# Functional order derivatives with applications to heat convection equations

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**Abstract:** The convection heat flow equation is expanded in this paper via the concept of the variational order derivative. The Crank-Nicholson technique will be used to solve the evolution problem. Within the discredited problem domain, the variable internal properties, boundaries, and stresses of the system are approximated. We study stability and convergence analysis of the numerical method. In particular we consider computational examples and discuss their simulations. [Oukouomi Noutchie, SC. **Functional order derivatives with applications to heat convection equations**. *Life Sci J* 2013;10(3):472-478] (ISSN:1097-8135). http://www.lifesciencesite.com. 69

**Keywords:** Heat flow equation; variational order derivative; Crank-Nicholson scheme; stability; convection.

## 1. Introduction

In recent years functional derivatives has been used to model physical and engineering processes. Areas of considerable interest include electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion, chemical physics, optics, engineering, acoustics, material science and signal processing. Space - time fractional differential equations obtained by replacing the first order time derivative and/or second-order space derivative in the standard diffusion equation by a generalized derivative of fractional order, respectively, were successfully used for modelling relevant physical processes [1-8]. These fractional diffusion equations arise quite naturally in continuous-time random walks. In this paper we extend the analysis by inserting fractional convection and fractional heat loss into the heat equations. Fractional derivatives may be introduced by different definitions. We consider the Caputo time fractional derivative for the diffusion convection equation with lateral heat loss

(1.1)

$$\begin{cases} D_t^{\alpha(x,t)}\mathrm{u}(x,t) = D_{xx}\mathrm{u}(x,t) + D_x\mathrm{u}(x,t) + u(x,t) \\ \quad 0 < \alpha(x,t) \le 1 \end{cases}$$

where the variational order differential operator is given by (2)

$$D_t^{\beta(x)}(f(x)) = \frac{1}{\Gamma(1-\beta(x))}\int_0^x (x-t)^{-\beta(t)}\frac{df(t)}{dt}dt.$$

Here $\beta(x)$ is a continuous function in (0, 1].

The paper is structure as follows: in section 2 we introduce Crank-Nicholson numerical scheme, in section 3 we study its convergence and in section 4 we study its stability with respect to our evolution equation. In section 5 we fully apply the scheme to the functional heat equation with mass loss and convection, section 6 is devoted to the conclusion.

## 2-Numerical Solution

The occurrence of integral and differential term in the functional heat equation with convection and lateral heat loss makes it difficult to solve the problem explicitely. The existence and uniqueness of solutions is known thanks to functional analytic methods. To understand the behavior of the solution, we are going to make use of a numerical approach that converges. It yields approximate solutions to the governing equation through the discretization of space and time. Within the discredited problem domain, the variable internal properties, boundaries, and stresses of the system are approximated. Deterministic, distributed-parameter, numerical models can relax the rigid idealised conditions of analytical models or lumped-parameter models, and they can therefore be more realistic and flexible for simulating fields conditions. The finite difference schemes for constant-order time or space fractional diffusion equations have been widely studied [9-18]. To establish the numerical schemes for the above equation, we let $x_l = lh$ $0 \le l \le M, Mh = L, t_k = k\tau, 0 \le k \le N, N\tau = T,$

$\mathrm{h}$ is the step and $\tau$ is the time size, $M$ and $N$ are grid points.

**2.1 Crank–Nicholson scheme** [2]

We introduce the Crank–Nicholson scheme [11] as follows. Firstly, the discretization of first and second order space derivative is stated as:

$$\frac{\partial u}{\partial x}=\frac{1}{2}\left(\left(\frac{u(x_{l+1},t_{k+1})-u(x_{l-1},t_{k+1})}{2(h)}\right)+\left(\frac{u(x_{l+1},t_{k})-u(x_{l-1},t_{k})}{2(h)}\right)\right)+O(h)$$

(2.1)

