Stability and Convergence of a Time-Fractional Blood flow equation in a Deformable medium

Suares Clovis Oukouomi Noutchie

Department of Mathematical Sciences, North-West University, Mafikeng, 2735, South Africa
23238917@nwu.ac.za

Abstract: The medium through which the blood moves varies in time and space. In this paper we capture the effects of the moving boundary conditions by including time-fractional derivatives into the evolution equation. The modified equation is solved numerically via the Crank-Nicholson scheme. The stability and convergence of the numerical scheme is highlighted.
[Oukouomi Noutchie, SC. **Stability and Convergence of a Time-Fractional Blood flow in a Deformable medium**. *Life Sci J* 2013;10(3):533-538] (ISSN:1097-8135). http://www.lifesciencesite.com. 77

Keywords: Nonlinear differential equations; blood flow; fractional order derivative; deformable medium.

1. Introduction

A cardiovascular system is roughly speaking a series of tubes (the blood vessels) filled with fluid (blood) and connected to a pump (the heart). Pressure generated in the heart propels blood through the system continuously. The blood picks up oxygen at the lungs and nutrients in the intestine and then delivers these substances to the body's cells while simultaneously removing cellular wastes and heat for excretion. In addition, the cardiovascular system plays an important role in cell-to-cell communication and in defending the body against foreign invaders.
Blood flow is in general modelled as a Newtonian or Non Newtonian fluid with rigid boundary conditions. However this is just an approximation since in reality Blood vessels are in motion in response to the pressure exerted by the fluid on the boundaries.

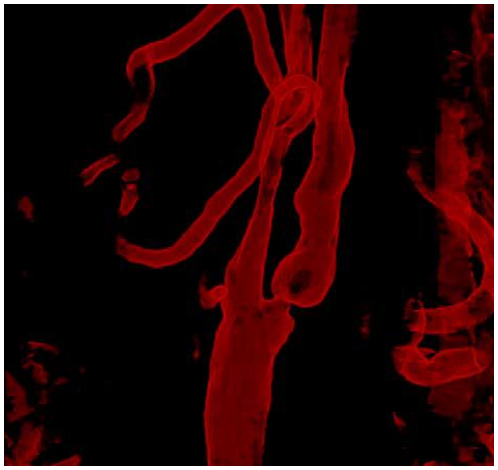

Fig. 1. Volume rendering of a carotid artery from a 3D MR dataset. [3]

In this paper we are going to analyse blood flow in the body taking into consideration the fact that the vessels are elastic and moving as a response of the pressure exerted on them by the blood pressure. In order capture the effects of the displacement [1,2,4,5] and contraction of the vessels, we introduce a time fractional order derivative model the effects of the artificial boundary conditions in the domain. This paper is divided as follows: in section 2 we introduce blood flow model and we consider its variational representation. In section 3 we analyse the model with artificial boundary conditions and we prove its well-posedness. In section a we introduce the fractional time derivative and approximate the solution. We perform a numerical simulation on our approximate solution. Chapter 5 is devoted to the conclusion.

2. Formulation of the problem

2. 1. Notations

In this subsection, we summarize some notations that will occur throughout the article.
Vectors and tensors are denoted by bold-face letters.
$\mathbf{x} = (x1, x2, x3)$: location of fluid particle.
$\mathbf{u}$: velocity field of the flow.
p: pressure.
∇p : the pressure gradient
$\mathbf{I}$: identity tensor.
$\mathbf{T}$: symmetric Cauchy stress tensor.
μ: dynamic viscosity.
ρ: fluid mass density.
R : the set of real numbers
$|\cdot|$: the absolute value of R and correspondingly, the norm of R3.
Ω denotes a bounded domain in R3.
Γ : denotes the boundary of Ω.
$\nabla\mathbf{u}$: the gradient of $\mathbf{u}$.
$\nabla\cdot\mathbf{u}$: the divergence of $\mathbf{u}$.
$\Delta\mathbf{u}$: the Laplacian of $\mathbf{u}$.
The spaces $C(\Omega)$ and $\mathbf{C}(\Omega)$, are defined as usual, the superscript indicating continuous derivatives to a certain order and the subscript zero indicating functions with compact support. Note that Poincar´e's inequality holds for φ *in* $\mathbf{X}$ The artificial sections

