Analysis of Coupled Korteweg-de-Vries Equations with fractional derivatives

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Abstract: A Coupled Korteweg-de Vries Equations with fractional derivatives is examined via the Homotopy decomposition method (HDM). We show that this method is more reliable and efficient than earlier numerical techniques. In particular we show the dependence of solutions properties towards the fractional order derivative values.

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1. Introduction

In this paper we study an initial-boundary value problem for a coupled system of two equations of Korteweg–de Vries (KdV)-type in a bounded domain . The original coupled system of KdV equations which we consider here is an extension of the model used in [9] to describe the strong interaction of weakly nonlinear long wave modes propagating in a density stratified fluid. This is described by the following equations:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + au(x,t)u_x(x,t) + bv(x,t)v_x(x,t) + au_{xxx}(x,t) = 0,$$

$$\frac{\partial^\beta v(x,t)}{\partial t^\beta} + bu(x,t)v_x(x,t) + au_x(x,t)v_x(x,t) + bv_{xxx}(x,t) = 0.$$

Here $0 < \alpha \leq 1$ and $0 < \beta \leq 1$. Furthermore this equation is subject to the initial conditions

$$u(x,0) = \frac{b}{a}\left(\tanh\left(\frac{1}{2}\sqrt{\frac{b}{a}}x\right)\right)^2, \quad v(x,0) = \frac{b}{\sqrt{a}}\left(\operatorname{sech}\left(\frac{1}{2}\sqrt{\frac{b}{a}}x\right)\right)^2.$$

In this paper, we make use of the Homotopy Decomposition Method (HDM) [1-6] in order to derive analytical approximate solutions to this coupled system of nonlinear time fractional KDV equations.

The structured as this paper is as follows: In section 2 we briefly introduce fractional order derivatives and theirs properties. Then in section 3 we present the basic ideal of the homotopy decomposition method for solving high order nonlinear fractional partial differential equations. Section 4 is devoted to the application of the HDM for solving the evolution system of fractional nonlinear differential equations and discuss numerical results.

2. Fractional derivative order [9]

2.1. Preliminaries

In the literature, one can find several definitions of fractional derivatives. The most common used are the Riemann–Liouville and the Caputo derivatives. For Caputo we have (2.1)

$${}_0^C D_x^\alpha\big(f(x)\big) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}\frac{d^n f(t)}{dt^n}dt$$

For the case of Riemann-Liouville we have the following definition (2.2)

$$D_x^\alpha\big(f(x)\big) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}f(t)dt$$

Each fractional derivative presents some advantages and disadvantages. The Riemann–Liouville derivative of a constant is not zero while Caputo's derivative of a constant is zero but demands higher conditions of regularity for differentiability: to compute the fractional derivative of a function in the Caputo sense, we must first calculate its derivative. Caputo derivatives are defined only for differentiable functions while functions that have no first order derivative might have fractional derivatives of all orders less than one in the Riemann–Liouville sense [1-6]. Recently, Guy Jumarie [7] proposed a simple alternative definition to the Riemann–Liouville derivative. (2.3)

$$D_x^\alpha\big(f(x)\big) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}\{f(t)-f(0)\}dt$$

His modified Riemann–Liouville derivative seems to have advantages of both the standard Riemann–

Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (non-differentiable) functions and the fractional derivative of a constant is equal to zero. However from it definition we do not actually give a fractional derivative of a function says $f(x)$ but the fractional derivative of $f(x)-f(0)$ and always leads to fractional derivative that is not defined for some function for which $f(0)$ does not exists [3]

We can point out that Caputo and Riemann-Liouville may have their disadvantages but they still remain the best definition of the fractional derivative. Every definition must be used accordingly [4].