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{2}\left(\left(\frac{u(x_{l+1},t_{k+1})-2u(x_l,t_{k+1})+u(x_{l-1},t_{k+1})}{(h)^2}\right)+\left(\frac{u(x_{l+1},t_{k})-2u(x_l,t_{k})+u(x_{l-1},t_{k})}{(h)^2}\right)\right)+O(h^2)$$

$$u=\frac{1}{2}\left(u(x_l,t_{k+1})+u(x_l,t_k)\right)$$

The Crank–Nicholson scheme for the time fractional diffusion with convection and lateral heat loss model can be stated as follows:

(2.2)

$$\frac{\partial^{\alpha_l^{k+1}}u(x_l,t_{k+1})}{\partial t^{\alpha_l^{k+1}}}=\frac{\tau^{-\alpha_l^{k+1}}}{\Gamma\left(2-\alpha_l^{k+1}\right)}\Bigg(u(x_l,t_{k+1})-u(x_l,t_k)+\sum_{j=1}^{k}\left[u(x_l,t_{k+1-j})-u(x_l,t_{k-j})\right]\left[(j+1)^{1-\alpha_l^{k+1}}-(j)^{1-\alpha_l^{k+1}}\right]+\sum_{j=1}^{k}\left[u(x_{l+1},t_{k+1-j})-u(x_{l-1},t_{k-j})\right]\left[(j+1)^{2+k}-(j)^{1+2k}\right]\Bigg).$$

Now replacing equations (2.1) and (2.2 in (1.1) we obtain the following: (2.3)

$$\left[\frac{S\tau^{-\alpha_l^{k+1}}}{\Gamma\left(2-\alpha_l^{k+1}\right)}\left(u(x_l,t_{k+1})-u(x_l,t_k)+\sum_{j=1}^{k}\left[u(x_l,t_{k+1-j})-u(x_l,t_{k-j})\right]\left[(j+1)^{1-\alpha_l^{k+1}}-(j)^{1-\alpha_l^{k+1}}\right]\right)\right]$$
$$=T\left[\frac{1}{2}\left(\left(\frac{u(x_{l+1},t_{k+1})-2u(x_l,t_{k+1})+u(x_{l-1},t_{k+1})}{(h)^2}\right)+\left(\frac{u(x_{l+1},t_{k})-2u(x_l,t_{k})+u(x_{l-1},t_{k})}{(h)^2}\right)\right)\right]$$
$$+\frac{1}{r_l}\left[\frac{1}{2}\left(\left(\frac{u(x_{l+1},t_{k+1})-u(x_{l-1},t_{k+1})}{2(h)}\right)+\left(\frac{u(x_{l+1},t_{k})-u(x_{l-1},t_{k})}{2(h)}\right)\right)\right]$$
$$+\frac{3}{r_l}\left[\frac{1}{2}\left(\left(\frac{u(x_l,t_{k+1})-u(x_l,t_{k+1})}{2(h)}\right)+\left(\frac{u(x_l,t_{k})-u(x_l,t_{k})}{2(h)}\right)\right)\right]$$
$$+\frac{1}{r_l}\left[\frac{1}{2}\left(\left(\frac{u(x_l,t_{k+1})-u(x_l,t_{k})}{2(h)}\right)+\left(\frac{u(x_l,t_{k})-u(x_l,t_{k-1})}{2(h)}\right)\right)\right]$$

Next we set

$b_j^{l,k+1}=(j+1)^{1-\alpha_l^{k+1}}-(j)^{1-\alpha_l^{k+1}}$; $T_l^{k+1}=\frac{\Gamma\left(2-\alpha_l^{k+1}\right)\tau^{\alpha_l^{k+1}}}{Sh^2}T$; $G_l^{k+1}=\frac{\Gamma\left(2-\alpha_l^{k+1}\right)\tau^{\alpha_l^{k+1}}}{Sh}$

and $\lambda_j^{l,k+1}=b_{j-1}^{l,k+1}-b_j^{l,k+1}$

Equation (2.3) becomes:

$$u_l^{k+1}\left(1+2T_l^{k+1}\right)=u_{l+1}^{k+1}\left(T_l^{k+1}+\frac{G_l^{k+1}}{r_l}\right)+u_{l-1}^{k+1}\left(T_l^{k+1}-\frac{G_l^{k+1}}{r_l}\right)+u\left(T_l^{k+1}-\frac{G_l^{k+1}}{r_l}\right)+u_l^{k+1}\left(1+2T_l^{k+1}\right)+\sum_{j=1}^{k}\left(u_l^{k+1-j}-u\right)\lambda_j^{l,k+1}G_l^{k+1}$$

