INTEGRATED SEMI GROUPS AND CAUCHY PROBLEM FOR SOME FRACTIONAL ABSTRACT DIFFERENTIAL EQUATIONS

Mahmoud M. El-Borai\textsuperscript{a} and Khairia El-Said El-Nadi\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, faculty of science, Alexandria university, Alexandria
\textsuperscript{b}Department of Mathematics, Faculty of science, Alexandria University, Alexandria

Email: m_m_elborai@yahoo.com; khairia_el_said@hotmail.com

Abstract: Let A be a linear closed operator defined on a dense set in a Banach space E to E. In this note it is supposed that A is the generator of a \( \alpha \) -times integrated semi group, where \( \alpha \) is a positive number. The abstract Cauchy problem of the fractional differential equation: \( \frac{d^{\beta}u(t)}{dt^{\beta}} = Au(t) + F(t) \), With the initial condition \( u_0 \in E \), is studied, where \( 0 < \beta \leq 1 \), and \( F \) is a given abstract function. An application is given.


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1. INTRODUCTION

The theory of integrated semi groups of operators on a Banach space were introduced by Arendt [1], [2]. Hieher [3] refined the theory by introducing \( \alpha \)-times integrated semi groups for positive numbers.

Integrated semi groups are a natural extension of semi group theory to deal with operators that have polynomially bounded resolvent in a half plane. It is well known that the Schrodinger operator:

\[
\int \left[ \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right] dt
\]

Generates a co-semi group on \( L^p(\mathbb{R}^n) \) if and only if \( p = 2 \), (see Hormander [4, 5 and 6]). But Hieber [3] showed that the Schrodinger operator generates an \( \alpha \)-times integrated semi group on \( L^p(\mathbb{R}^n) \) for \( \alpha > n \frac{1}{1+\frac{\beta}{p}} \), where \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space and \( L^p(\mathbb{R}^n) \) is the set of all measurable functions \( f \) such that the integral \( \int_{\mathbb{R}^n} |f(x)|^p dx \) exists.

Denote by \( E \) a Banach space. Let \( L(E) = L(E, E) \) be the space of bounded linear operators from \( E \) to \( E \). Let \( \{S(t), t \geq 0\} \) be a family of operators in \( L(E) \). Suppose that \( A \) is a linear closed operator defined on a dense set \( D(A) \) in \( E \). The family \( \{S(t), t \geq 0\} \) is called exponential bounded \( \alpha \)-times integrated semi group generated by \( A \) if the following conditions are satisfied:

\( C_1 \): \( \{S(t), t \geq 0\} \) is strongly continuous,
\( C_2 \): There exists \( M > 0 \) and a real number \( c \) such that

\[
\|S(t)\| \leq Me^{ct}, \quad t \geq 0,
\]
\( C_3 \): The interval \( (c, \infty) \) is contained in the resolvent set \( \rho(A) \) of \( A \) and
\( C_4 \):

\[
(\lambda I - A)^{-1} = \lambda^c \int_0^\infty e^{-\lambda t} S(t) dt,
\]

For all \( \lambda > c \), where \( I \) is the identity operator (see [7], [8], and [9]).

Under the conditions \( C_1, ..., C_4 \) we shall solve in section 2 the following Cauchy problem:

\[
\frac{d^\beta u(t)}{dt^\beta} = Au(t), \quad t > 0, \quad u(0) = u_0 \in D(A),
\]

Where \( 0 < \beta \leq 1 \).

Recall the definition of fractional derivatives, one of the definitions of the fractional derivative \( \frac{d^\beta f}{dt^\beta} \) is given by

\[
\frac{d^\beta f(t)}{dt^\beta} = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^n(s)}{(t-s)^{n-1+\beta}} ds + \sum_{k=0}^{n-1} f^k(0^+) \Phi_{k-\beta+1}(t),
\]

Where \( n - 1 \leq \beta < n, \Phi_k(t) = \frac{t^k}{\Gamma(k+1)}, t_{\beta}^c = \int H(t)dt, H(t) \) being the Heaviside function and \( \Gamma(c) \) is the gamma function (see [5], [6]).