2.2 Properties and definitions

Definition 1 A real function $f(x), x > 0$, is said to be in the space C_μ, $\mu\in\mathbb{R}$ if there exists a real number $p>\mu$, such that $f(x) = x^p h(x)$, where $h(x) \in C[0,\infty)$, and it is said to be in space C_μ^m if $f^{(m)} \in C_\mu$, $m\in\mathbb{N}$

Definition 2 The Riemann-Liouville fractional integral operator of order $\alpha\geq0$, of a function $f \in C_\mu$, $\mu\geq-1$, is defined as (2.4)

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt,\ \ \alpha>0, x>0$$

$$J^0 f(x) = f(x).$$

Properties of the operator can be found in [14][15] we mention only the following:

For

$f \in C_\mu$, $\mu \geq -1$, α, $\beta \geq 0$ and $\gamma > -1$:

(2.5)

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$$

Lemma 1 If

$m-1 < \alpha \leq m, m \in \mathbb{N}$ and $f \in C_\mu^m, \mu \geq -1,$

then

$D^\alpha J^\alpha f(x) = f(x)$ and,

$$J^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!},\ \ x>0$$

(2.6)

Definition 4: Partial Derivatives of Fractional order

Assume now that $f(x)$ is a function of n variables x_i $i = 1, \dots\dots, n$ also of class C on $D\in \mathbb{R}_n$. As an extension of definition 3 we define partial derivative of order α for f respect to x_i the function (2.7)

$$a\partial_{\underline{x}}^\alpha f = \frac{1}{\Gamma(m-\alpha)}\int_a^{x_i} (x_i - t)^{m-\alpha-1}\, \partial_{x_i}^m f(x_j)|_{x_j=t} dt$$

If it exists, where $\partial_{x_i}^m$ is the usual partial derivative of integer order m.

3. Basic idea of the HDM [9]

To illustrate the basic idea of this method we consider a general nonlinear non-homogeneous fractional partial differential equation with initial conditions of the following form

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = L(U(x,t)) + N(U(x,t)) + f(x,t),\ \ \alpha>0 \quad (3.1)$$

Subject to the initial condition

$$D_0^{\alpha-k} U(x,0) = f_k(x),\quad (k=0,\dots\dots n-1), D_0^{\alpha-n}U(x,0) = 0 \ and\ n=[\alpha]$$

$$D_0^{k} U(x,0) = g_k(x),\quad (k=0,\dots\dots n-1), D_0^{n}U(x,0) = 0 \ and\ n=[\alpha]$$

Where, $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the Caputo or Riemann-Liouville fraction derivative operator, f is a known function, N is the general nonlinear fractional differential operator and L represents a linear fractional differential operator. The method first step here is to transform the fractional partial differential equation to the fractional partial integral equation by applying the inverse operator $\frac{\partial^\alpha}{\partial t^\alpha}$ of on both sides of equation (3.1) to obtain: In the case of Riemann-Liouville fractional derivative (3.2)

$$U(x,t) = \sum_{j=1}^{n-1} \frac{f_j(x)}{\Gamma(\alpha-j+1)} t^{\alpha-j} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[L(U(x,\tau)) + N(U(x,\tau)) + f(x,\tau)]d\tau$$

In the case of Caputo fractional derivative

$$U(x,t) = \sum_{j=1}^{n-1} \frac{g_j(x)}{\Gamma(\alpha-j+1)} t^{j} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[L(U(x,\tau)) + N(U(x,\tau)) + f(x,\tau)]d\tau$$

Or in general by putting

$$\sum_{j=1}^{n-1} \frac{f_j(x)}{\Gamma(\alpha-j+1)} t^{\alpha-j} = f(x,t)\ \ or\ \ f(x,t) = \sum_{j=1}^{n-1} \frac{g_j(x)}{\Gamma(\alpha-j+1)} t^{j}$$

We obtain: (3.3)

$$U(x,t) = T(x,t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[L(U(x,\tau)) + N(U(x,\tau)) + f(x,\tau)]d\tau$$