**3- Stability analysis of the Crank–Nicholson scheme [2]**

In this section, we follow [2] where we will analyze the stability conditions of the Crank–Nicholson scheme for the generalized advection dispersion equation.

Let $\zeta_l^k = u_l^k - \Theta_l^k$, here $\Theta_l^k$ is the approximate solution at the point $(x_l, t_k), (k = 1,2 \ldots. . N, l = 1,2, \ldots \ldots. M - 1)$

and in addition $\zeta^k = [\zeta_1^k, \zeta_2^k, \quad , \zeta_{M-1}^k]^T$ and the function $\zeta^k(x)$ is chosen to be :

(3.1)

$$\zeta^k(x) = \begin{cases} \zeta_l^k \ if \ x_l - \frac{h}{2} < x \leq x_l + \frac{h}{2}, l = 1,2, \ldots. M-1 \\ 0 \ if \ L - \frac{h}{2} < x \leq L \end{cases}$$

Then, the function $\zeta^k(x)$ can be expressed in Fourier series as follows:

$$\zeta^k(x) = \sum_{m=-\infty}^{m=\infty} \delta_m(m) exp[2i\pi mk/L]$$

(3.2)

$$\delta_k(x) = \frac{1}{L}\int_0^L \rho^k(x) exp[2i\pi mx/L]dx$$

. Under this situation, the error committed while approximating the solution of the generalized advection dispersion equation with Crank–Nicholson scheme can be presented as follows:

$$\zeta_l^{k+1}(1 + 2T_l^{k+1}) = \zeta_{l+1}^{k+1}(T_l^{k+1}) + \zeta_{l-1}^{k+1}(T_l^{k+1}) + \zeta_{l+1}^{k}(T_l^{k+1}) + \zeta_l^k(1 + 2T_l^{k+1}) + \sum_{j=1}^{k}(\zeta_l^{k+1-j} - \zeta_l^{k-j})\lambda_j^{l,k+1} G_l^{k+1}$$

If we assume that: $\zeta_l^k$ in equation (2.3) can be put in the delta-exponential form as follows:

$$\zeta_l^k = \delta_k exp[i\varphi lk] \quad (3.3)$$

where $\varphi$ is a real spatial wave number, new replacing the above equation (2.2) in (2.3) we obtain:

$$\left[1 + 4T_l^1 sin^2\left(\frac{\varphi h}{2}\right)\right]\delta_1 = \left[1 - 4T_l^1 sin^2\left(\frac{\varphi h}{2}\right)\right]\delta_0 \ for \ k = 0$$

(3.4)

$$\left[1 + 4T_l^{1+k} sin^2\left(\frac{\varphi h}{2}\right)\right]\delta_{k+1} = \left[1 - 4T_l^{1+k} sin^2\left(\frac{\varphi h}{2}\right) - \lambda_1^{l,k+1}\right]\delta_k + \sum_{j=0}^{k-1}\lambda_{j+1}^{l,k+1}\delta_{k-j} + \lambda_k^{l,k+1}\delta_0 \ for \ k = 1,2, \ldots, N-1$$

Equation (4.6) can be writing in the following form:

$$\delta_1 = \frac{\left[1 - 4T_l^1 sin^2\left(\frac{\varphi h}{2}\right)\right]\delta_0}{\left[1 + 4T_l^1 sin^2\left(\frac{\varphi h}{2}\right)\right]}$$

