

On the Solutions of Some Systems of Second Order Rational Difference Equations

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Abstract: In this paper we deal with the form of the solutions of some systems of rational difference equations of order two with a nonzero real numbers initial conditions.

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1. Introduction

Our aim in this paper is to get the form of the solutions of some systems of the following rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{\alpha - x_{n-1}y_n}, y_{n+1} = \frac{y_{n-1}}{\beta + \gamma y_{n-1}x_n},$$

with a nonzero real numbers initial conditions and α, β, γ integers numbers.

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics, and so on. So, recently there has been an increasing interest in the study of qualitative analysis of rational difference equations and systems of difference equations. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solutions. See [1]-[15] and the references cited therein.

Periodic solutions of a difference equation have been investigated by many researchers, and various methods have been proposed for the existence and qualitative properties of the solution.

The periodicity of the positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{m}{y_n}, y_{n+1} = \frac{py_n}{x_{n-1}y_{n-1}},$$

was studied by Cinar in [5].

Elabbasy et al. [6] has obtained the solution of particular cases of the following general system of difference equations

$$x_{n+1} = \frac{a_1 + a_2 y_n}{a_3 z_n + a_4 x_{n-1} z_n}, y_{n+1} = \frac{b_1 z_{n-1} + b_2 z_n}{b_3 x_n y_n + b_4 x_n y_{n-1}},$$

$$z_{n+1} = \frac{c_1 z_{n-1} + c_2 z_n}{c_3 x_{n-1} y_{n-1} + c_4 x_{n-1} y_n + c_5 x_n y_n}.$$

Elsayed [10] has obtained the solutions of the following system of the difference equations

$$x_{n+1} = \frac{1}{y_{n-k}}, y_{n+1} = \frac{y_{n-k}}{x_n y_n}.$$

In [11] Kurbanli et al. studied the behavior of positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1}y_n}, y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1}x_n}.$$

Özban [21] has investigated the positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{1}{y_{n-k}}, y_{n+1} = \frac{y_n}{x_{n-m} y_{n-m-k}}.$$

In [22] Yalçinkaya investigated the sufficient condition for the global asymptotic stability of the following system of difference equations

$$z_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, t_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}.$$

Also, Yalçinkaya [23] has obtained the sufficient conditions for the global asymptotic stability of the system of two nonlinear difference equations

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1}, y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}.$$

Yang et al. [24] has investigated the positive solutions of the systems

$$x_n = \frac{a}{y_{n-p}}, y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}.$$

Similar nonlinear systems of rational difference equations were investigated see [16]-[25].

Definition (Periodicity)

A sequence $\{x_n\}_{n=-k}^\infty$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2. On the Solution of System:

$$x_{n+1} = \frac{x_{n-1}}{1 - x_{n-1}y_n}, y_{n+1} = \frac{y_{n-1}}{-1 + y_{n-1}x_n}.$$

In this section, we study the solutions of the system of two difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 - x_{n-1}y_n}, y_{n+1} = \frac{y_{n-1}}{-1 + y_{n-1}x_n}, \quad (1)$$

with a nonzero real numbers initial conditions with $x_{-1}y_0 \neq 1, x_{-1}y_0 \neq \frac{1}{2}, x_0y_{-1} \neq \pm 1$.

Theorem 1 Suppose that $\{x_n, y_n\}$ are solutions of system (1). Also, assume that x_{-1}, x_0, y_{-1} and y_0 are arbitrary nonzero real numbers and let $x_{-1} = a, x_0 = b, y_{-1} = A, y_0 = B$. Then