2. REPRESENTATION OF THE SOLUTION

Let us solve the Cauchy problem (1.1), (1.2) under the conditions \( C_1, ..., C_4 \). It is suitable to rewrite the Cauchy problem (1.1), (1.2) in the form:

\[
u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t \frac{Au(s)}{(t-s)^{1-\beta}} ds.
\]

By a solution (2.1), we mean a function \( u \) such that:

http://www.lifesciencesite.com 793 lifesciencej@gmail.com
1. $u$ is continuous on $J = [0, T]$, $u$ is an element of $D(A)$, for each $t$ in $J$ and $Au(t)$ is continuous on $J$.

2. $u$ satisfies equation (2.1).

Theorem 2.1. If the conditions $C_1, ..., C_4$ are satisfied and $u_0$ is a given element in $D(A)$, then the unique solution of (2.1) is represented by:

$$u(t) = \frac{d^\alpha}{dt^\alpha} \int_0^t \xi_\beta(s)S(t^\beta s)u_0 ds,$$  (2.2)

where $\xi_\beta(s)$ is a probability density function defined on $(0, \infty)$, $0 < \beta \leq 1$ and $n - 1 < \alpha \leq n$.

Proof. Applying formally the Laplace transform

$$v(p) = \int_0^\infty e^{-pt}u(t)dt, \quad P > 0$$

To (2.1) yields

$$v(p) = p^{\alpha + \beta - 1} \int_0^\infty e^{-pt}S(t)u_0 dt.$$  (2.3)

Consider the one-sided stable probability density function $\rho_\beta(t)$, whose Laplace transform is given by

$$\int_0^\infty \rho_\beta(t)e^{-pt}dt = e^{-p^\beta},$$

Consequently

$$\int_0^\infty \rho_\beta(t)e^{-p^\beta}dt = e^{-p^\beta}.$$  (2.4)

Differentiating both sides of (2.4) with respect to $p$, we get

$$\int_0^\infty t\rho_\beta(t)e^{-p^\beta}dt = \beta \theta^{\beta - 1}p^{\beta - 1}e^{-p^\beta}.$$  (2.5)

From (2.3) and (2.5), one gets:

$$v(p) = p \int_0^\infty e^{-pt} \left[ \int_0^\infty e^{\beta \theta}S(t^\beta \theta)u_0 d\theta \right] dt.$$  (2.6)

Where

$$\xi_\beta(t) = \frac{1}{\beta} t^{-1} \rho_\beta \left( \frac{t}{\beta} \right).$$

Notice that $\xi_\alpha(t)$ is a probability density function defined on $[0, \infty]$. The Laplace transform of $\xi_\beta$ is given by

$$\int_0^\infty e^{-pt}\xi_\beta(t)dt = \sum_{j=0}^\infty \frac{(-p)^j}{\Gamma(1 + j\beta)}.$$

We have

$$S(t)u_0 = \frac{t^\alpha}{\Gamma(\alpha + 1)}u_0 + \int_0^t S(s)Au_0ds,$$  (2.7)

For all $t > 0$ and $u_0 \in D(A)$.

Since $u_0 \in D(A^\alpha)$, $n - 1 < \alpha \leq n$, one gets from (2.7)

$$\frac{d^\beta S(t)}{dt^\beta}u_0 = 0, \quad at \ t = 0, \quad k = 0, 1, ..., n - 1$$  (2.8)

Remembering the simple fact about the Laplace transform of the fractional derivatives and using (2.6), (2.8), one get

$$u(t) = \frac{d^\alpha}{dt^\alpha} \int_0^\infty \xi_\beta(s)S(t^\beta s)u_0 ds.$$  

Hence the required result.