consist of the upstream section on the side of the heart and the downstream sections on the side of the peripheral vessels. Rather than giving serious thought to the artificial sections boundary conditions, in seeking a variational formulation, the test space is left free on these portions of the boundary. Accordingly we introduce

$\mathbf{V} \equiv \{\boldsymbol{\varphi}\ in\ \mathbf{H}1(\Omega) : \boldsymbol{\varphi}|\Gamma\text{wall} = 0, \nabla\cdot \boldsymbol{\varphi} = 0\}$.

as the test space. To prove an existence theorem for a Navier-Stokes problem, eithersteady or non-steady, it is convenient to construct the solution as a limit of Galerkin approximations in terms of the eigenfunctions of the corresponding steady Stokes problem.

We make use of HDM to solve the main evolution system.

3. Solvability

Following the steps involved in the method, we arrive at the following equations

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) - \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(a\sum_{n=0}^{\infty} p^n u_n\left(\sum_{n=0}^{\infty} p^n u_n\right)_x - b\sum_{n=0}^{\infty} p^n v_n\left(\sum_{n=0}^{\infty} p^n v_n\right)_x\right) + \left(\sum_{n=0}^{\infty} p^n u_n\right)_{xxx}$$

(4.1)

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = v(x,0) - \frac{p}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\left(\left(a\sum_{n=0}^{\infty} p^n u_n\left(\sum_{n=0}^{\infty} p^n u_n\right)_x + a\sum_{n=0}^{\infty} p^n\left(u_n\sum_{n=0}^{\infty} p^n v_n\right)_x + b\sum_{n=0}^{\infty} p^n u_n\left(\sum_{n=0}^{\infty} p^n v_n\right)_x\right)\right)$$

If we compare the terms of the same power of p we obtain the following integral equations. Note that when comparing this approach with the methodology of the Homotopy perturbation method, one will obtain in this step, a set of ordinary differential equations which something need to be also solving with care, because one will need to chose an appropriate initial guest. But with the current approach, the initial guess is straightforward obtained as Taylor series of the exact solution of the problem under investigation, this is one of the advantages, the approach has over the HPM. On the other hand when comparing this approach with the Variational Iteration Method, one will find out that, we do need the Lagrange multiplier here or the correctional function. Also this approach provides us with a convenient way to control the convergence of approximation series without adapting h, as in the case of [8] which is a fundamental qualitative difference in analysis between HDM and other methods.

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Therefore comparing the terms of the same power we obtain:

$$p^0: u_0(x,t) = u(x,0), u_0(x,0) = u(x,0)$$

(4.2)

$$p^0: u_0(x,t) = u(x,0), u_0(x,0) = u(x,0)$$

$$p^1: u_1(x,t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(6au_0(u_0)_x - 2bv_0(v_0)_x + a(u_0)_{xxx}\right)d\tau, u_1(x,0) = 0$$

$$p^1: v_1(x,t) = -\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\left(3bu_0(v_0)_x + b(u_0)_{xxx}\right)d\tau, v_1(x,0) = 0$$

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$$p^n: u_n(x,t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(6a\sum_{i=0}^{n-1} u_i(u_{n-i-1})_x - 2b\sum_{i=0}^{n-1} v_i(v_{n-i-1})_x + a\,(u_{n-1})_{xxx}\right)d\tau, u_n(x,t) = 0$$