$$x_{4n-1} = \frac{a(1-2aB)^n}{(1-aB)^{2n}}, x_{4n} = b(1-b^2A^2)^n,$$

$$x_{4n+1} = \frac{a(1-2aB)^n}{(1-aB)^{2n+1}}, x_{4n+2} = b(1-bA)(1-b^2A^2)^n,$$

$$y_{4n-1} = \frac{A}{(1-b^2A^2)^n}, y_{4n} = \frac{B(1-aB)^{2n}}{(1-2aB)^n},$$

$$y_{4n+1} = \frac{-A}{(1-bA)(1-b^2A^2)^n}, y_{4n+2} = \frac{-B(1-aB)^{2n+1}}{(1-2aB)^{n+1}}.$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$x_{4n-5} = \frac{a(1-2aB)^{n-1}}{(1-aB)^{2n-2}}, x_{4n-4} = b(1-b^2A^2)^{n-1},$$

$$x_{4n-3} = \frac{a(1-2aB)^{n-1}}{(1-aB)^{2n-1}}, x_{4n-2} = b(1-bA)(1-b^2A^2)^{n-1},$$

$$y_{4n-5} = \frac{A}{(1-b^2A^2)^{n-1}}, y_{4n-4} = \frac{B(1-aB)^{2n-2}}{(1-2aB)^{n-1}},$$

$$y_{4n-3} = \frac{-A}{(1-bA)(1-b^2A^2)^{n-1}}, y_{4n-2} = \frac{-B(1-aB)^{2n-1}}{(1-2aB)^n}.$$

Now, it follows from Eq.(1) that

$$\begin{aligned} x_{4n-1} &= \frac{x_{4n-3}}{1 - x_{4n-3}y_{4n-2}} \\ &= \frac{\frac{a(1-2aB)^{n-1}}{(1-aB)^{2n-1}}}{1 - \left(\frac{a(1-2aB)^{n-1}}{(1-aB)^{2n-1}}\right)\left(\frac{-B(1-aB)^{2n-1}}{(1-2aB)^n}\right)} \\ &= \frac{\frac{a(1-2aB)^{n-1}}{(1-aB)^{2n-1}}}{1 + \left(\frac{aB}{(1-2aB)}\right)\left(\frac{(1-2aB)}{(1-2aB)}\right)} \\ &= \frac{a(1-2aB)^{n-1}(1-2aB)}{(1-aB)^{2n-1}(1-2aB+aB)} \\ &= \frac{a(1-2aB)^n}{(1-aB)^{2n-1}(1-aB)} = \frac{a(1-2aB)^n}{(1-aB)^{2n}}. \end{aligned}$$

Also, we have

$$y_{4n-1} = \frac{y_{4n-3}}{-1 + y_{4n-3}x_{4n-2}}$$

$$\begin{aligned}
 &= \frac{-A}{(1-bA)(1-b^2A^2)^{n-1}} \\
 &= \frac{-A}{(1-bA)(1-b^2A^2)^{n-1}} \left(\frac{-A}{(1-bA)(1-b^2A^2)^{n-1}} \right) (b(1-bA)(1-b^2A^2)^{n-1}) \\
 &= \frac{-A}{(1-bA)(1-b^2A^2)^{n-1}} \\
 &= \frac{-A}{(1-bA)(1-b^2A^2)^{n-1}(-1+(-Ab))} \\
 &= \frac{A}{(1-bA)(1-b^2A^2)^{n-1}(1+Ab)} = \frac{A}{(1-b^2A^2)^n}.
 \end{aligned}$$

Also, it follows from Eq.(1) that

$$\begin{aligned}
 x_{4n+2} &= \frac{x_{4n}}{1-x_{4n}y_{4n+1}} \\
 &= \frac{b(1-b^2A^2)^n}{1-(b(1-b^2A^2)^n) \left(\frac{-A}{(1-bA)(1-b^2A^2)^n} \right)} \\
 &= \frac{b(1-b^2A^2)^n}{1-\left(\frac{-Ab}{(1-bA)}\right) \left(\frac{1-bA}{1-bA}\right)} = \frac{b(1-b^2A^2)^n(1-bA)}{1-bA+bA} \\
 &= b(1-bA)(1-b^2A^2)^n,
 \end{aligned}$$