Noticing that:

$$\frac{d^\beta}{dt^\beta} t^{\alpha\beta} = \Gamma(\alpha\beta + 1), \quad \int_0^\infty \theta^\alpha \xi_\beta(\theta)d\theta = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha\beta + 1)}$$

And using (2.2), (2.7), we get

$$u(t) = u_0 + \frac{d^\alpha}{dt^\alpha} \int_0^\infty \xi_\beta S(s)Au_0 dsd\theta.$$  (2.9)

3. NON HOMOGENEOUS EQUATIONS

Let us consider the nonhomogeneous equation

$$\frac{d^\beta u(t)}{dt^\beta} = Au(t) + f(t),$$  (3.1)

With the initial condition

$$u(0) = u_0 \in D(A^\alpha),$$  (3.2)

Where $f$ is a given abstract function defined on $J$ and with values in $E$.

**Theorem 3.1.**

If the conditions $C_1, ..., C_4$ are satisfied, $u_0 \in D(A^\alpha)$, $n = 1, 2, ..., \quad and \quad (t) \in D(A^\alpha)$ for every $t \in J$, $n - 1 < \alpha \leq n$, then the solution of the Caushy problem (3.1), (3.2) is given by:

$$u(t) = u^*(t) + F(t),$$

Where $u^*(t)$ is given by formula (2.2) or (2.9) and

$$F(t) = \int_0^t \int_0^r \frac{d^\beta}{d\theta^\beta} \theta \xi_\beta(\theta)\eta^{\beta - 1}S(\eta^\beta \theta) f(t - \eta)d\theta d\eta.$$  

Proof. If $v$ and $g$ are the Laplace transform of $u$ and $f$, respectively, then

$$v(p) = p^{\beta - 1}(p^\beta I - A)^{-1}u_0 + (p^\beta I - A)^{-1}g(p),$$

So

$$v(p) = p^{\beta - 1} \int_0^\infty e^{-p^\beta t}S(t)g(p)dt$$

$$+ p^{\beta - 1} \int_0^\infty e^{-p^\beta t}S(t)g(p)dt.$$  

Using techniques similar to the techniques which are used in theorem (1.1), we get
\[ L^{-1} \left[ \int_0^\infty e^{-pt}S(t)dt \right] = \frac{d^\alpha}{dt^\alpha} \int_0^\infty \theta^\alpha \mathcal{S}(t^\alpha \theta)xd\theta, \]

For every element \( x \in D(\mathcal{A}^{\alpha+1}) \), where \( L^{-1} \) is the inverse Laplace transform of \( L \).

Thus
\[ F(t) = \beta \int_0^t \frac{d^\alpha}{dt^\alpha} \theta^\alpha \mathcal{S}(t^\alpha \theta) \eta^{1-\alpha} \mathcal{S}(t^\alpha \eta)f(t - \eta)d\eta d\theta. \]

Hence required result, see [10-16].

4. APPLICATION

Let \( p > 1, 0 < \alpha \leq \frac{p-1}{p} \), \( E = L^p[0,1] \).

Define the operator \( \mathcal{A} \) by
\[ (\mathcal{A}g)(x) = -\frac{d^\alpha g(x)}{dx^\alpha} + \alpha \frac{g(x)}{x}, \]

Where \( D(\mathcal{A}) \) is the set of all absolutely continuous functions \( g \) defined on the interval \([0,1]\) with \( g(0) = 0 \) and \( \frac{dg(x)}{dx} \in L^p[0,1] \).

The considered operator \( \mathcal{A} \) generates the integrated semi group \( S(t) \), where
\[ [S(t)g](x) = \int_0^t x^\sigma(x-s)^{-\sigma}g(x-s)H(x-s)ds, \quad (4.1) \]

\( x \in [0,1], H \) is the Heaviside function, see [7], notice that \( S(t) \) is not a semi group.

Consider now the following Cauchy problem
\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = -\frac{\partial u(x,t)}{\partial x} + \alpha \frac{u(x,t)}{x}, \quad (4.2) \]
\[ u(x,0) = u_0(x), \quad (4.3) \]

Where \( u_0(x) \in D(A) \).

Using formula (2.2), (3.1), we can solve the Cauchy problem (4.2), (4.3) in \( D(A) \).

REFERENCES


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