In the homotopy decomposition method, the basic assumption is that the solutions can be written as a power series in p (3.4 a)

$$U(x,t,p) = \sum_{n=0}^{\infty} p^n U_n(x,t),$$

$$U(x,t) = lim_{p\to1} U(x,t,p) \quad (3.4\ b)$$

and the nonlinear term can be decomposed as (3.5)

$$NU(x,t) = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(U)$$

Where $p\epsilon\ (0,\ 1]$ is an embedding parameter.
$\mathcal{H}_n(U)$ is the He's polynomials that can be generated by (3.6)

$$\mathcal{H}_n(U_0, \cdots\cdots, U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{j=0}^{\infty} p^j U_j(x,t)\right)\right], n = 0,1,2 \cdots\cdots$$

The homotopy decomposition method is obtained by the graceful coupling of homotopy technique with Abel integral and is given by (3.7)

$$\sum_{n=0}^{\infty} p^n U_n(x,t) - T(x,t) = \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[f(x,\tau) + L\left(\sum_{n=0}^{\infty} p^n U_n(x,\tau)\right) + N\left(\sum_{n=0}^{\infty} p^n U_n(x,\tau)\right)\right]d\tau$$

Comparing the terms of same powers of p gives solutions of various orders with the first term:
$U_0(x,t) = T(x,t)$ (3.8)

3.1 Convergence of the method and unicity of the solution

Theorem 1[2]: Assuming that $X \times T \subset \mathbb{R} \times \mathbb{R}^+$ is a Banach space with a well defined norm $\|\ \|$, over which the series sequence of the approximate solution of (1.1) is defined, and the operator $G(U_n(x,t)) = U_{n+1}(x,t)$ defining the series solution of (1.4b) satisfies the Lipschitzian conditions that is
$\|G(U_k{}^*) - G(U_k)\| \le \varepsilon\|U^*{}_k(x,t) - U_k(x,t)\|$
for all $(x,t,k) \in X \times T \times \mathbb{N}$, then series solution obtained (1.5) is unique.

Proof : Assume that $U(x,t)$ and $U^*(x,t)$ are the series solution satisfying equation (1.1) then:

$U^*(x,t,p) = \sum_{n=0}^{\infty} p^n U^*{}_n(x,t)$ with initial guess $T(x,t)$
$U(x,t,p) = \sum_{n=0}^{\infty} p^n U_n(x,t)$ also with initial guess $T(x,t)$ therefore
$\|U^*{}_n(x,t) - U_n(x,t)\| = 0\ , n = 0,1,2, \cdots\cdots\cdots$
By the recurrence for
$n = 0, U^*{}_n(x,t) = U_n(x,t) = T(x,t)$
, assume that for
$n > k \ge 0 \|U^*{}_k(x,t) - U_k(x,t)\| = 0$. Then
$\|U^*{}_{k+1}(x,t) - U_{k+1}(x,t)\| = \|G(U_k{}^*) - G(U_k)\| \le \varepsilon\|U^*{}_k(x,t) - U_k(x,t)\| = 0$

which completes the proof.

3.2 Complexity of the homotopy decomposition method

It is very important to test the computational complexity of a method or algorithm. Complexity of an algorithm is the study of how long a program will take to run, depending on the size of its input and long of loops made inside the code. We compute a numerical example which is solved by the homotopy decomposition method. The code has been presented with Mathematica 8 according to the following code [18].

Step 1: Set $m \leftarrow 0$
Step 2: Calculated the recursive relation after the comparison of the terms of the same power is done.
Step 3: If $\|U_{n+1}(x,t) - U_n(x,t)\| < r$ with r the ratio of the neighbourhood of the exact solution [4] then go to step 4, else $m \leftarrow m+1$ and go to step 2
Step 4: Print out:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t)$$

as the approximate of the exact solution.