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Integrating the above we obtain the following series solutions: (4.3)

$$u(x,0) = \frac{b}{a}\left(\tanh\left(\frac{1}{2}\sqrt{\frac{b}{a}}x\right)\right)^2, \quad v(x,0) = \frac{b}{\sqrt{a}}\left(\mathrm{sech}\left(\frac{1}{2}\sqrt{\frac{b}{a}}x\right)\right)^2.$$

Next we set

$$d = \frac{b}{a}; d_1 = \frac{b}{\sqrt{2a}} \text{ and } m = \frac{1}{2}\sqrt{\frac{b}{a}}.$$

Then

$$u_1(x,t) = \frac{(d+m)mt^{\alpha}}{\Gamma(1+\alpha)}\Big(-dd_1{}^2 + ad\,(d-m^2) + adm^2 sech(2mx)\Big)(\mathrm{cosech}\,(mx))^4 \tanh\,(mx)$$

$$v_1(x,t) = \frac{2dd_1mt^{\beta}}{\Gamma(1+\beta)}\left(d - m^2 + 2m^2 tanh(2mx)\right)(\mathrm{sech}\,(mx))^4 \tanh\,(mx)$$

$$u_2(x,t) = \frac{m}{\Gamma(0.8+\alpha)\Gamma(0.9+\beta)\Gamma(1+\alpha)\Gamma(\beta)}\Big(2^{1-2\alpha}m^2\sqrt{\pi}t^{\alpha}\Gamma(1 + \beta)\Big(-2b^2d_1^2t^{\beta}\Big(-12d + 44m^2 + (d-m)cosh(2mx) + 2m^2cosh(4mx)\Gamma(1+2\alpha) + ad_1{}^2\Big(-8\big(2bd_1^2(-3d+13m^2) + ad(18d^2 - 111dm^2 + 151m^4)\big) + \big(4bd_1^2(-9d+49m^2) + 3ad(36d^2 - 272dm^2 + 397m^4)\big)cosh(2mx) - 4m^2\big(4bd_1^2 - ad(m-2m^2)\big)cosh(4mx) + adm^4cosh(6xm)\Big)\Gamma(1+\alpha + \beta)\Big)(tanh(mx))^8\Big)\Big)$$

$$v_2(x,t) = \frac{dm}{\Gamma(1+\alpha)\Gamma(\alpha+\beta)\Gamma(0.6+\alpha)\Gamma(1+\alpha+\beta)}\Big(2^{1-2\beta}m^2\sqrt{\pi}t^{\beta}\Gamma(1 + \alpha)(sech(mx))^8\Big(bt^{\beta}\big(-27d^2 + 411dm^2 - 1208m^4 + 3(6d^2 - 124dm^2 + 397m^4)cosh(2mx) + 3m^2(9d - 40m^2)cosh(4mx) + m^4cosh(6mx)\big)\Gamma(1+\alpha+\beta) + 12t^{\alpha}\big(-bd_1^2 + ad(3d-5m^2) + adm^2cosh(2mx)\big)\Gamma(1 + 2\beta)(\sinh\,(mx))^2\Big)\Big)$$

And so on, using the package Mathematica, in the same manner one can obtain the rest of the components. But, here, few terms were computed and the asymptotic solution is given by:

(4.4)

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$
$$v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots$$

In this section, we make use of HDM to solve the main evolution system. Following the steps involved in the method, we arrive at the following equations

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) - \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(a\sum_{n=0}^{\infty} p^n u_n\left(\sum_{n=0}^{\infty} p^n u_n\right)_x - b\sum_{n=0}^{\infty} p^n v_n\left(\sum_{n=0}^{\infty} p^n v_n\right)_x\right) + \left(\sum_{n=0}^{\infty} p^n u_n\right)_{xxx}$$

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5. **Conclusion**

We introduced a nonlinear fractional order system describing blood flow in Areas that continuously move under the influence of pressure from the heart pump. We used functional analytic tools to describe the problem with accuracy, we performed some numerical simulations.

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7/14/2013