and

$$\begin{aligned}
 y_{4n+2} &= \frac{y_{4n}}{-1+y_{4n}x_{4n+1}} \\
 &= \frac{\frac{B(1-aB)^{2n}}{(1-2aB)^n}}{-1+\left(\frac{B(1-aB)^{2n}}{(1-2aB)^n}\right) \left(\frac{a(1-2aB)^n}{(1-aB)^{2n+1}}\right)} \\
 &= \frac{\frac{B(1-aB)^{2n}}{(1-2aB)^n}}{-1+\left(\frac{aB}{(1-aB)}\right) \left(\frac{1-aB}{1-aB}\right)} \\
 &= \frac{B(1-aB)^{2n}(1-aB)}{(1-2aB)^n(-1+aB+aB)}
 \end{aligned}$$

$$= \frac{-B(1-aB)^{2n+1}}{(1-2aB)^{n+1}}.$$

Similarly one can prove the other relations. The proof is complete.

Lemma 1. If a, b, A and B arbitrary real numbers and let $\{x_n, y_n\}$ are solutions of system (1) then the following statements are true:-

- (i) If $a = 0, B \neq 0$, then we have $x_{4n-1} = x_{4n+1} = 0$ and $y_{4n} = B, y_{4n+2} = -B$.
- (ii) If $b = 0, A \neq 0$, then we have $x_{4n} = x_{4n+2} = 0$ and $y_{4n-1} = A, y_{4n+1} = -A$.
- (iii) If $A = 0, b \neq 0$, then we have $y_{4n-1} = y_{4n+1} = 0$ and $x_{4n} = x_{4n+2} = b$.
- (iv) If $B = 0, a \neq 0$, then we have $y_{4n} = y_{4n+2} = 0$ and $x_{4n-1} = x_{4n+1} = a$.

Proof: The proof follows from the form of the solutions of system (1).

3. On the Solution of System:

$$x_{n+1} = \frac{x_{n-1}}{1-x_{n-1}y_n}, y_{n+1} = \frac{y_{n-1}}{1+y_{n-1}x_n}.$$

In this section, we study the solutions of the system of two difference equations

$$x_{n+1} = \frac{x_{n-1}}{1-x_{n-1}y_n}, y_{n+1} = \frac{y_{n-1}}{1+y_{n-1}x_n}, \tag{2}$$

with a nonzero real numbers initial conditions with $x_{-1}y_0 \neq 1, x_0y_{-1} \neq -1$.

Theorem 2 Suppose that $\{x_n, y_n\}$ are solutions of system (2). Also, assume that x_{-1}, x_0, y_{-1} and y_0 are arbitrary nonzero real numbers and let $x_{-1} = a, x_0 = b, y_{-1} = A, y_0 = B$. Then

$$\begin{aligned}
 x_{2n-1} &= \frac{a}{(1-aB)^n}, x_{2n} = b(1+bA)^n, \\
 y_{2n-1} &= \frac{A}{(1+bA)^n}, y_{2n} = B(1-aB)^n.
 \end{aligned}$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$x_{2n-3} = \frac{a}{(1-aB)^{n-1}}, \quad x_{2n-2} = b(1+bA)^{n-1},$$

$$y_{2n-3} = \frac{A}{(1+bA)^{n-1}}, \quad y_{2n-2} = B(1-aB)^{n-1}.$$

Now, it follows from Eq.(2) that

$$x_{2n-1} = \frac{x_{2n-3}}{1 - x_{2n-3}y_{2n-2}}$$

$$= \frac{\frac{a}{(1-aB)^{n-1}}}{1 - \left(\frac{a}{(1-aB)^{n-1}}\right)(B(1-aB)^{n-1})}$$