Lemma 1: If the exact solution of the fractional partial differential equation (3.1) exists, then $\|U_{n+1}(x,t) - U_n(x,t)\| < r$ for all $(x,t) \in X \times T$

Proof: Let $(x,t) \in X \times T$, then since the exact solution exists, then we have that following

$$\|U_{n+1}(x,t) - U_n(x,t)\| = \|U_{n+1}(x,t) - U(x,t) + U(x,t) - U_n(x,t)\|$$
$$\le \|U_{n+1}(x,t) - U(x,t)\| + \|-U_n(x,t) + U(x,t)\|$$
$$\le \frac{r}{2} + \frac{r}{2} = r$$

The last inequality follows from [1].

Lemma 2: The complexity of the homotopy decomposition method is of order $O(n)$

Proof: The number of computations including product, addition, subtraction and division are
In step 2
U_0: 0 because, obtains directly from the initial guess [18]
U_1: 3
$\vdots$
U_n:3
Now in step 4 the total number of computations is equal to $\sum_{j=0}^{n} U_j(x,t) = 3n = O(n)$.

4-Application

In this section, we make use of HDM to solve the main evolution system. Following the steps involved in the method, we arrive at the following equations

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) - \frac{p}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(a\sum_{n=0}^{\infty} p^n u_n\left(\sum_{n=0}^{\infty} p^n u_n\right)_x - b\sum_{n=0}^{\infty} p^n v_n\left(\sum_{n=0}^{\infty} p^n v_n\right)_x\right) + \left(\sum_{n=0}^{\infty} p^n u_n\right)_{xxx}$$

(4.1)

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = v(x,0) - \frac{p}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\left(\left(a\sum_{n=0}^{\infty} p^n u_n\left(\sum_{n=0}^{\infty} p^n u_n\right)_x + a\sum_{n=0}^{\infty} p^n\left(u_n\sum_{n=0}^{\infty} p^n v_n\right)_x + b\sum_{n=0}^{\infty} p^n u_n\left(\sum_{n=0}^{\infty} p^n v_n\right)_x\right)\right)$$

If we compare the terms of the same power of p we obtain the following integral equations. Note that when comparing this approach with the methodology of the Homotopy perturbation method, one will obtain in this step, a set of ordinary differential equations which something need to be also solving with care, because one will need to chose an appropriate initial guest. But with the current approach, the initial guess is straightforward obtained as Taylor series of the exact solution of the problem under investigation, this is one of the advantages, the approach has over the HPM. On the other hand when comparing this approach with the Variational Iteration Method, one will find out that, we do need the Lagrange multiplier here or the correctional function. Also this approach provides us with a convenient way to control the convergence of approximation series without adapting h, as in the case of [8] which is a fundamental qualitative difference in analysis between HDM and other methods. Therefore comparing the terms of the same power we obtain:

$$p^0: u_0(x,t) = u(x,0), u_0(x,0) = u(x,0)$$

(4.2)

$$p^0: u_0(x,t) = u(x,0), u_0(x,0) = u(x,0)$$

$$p^1: u_1(x,t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(6au_0(u_0)_x - 2bv_0(v_0)_x + a(u_0)_{xxx}\right)d\tau, u_1(x,0) = 0$$

$$p^1: v_1(x,t) = -\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\left(3bu_0(v_0)_x + b(u_0)_{xxx}\right)d\tau, v_1(x,0) = 0$$

$$\vdots$$

$$p^n: u_n(x,t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(6a\sum_{i=0}^{n-1} u_i(u_{n-i-1})_x - 2b\sum_{i=0}^{n-1} v_i(v_{n-i-1})_x + a(u_{n-1})_{xxx}\right)d\tau, u_n(x,t) = 0$$

$$p^n: v_n(x,t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(3b\sum_{i=0}^{n-1} u_i(v_{n-i-1})_x + b(v_{n-1})_{xxx}\right)d\tau, v_n(x,t) = 0$$