(3.5)

$$\delta_{k+1} = \frac{\left[1 - 4T_l^{1+k} sin^2\left(\frac{\varphi h}{2}\right) - e_1^{l,k+1}\right]\delta_k + \sum_{j=0}^{k-1}\lambda_{j+1}^{l,k+1}\delta_{k-j} + \lambda_k^{l,k+1}\delta_0}{\left[1 + 4T_l^{1+k} sin^2\left(\frac{\varphi h}{2}\right)\right]}$$

Our next concern here is to show that for all $k = 1,2, \ldots \ldots. N - 1$ the solution of equation (4.7) satisfies the following condition:

$|\delta_k| < |\delta_0|$,

To achieve this we make use of the recurrence technique on the natural number $k$

For $k = 1$ and remembering that $d_l^{k+1}$, $b_l^{k+1}$ are positive for all $l = 1,2 \ldots \ldots M - 1$, then we obtain:

(3.6)

$$\frac{|\delta_1|}{|\delta_0|} = \left|\frac{\left[1 - 4T_l^1 sin^2\left(\frac{\varphi h}{2}\right)\right]}{\left[1 + 4T_l^1 sin^2\left(\frac{\varphi h}{2}\right)\right]}\right| < 1$$

Assuming that for $m = 2,3 \ldots \ldots k$ the property is verified. Then (3.7)

$$|\delta_{k+1}| = \left|\frac{\left[1 - 4T_l^{1+k} sin^2\left(\frac{\varphi h}{2}\right) - e_1^{l,k+1}\right]\delta_k + \sum_{j=0}^{k-1}\lambda_{j+1}^{l,k+1}\delta_{k-j} + \lambda_k^{l,k+1}\delta_0}{\left[1 + 4T_l^{1+k} sin^2\left(\frac{\varphi h}{2}\right)\right]}\right|$$

Making use of the triangular inequality we obtain: (3.8)

$$|\delta_{k+1}| \le \frac{\left|1-4T_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right)-e_1^{l,k+1}\right||\delta_k| + \left|\sum_{j=0}^{k-1}\lambda_{j+1}^{l,k+1}\delta_{k-j}\right| + \left|e_k^{l,k+1}\delta_0\right|}{\left|1+4T_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right)\right|}$$

Using the recurrence hypothesis we have:

$$|\delta_{k+1}| \le \left(\frac{\left|1-4T_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right)\right| + \left|\sum_{j=0}^{k-1}\lambda_{j+1}^{l,k+1}\right|}{\left|1+4T_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right)\right|}\right)|\delta_0|$$

(3.9)

$$|\delta_{k+1}| \le \left(\frac{\left|1+4T_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right)\right|}{\left|1+4T_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right)\right|}\right)|\delta_0|$$

$|\delta_{k+1}| < |\delta_0|$.

**4. Convergence analysis of the Crank–Nicholson scheme [2]**

If we assume that,

$u(r_l,t_k)(l = 1,2,\dots\ M, k = 1,2\dots\dots N-1)$ is the exact solution of our problem at the point $(r_l,t_k)$, by letting $\Omega_l^k = u(r_l,t_k) - u_l^k$ and $\Omega^k = \left(0,\Omega_1^k,\Omega_2^k,\dots.\Omega_{M-1}^k\right)$ substituting this in equation (3.7), we obtain:

$$\zeta_l^1(1+2T_l^1) - \zeta_{l+1}^1(T_l^1) - \zeta_{l-1}^1(T_l^1) = R_l^1 \; for\; k=0 \quad (4.1)$$

It follows that (5.2)

$$R_l^{1+k} = u(r_l,t_{k+1}) - \sum_{j=0}^{k-1} u(r_l,t_{k-j})\lambda_{j+1}^{l,k+1} + b_1^{l,k+1}u(r_l,t_0) - T_l^{1+k}[u(r_{l+1},t_{k+1}) - 2u(r_l,t_{k+1}) + u(r_{l-1},t_{k+1})]$$