$$= \frac{a}{1-aB} = \frac{a}{(1-aB)^n},$$

$$y_{2n-1} = \frac{y_{2n-3}}{1 + y_{2n-3}x_{2n-2}}$$

$$= \frac{\frac{A}{(1+bA)^{n-1}}}{1 + \left(\frac{A}{(1+bA)^{n-1}}\right)b(1+bA)^{n-1}}$$

$$= \frac{A}{1+bA} = \frac{A}{(1+bA)^n}.$$

Also, it follows from Eq.(2) that

$$x_{2n} = \frac{x_{2n-2}}{1 - x_{2n-2}y_{2n-1}}$$

$$= \frac{b(1+bA)^{n-1}}{1 - b(1+bA)^{n-1}\left(\frac{A}{(1+bA)^n}\right)}$$

$$= \frac{b(1+bA)^{n-1}}{1 - \left(\frac{bA}{1+bA}\right)\left(\frac{1+bA}{1+bA}\right)}$$

$$= \frac{b(1+bA)^n}{1+bA-bA} = b(1+bA)^n,$$

$$y_{2n} = \frac{y_{2n-2}}{1 + y_{2n-2}x_{2n-1}}$$

$$= \frac{B(1-aB)^{n-1}}{1 + (B(1-aB)^{n-1})\left(\frac{a}{(1-aB)^n}\right)}$$

$$= \frac{B(1-aB)^{n-1}}{1 + \left(\frac{aB}{1-aB}\right)\left(\frac{1-aB}{1-aB}\right)}$$

$$= \frac{B(1-aB)^n}{1-aB+aB} = B(1-aB)^n.$$

The proof is complete.

Lemma 2. Let $\{x_n, y_n\}$ be a positive solution of system (2), then $\{y_n\}$ is bounded and converges to zero.

Proof: It follows from Eq.(2) that

$$y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1}x_n} \leq y_{n-1}.$$

Then the subsequences $\{y_{2n-1}\}_{n=0}^\infty, \{y_{2n}\}_{n=0}^\infty$ are decreasing and so are bounded from above by $M = \max\{y_{-1}, y_0\}$.

Theorem 3 The system (2) has a periodic solutions of period four iff $aB = 2, bA = -2$ and will be take the form

$$\{x_n\} = \{a, b, -a, -b, \dots\},$$

$$\{y_n\} = \{A, B, -A, -B, \dots\}.$$

Lemma 3. Assume that $aB \neq 2, bA \neq -2$. Then the solutions of system (2) are unbounded solutions.

Lemma 4. If a, b, c, d, A, B, C and D arbitrary real numbers and let $\{x_n, y_n\}$ are solutions of system (2) then the following statements are true:-

- (i) If $a = 0, B \neq 0$, then we have $x_{2n-1} = 0$ and $y_{2n} = B$.

(ii) If $b = 0, A \neq 0$, then we have $x_{2n} = 0$ and $y_{2n-1} = A$.

(iii) If $A = 0, b \neq 0$, then we have $y_{2n-1} = 0$ and $x_{2n} = b$.

(iv) If $B = 0, a \neq 0$, then we have $y_{2n} = 0$ and $x_{2n-1} = a$.

The following systems can be proved similarly:

4. On the Solution of System:

$$x_{n+1} = \frac{x_{n-1}}{1 - x_{n-1}y_n}, y_{n+1} = \frac{y_{n-1}}{-1 - y_{n-1}x_n}.$$

In this section, we study the solutions of the system of two difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 - x_{n-1}y_n}, y_{n+1} = \frac{y_{n-1}}{-1 - y_{n-1}x_n}, \quad (3)$$

with a nonzero real numbers initial conditions with $x_{-1}y_0 \neq \pm 1, x_0y_{-1} \neq -1, -\frac{1}{2}$.