Integrating the above we obtain the following series solutions: (4.3)

$$u(x,0) = \frac{b}{a}\left(\tanh\left(\frac{1}{2}\sqrt{\frac{b}{a}}x\right)\right)^2, \quad v(x,0) = \frac{b}{\sqrt{a}}\left(\operatorname{sech}\left(\frac{1}{2}\sqrt{\frac{b}{a}}x\right)\right)^2.$$

Next we set

$$d = \frac{b}{a}; d_1 = \frac{b}{\sqrt{2a}} \text{ and } m = \frac{1}{2}\sqrt{\frac{b}{a}}.$$

Then

$$u_1(x,t) = \frac{(d+m)mt^{\alpha}}{\Gamma(1+\alpha)}\Big(-dd_1{}^2 + ad\,(d-m^2) + adm^2 sech(2mx)\Big)(\text{cosech}\,(mx))^4 \tanh\,(mx)$$

$$v_1(x,t) = \frac{2dd_1 mt^{\beta}}{\Gamma(1+\beta)}\left(d - m^2 + 2m^2 tanh(2mx)\right)(\text{sech}\,(mx))^4 \tanh\,(mx)$$

$$\begin{aligned} u_2(x,t) = \frac{m}{\Gamma(0.8+\alpha)\Gamma(0.9+\beta)\Gamma(1+\alpha)\Gamma(\beta)}\Bigg(& 2^{1-2\alpha} m^2 \sqrt{\pi} t^{\alpha} \Gamma(1 \\ &+ \beta)\Big(-2b^2 d_1^2 t^{\beta}\Big(-12d + 44m^2 + (d-m)cosh(2mx) \\ &+ 2m^2 cosh(4mx)\Gamma(1+2\alpha) \\ &+ ad_1{}^2\Big(-8\big(2bd_1^2(-3d+13m^2) + ad(18d^2 - 111dm^2 + 151m^4)\big) \\ &+ \big(4bd_1^2(-9d+49m^2) + 3ad(36d^2 - 272dm^2 + 397m^4)\big)cosh(2mx) \\ &- 4m^2\big(4bd_1^2 - ad(m-2m^2)\big)cosh(4mx) + adm^4 cosh(6xm)\Big)\Gamma(1+\alpha \\ &+ \beta)\Big)\left(tanh(mx)\right)^8\Big)\Bigg) \end{aligned}$$

$$\begin{aligned} v_2(x,t) = \frac{dm}{\Gamma(1+\alpha)\Gamma(\alpha+\beta)\Gamma(0.6+\alpha)\Gamma(1+\alpha+\beta)}\Big(& 2^{1-2\beta} m^2 \sqrt{\pi} t^{\beta} \Gamma(1 \\ &+ \alpha)\left(sech(mx)\right)^8\Big(bt^{\beta}\big(-27d^2 + 411dm^2 - 1208m^4 \\ &+ 3(6d^2 - 124dm^2 + 397m^4)cosh(2mx) + 3m^2(9d - 40m^2)cosh(4mx) \\ &+ m^4 cosh(6mx)\big)\Gamma(1+\alpha+\beta) \\ &+ 12t^{\alpha}\big(-bd_1^2 + ad(3d - 5m^2) + adm^2 cosh(2mx)\big)\Gamma(1 \\ &+ 2\beta)(\sinh\,(mx))^2\Big)\Big) \end{aligned}$$

And so on, using the package Mathematica, in the same manner one can obtain the rest of the components. But, here, few terms were computed and the asymptotic solution is given by:

(4.4)

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$
$$v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots$$

4.1 Numerical solutions

The following figures show the graphical representation of the approximated solution of the system of time-fractional Coupled- the Korteweg-de Vries Equations for $\lambda = 1, a = b = 1$.