From equation (3.1) and (3.4) we have

$$\frac{\partial^2 u(r_l,t_{k+1})}{\partial r^2} + h^2V_1 = \frac{1}{2}\left(\frac{\left(u(r_{l+1},t_{k+1}) - 2u(r_l,t_{k+1}) + u(r_{l-1},t_{k+1})\right)}{h^2} + \frac{\left(u(r_{l+1},t_k) - 2u(r_l,t_k) + u(r_{l-1},t_k)\right)}{h^2}\right)$$

,

and

$$\frac{\partial^{\alpha_l^{k+1}} u(r_l,t_{k+1})}{\partial t^{\alpha_l^{k+1}}} + \tau V_2 = \frac{\tau^{-\alpha_l^{k+1}}}{\Gamma(2-\alpha_l^{k+1})}\left(u(r_l,t_{k+1}) - u(r_l,t_k) + \sum_{j=1}^{k}[u(r_l,t_{k+1-j}) - u(r_l,t_{k-j})]\lambda_j^{l,k}\right).$$

From the above we have that: (5.3)

$$R_l^{k+1} \le K\left(\tau^{1+\alpha_l^{k+1}} + h^2\tau^{\alpha_l^k}\right)$$

where $K_1$, $K_2$, and $K$ are constants. Taking into account Caputo type fractional derivative, the detailed error analysis on the above schemes can refer to the work by Diethelm *et al.* [25] and further work by Li and Tao [26].

**Lemma 1:**

$$\|\Omega^{k+1}\|_\infty \le K\left(\tau^{1+\alpha_l^{k+1}} + h^2\tau^{\alpha_l^k}\right)\left(\Omega_j^{l,k+1}\right)^{-1}$$

is true for $(\ k = 0,1,2\dots\dots N-1\ )$ where $\|w^k\|_\infty = \max_{1<l<M-1}(\Omega^k)$, K is a constant. In addition,

$$\alpha^{k+1} = \begin{cases} \min\limits_{1\le l\le M-1} \alpha_l^{k+1}, if\ \tau < 1 \\ \max\limits_{1\le l\le M-1} \alpha_l^{k+1}, if\ \tau > 1 \end{cases}$$

This can be achieved via the recurrence technique on the natural number k.

When $k=0$, we have the following: (5.4)

$$|\Omega_l^1| \le (c_l^1 + b_l^1)|w_{l+1}^1| + (c_l^1 - 2b_l^1)|w_{l-1}^1| = |F_l^1| \le V\left(\tau^{1+\alpha_l^{k+1}} + h^2\tau^{\alpha_l^k}\right)\left(\lambda_j^{l,k+1}\right)^{-1}$$

Now suppose that

$$\|\Omega^{i+1}\|_\infty \le K\left(\tau^{1+\alpha_l^{i+1}} + h^2\tau^{\alpha_l^i}\right)\left(\lambda_j^{l,i+1}\right)^{-1}, i = 1,\dots.N-2$$

. Then

$$\left|w_l^{1+k}\right| \le \left|b_l^{k+1}\left[\left(w_{l+1}^{k+1} - 2w_l^{k+1} + w_{l-1}^{k+1}\right)\right] + c_l^{k+1}\left[\left(w_{l+1}^{k+1} - w_{l-1}^{k+1}\right)\right] + d_l^{k+1}w_l^{k+1}\right|$$

$$\le \left(b_l^{k+1} + c_l^{k+1}\right)\left|w_{l+1}^{k+1}\right| + \left(b_l^{k+1} - c_l^{k+1}\right)\left|w_{l-1}^{k+1}\right| + \left(d_l^{k+1} - 2b_l^{k+1}\right)\left|w_l^{k+1}\right|$$

$$= \left| R_l^{k+1} + \sum_{i=1}^{k}\left(\Omega_l^{k-i}\right)\lambda_j^{l,k+1}\right|$$

$$\le \left| R_l^{k+1}\right| + \sum_{i=1}^{k}\left|\Omega_l^{k-i}\right|\lambda_j^{l,k+1}$$

$$\le K\left(\tau^{1+\alpha_l^{k+1}} + h^2\tau^{\alpha_l^k}\right) + \sum_{i=1}^{k}\left\|\Omega_l^{k-i}\right\|_\infty \lambda_j^{l,k+1}$$

