverse problem (1.1)-(1.3) and (2.1).

Theorem 2.1. Suppose that the conditions $(A_1 - A_4)$ are satisfied. Then the following assertions are valid :

(I) If the nonlocal inverse problem (1.1)-(1.3) is solvable, then equation (2.1) has a solution $f \in C(J)$,

(II) If equation (2.1) has a solution $f \in C(J)$ and the compatibility condition

$$(u(0), v) = w(0), (2.3)$$

holds, then the nonlocal inverse problem (1.1) - (1.3) is solvable.

Proof. Assume that the inverse problem (1.1) - (1.3) is solvable. Multiplying both sides of (1.1) by v scalarly in H, we obtain the relation

$$\frac{d^\alpha}{dt^\alpha}(u(t), v) = (Au(t), v) + f(t) g_1(t). (2.4)$$

From (2.2) and (2.4), one gets

$$f = Pf + \frac{1}{g_1} \frac{d^\alpha w}{dt^\alpha}$$

This means that f solves equation (2.1).

To prove assertion (II), we notice that by assumption, equation (2.1) has a solution $f \in C(J)$. When inserting this function in (1.1), the resulting problem (1.1), (1.2) can be treated as a direct nonlocal problem having a unique solution u. Using results from my papers [7],[11] this solution is given by

$$u(t) = \psi(t)\varphi u_0 + \psi(t)\sum_{k=1}^{p} c_k \int_0^{t_k} \psi^*(t_k - s) f(s) g(s) ds + \int_0^t \psi^*(t-s) f(s) g(s)\, ds (2.5)$$

Let us prove now that u satisfies the overdetermination condition (1.3). In this case u and f are known , consequently (2.1) will represent the following identity:

$$f(t)\, g_1(t) = \frac{d^\alpha w(t)}{dt^\alpha} - (Au(t), v).(2.6)$$

Subtracting equation (2.4) from (2.6), one gets

$$\frac{d^\alpha w(t)}{dt^\alpha} = \frac{d^\alpha}{dt^\alpha}\ (u(t), v)).$$

applying the fractional integral of order α and taking into account the compatibility condition (2.3), we find out that u satisfies the overdetermination condition (1.3) and that the pair $\{u, f\}$ is a strict solution of the inverse problem (1.1) - (1.3). This completes the proof of the theorem.

Theorem 2.2. Let the conditions $(A_1 - A_4)$ and the compatibility condition (2.3) hold, then there exists a unique strictly solution of the nonlocal inverse problem (1.1) - (1.3).

Proof. Using (2.1), (2.2) and (2.5), one obtains (formally) the following Volterra integral equation:

$$f(t) = h(t) - \frac{1}{g_1(t)}(\psi(t)\varphi u_o, A^* v) - \frac{1}{g_1(t)}\int_0^t K(t,s) f(s) ds, (2.7)$$

where:

$$K(t,s) = -\sum_{k=1}^{p}(\psi(t)\varphi\psi^*(t_k - s)g(s), A^* v) - (\psi^*(t-s)g(s), A^* v),$$

and A^* is the adjoint of the operator A.

Using similar techniques as in our papers [7] , [11-17], we can see that the functions $g_1^{-1}(t)$, $h(t)$ and the kernel $K(t,s)$ are continuous functions of t, s in J. Consequently the integral Volterra equation (2.7) has a unique continuous solution f on J.According to theorem (2.1) this confirms that the considered nonlocal inverse problem is solvable. To prove the uniqueness, we assume to the contrary there were two different solutions $\{u_1, f_1\}$ and $\{u_2, f_2\}$ of the considered problem. We claim that in this case $f_1 \neq f_2$ for all points of J. In fact if $f_1 = f_2$ on J, we would have $u_1 = u_2$. Since both pairs satisfy identity (2.4), the functions f_1 and f_2 give two different solutions of equation (2.7). But this contradicts the uniqeness of solutions of (2.7). This completes the proof of the theorem.

3. Inverse nonlocal mixed problem

Let $W^m(G)$ be the completion of the space $C^m(G)$ with respect to the norm

$$\|v\|_m^2 = \sum_{|q|\le m}\int_G |D^q v(x)|^2\, dx.(3.1)$$

Denote by $W_0^m(G)$ the completion of the space $C_0^m(G)$ with respect to the norm (3.1), (where $C_0^m(G)$) is the set of all functions $f \in C^m(G)$ with compact supports in G.

Let $L^2(G)$ be the space of all square integrable functions on G.

The inverse problem (1.4)-(1.6) can be written in the abstract form (1.1)-(1.3), where A is the operator defined by $Au = u_1$,

$$u_1(x,t) = -\sum_{|q|\le 2m} a_q(x) D^q u(x,t)(3.2)$$

The domain of definition of A is given by

$$S = W^{2m}(G) \cap W_0^m(G)$$

The considered set S is dense in $L_2(G)$ and the closed operator A defined by (3.2) generates a bounded semigroup [14-16]. The adjoint operator A^* is given by $A^* u = u_2$, where

$$u_2(x,t) = -\sum_{|q|\le 2m}(-1)^{|q|} D^q [a_q(x) u(x,t)].$$

Applying theorems (2.1) and (2.2) we can see that the considered nonlocal inverse mixed problem is uniquely solvable,see [18,19].

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