The below figures show that the coupled solution of KDV equations is not only the function of time and space but also an increasing function of the fractional order derivative, which are α and β. Similar figure was obtained in [10]

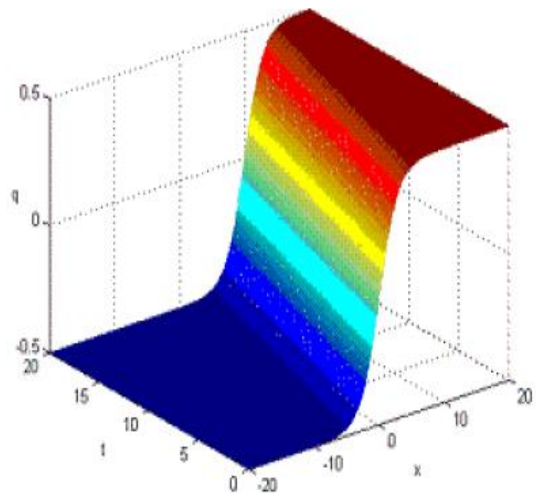

Figure 1: Approximated solution for

$\alpha = 0.75$ and $\beta = 0.45$

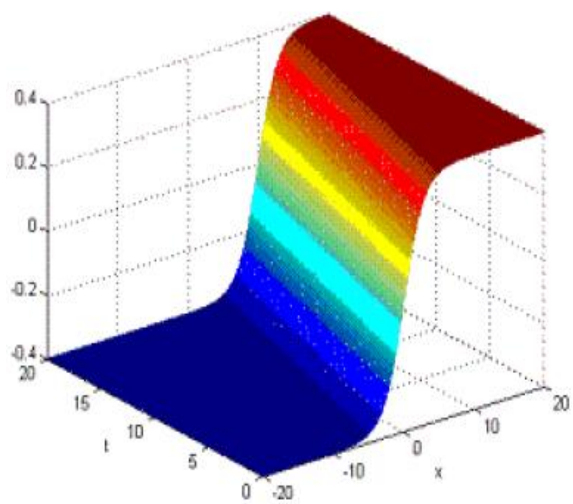

Figure 2: Approximated solution for

$\alpha = 1$ *and* $\beta = 0.95$

To test the accuracy of the used method, we represent in the below table the numerical values of the approximate and the exact solutions.

Table 1: Numerical values of the approximate and exact solutions.

x	t	$u(x,t)$ exact	$u(x,t)$ approxi	Error
-9	0.1 0.2	0.000164306 0.00014868	0.000164335 0.000148902	$2.95038\text{x}10^{-8}$ $2.30336\text{x}10^{-7}$
-4	0.1 0.2	0.0240924 0.0218249	0.0240964 0.0218557	$3.93593\text{x}10^{-6}$ 0.000030804 8
4	0.1 0.2	0.0240924 0.0218249	0.0240964 0.0218557	$3.93593\text{x}10^{-6}$ 0.000030805 0
9	0.1 0.2	0.000164306 0.00014868	0.000164334 0.000148901	$2.95039\text{x}10^{-8}$ $2.30335\text{x}10^{-7}$
x	t	$v(x,t)$ exact	$v(x,t)$ approxi	Error
-9	0.1 0.2	0.000116182 0.000105127	0.000116203 0.000105290	$2.08625\text{x}10^{-8}$ $1.62873\text{x}10^{-7}$
-4	0.1 0.2	0.170359 0.0154326	0.0170387 0.0154543	$2.78313\text{x}10^{-6}$ 0.000022782 4
4	0.1 0.2	0.170357 0.0154326	0.0170387 0.0154543	$2.78313\text{x}10^{-6}$ 0.000021782 5
9	0.1 0.2	0.000116182 0.000105127	0.000116203 0.000105290	$2.08625\text{x}10^{-8}$ $1.62873\text{x}10^{-7}$

5. Conclusions

We introduced a nonlinear fractional order system of KDV equations. We used HDM to analyze the differential equations and the method proved to be very liable as the error term was negligible. Our numerical scheme converged and we performed numerical simulations to support our analysis. This work shows that indeed HDM is a very powerful in investigating fractional systems of partial differential equations.

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