$$\le K\left(\tau^{1+\alpha_l^{k+1}} + h^2\tau^{\alpha_l^k}\right)\left(\lambda_j^{l,k+1} + \lambda_0^{l,k+1} - \lambda_j^{l,k+1}\right)\left(\lambda_j^{l,k+1}\right)^{-1}$$

$$\le V\left(\tau^{1+\alpha_l^{k+1}} + h^2\tau^{\alpha_l^k}\right)\left(\lambda_0^{l,k+1}\right)\left(\lambda_j^{l,k+1}\right)^{-1}$$

$$\left|\Omega_l^{1+k}\right| \le V\left(\tau^{1+\alpha_l^{k+1}} + h^2\tau^{\alpha_l^k}\right)\left(\lambda_j^{l,k+1}\right)^{-1}$$

which completes the proof.

**Theorem 1:** *The Crank-Nicholson scheme is convergent, and there exists a positive constant V such that:*

$$\left|u_l^k - u(x_l, t_k)\right| \le K(\tau + h^2), l = 1,2 \ldots, M-1, k = 1,2, \ldots N$$

An interested can find the solvability of the Crank-Nicholson scheme in the work done by [24]. Therefore the details of the proof will not be presented in this paper.

**5-Numerical results [3]**

We consider the fractional heat equation with convection and lateral heat loss where we specify some values for the coefficients. Note that the initial condition can be any continuous function and not necessarily differentiable. We can relax the assumption on continuity to accommodate a larger class of function without altering the intrinsic nature of the problem at hand.

**Example 1:**

$$\begin{cases} D_t^{\alpha(x,t)}u(x,t) = D_{xx}u(x,t) + D_x u(x,t) + u(x,t) \\ \quad 0 < \alpha(x,t) \le 1 \\ \quad \alpha(x,t) = 1 - \sin(x\,t) \\ \quad 0 < x < 1. \end{cases}$$

The numerical simulations of the approximate solution is displayed below

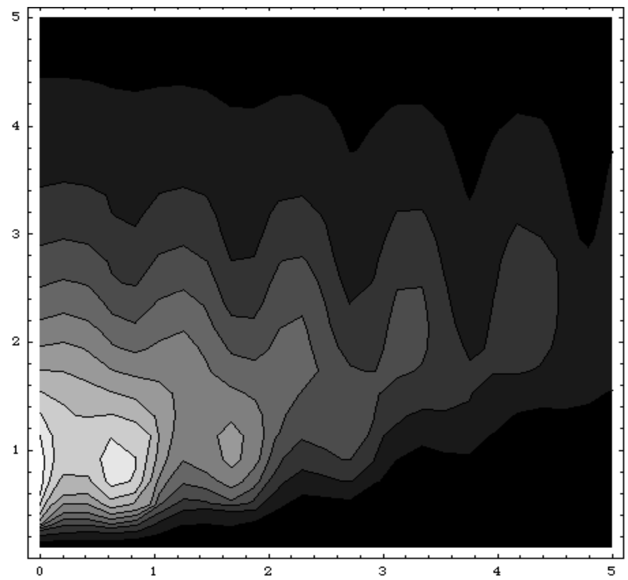


***Figure 1***

The next figure shows a 3dimentional visualization of example 1 with a parameterized variable.

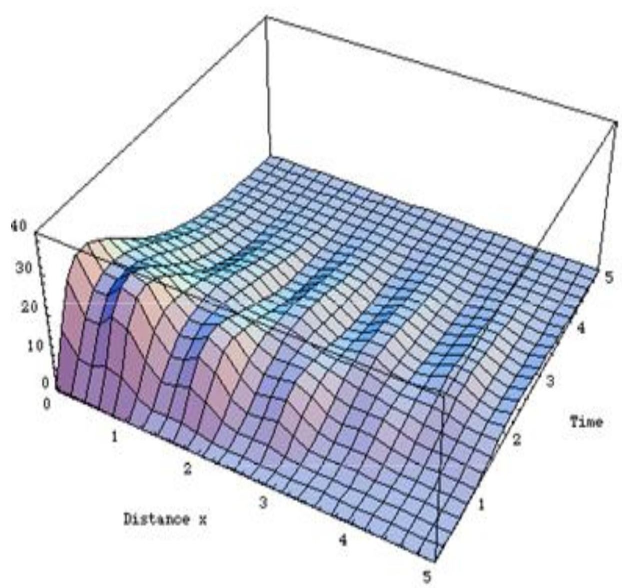


***Figure 2***

**Example 2:**

$$\begin{cases} D_t^{\alpha(x,t)}u(x,t) = D_{xx}u(x,t) + D_x u(x,t) + u(x,t) \\ 0 < \alpha(x,t) \le 1 \\ \alpha(x,t) = 1 - \cos(x^3 t) \\ 0 < x < 1. \end{cases}$$

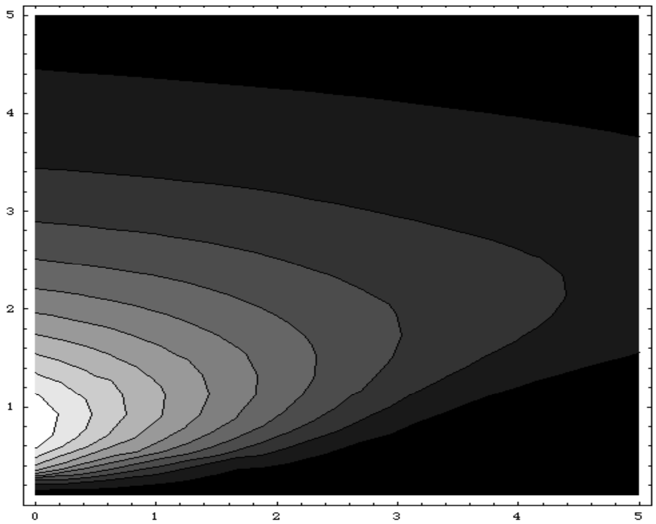


***Figure 3***

**Example 3:**

$$\begin{cases} D_t^{\alpha(x,t)}u(x,t) = D_{xx}u(x,t) + D_x u(x,t) + u(x,t) \\ 0 < \alpha(x,t) \le 1 \\ \alpha(x,t) = 1 - \cos(x\,t + 1) \\ 0 < x < 1. \end{cases}$$

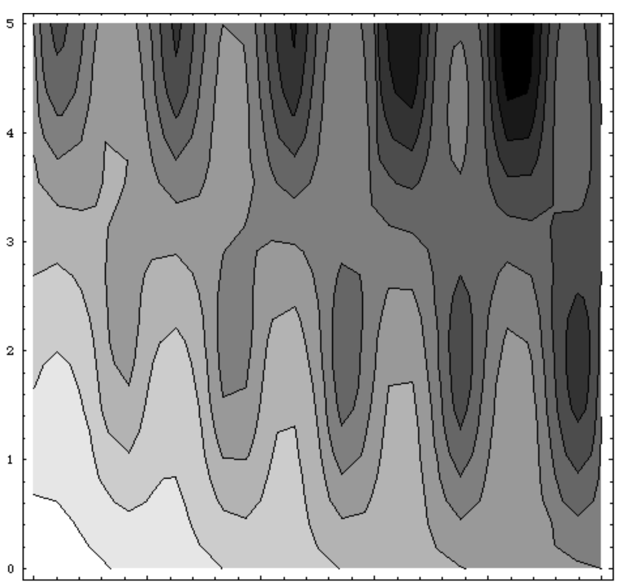


***Figure 4***

## 6-Conclusion

In this paper, we used numerical methods to study a functional heat equation with convection and heat loss. We showed that the Crank-Nicholson scheme used to analyze the evolution problem was stable and convergent. We derived analytic solutions and perform some numerical simulations for some specific expressions of the functional order derivative.

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