Theorem 4 Suppose that $\{x_n, y_n\}$ are solutions of system (3). Also, assume that x_{-1}, x_0, y_{-1} and y_0 are arbitrary nonzero real numbers and let $x_{-1} = a, x_0 = b, y_{-1} = A, y_0 = B$. Then

$$x_{4n-1} = \frac{a}{(1 - a^2B^2)^n}, \quad x_{4n} = \frac{b(1 + bA)^{2n}}{(1 + 2bA)^n},$$

$$x_{4n+1} = \frac{a}{(1 - aB)(1 - a^2B^2)^n}, \quad x_{4n+2} = \frac{b(1 + bA)^{2n+1}}{(1 + 2bA)^{n+1}},$$

$$y_{4n-1} = \frac{A(1 + 2bA)^n}{(1 + bA)^{2n}}, \quad y_{4n} = B(1 - a^2B^2)^n,$$

$$y_{4n+1} = \frac{-A(1 + 2bA)^n}{(1 + bA)^{2n+1}},$$

$$y_{4n+2} = -B(1 - aB)(1 - a^2B^2)^n.$$

Lemma 5. If a, b, c, d, A, B, C and D arbitrary real numbers and let $\{x_n, y_n\}$ are solutions of system (3) then the following statements are true:-

(i) If $a = 0, B \neq 0$, then we have $x_{4n-1} = x_{4n+1} = 0$ and $y_{4n} = B, y_{4n+2} = -B$.

(ii) If $b = 0, A \neq 0$, then we have $x_{4n} = x_{4n+2} = 0$ and $y_{4n-1} = A, y_{4n+1} = -A$.

(iii) If $A = 0, b \neq 0$, then we have $y_{4n-1} = y_{4n+1} = 0$ and $x_{4n} = x_{4n+2} = b$.

(iv) If $B = 0, a \neq 0$, then we have $y_{4n} = y_{4n+2} = 0$ and $x_{4n-1} = x_{4n+1} = a$.

5. On the Solution of System:

$$x_{n+1} = \frac{x_{n-1}}{-1 - x_{n-1}y_n}, y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1}x_n}.$$

In this section, we study the solutions of the system of two difference equations

$$x_{n+1} = \frac{x_{n-1}}{-1 - x_{n-1}y_n}, y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1}x_n}, \quad (4)$$

with a nonzero real numbers initial conditions with $x_{-1}y_0 \neq \pm 1, x_0y_{-1} \neq -1, -\frac{1}{2}$.

Theorem 5 Suppose that $\{x_n, y_n\}$ are solutions of system (4). Also, assume that x_{-1}, x_0, y_{-1} and y_0 are arbitrary nonzero real numbers and let $x_{-1} = a, x_0 = b, y_{-1} = A, y_0 = B$. Then

$$x_{4n-1} = \frac{a}{(1 - a^2B^2)^n}, \quad x_{4n} = \frac{b(1 + bA)^{2n}}{(1 + 2bA)^n},$$

$$x_{4n+1} = \frac{-a}{(1 + aB)(1 - a^2B^2)^n}, \quad x_{4n+2} = \frac{-b(1 + bA)^{2n+1}}{(1 + 2bA)^{n+1}},$$

$$y_{4n-1} = \frac{A(1 + 2bA)^n}{(1 + bA)^{2n}}, \quad y_{4n} = B(1 - a^2B^2)^n,$$

$$y_{4n+1} = \frac{A(1+2bA)^n}{(1+bA)^{2n+1}}, \quad y_{4n+2} = B(1+aB)(1-a^2B^2)^n.$$

Lemma 6. Let $\{x_n, y_n\}$ be a positive solution of system (4), then $\{y_n\}$ is bounded and converges to zero.

Lemma 7. If a, b, c, d, A, B, C and D arbitrary real numbers and let $\{x_n, y_n\}$ are solutions of system (4) then the following statements are true:-

(i) If $a = 0, B \neq 0$, then we have $x_{4n-1} = x_{4n+1} = 0$ and $y_{4n} = y_{4n+2} = B$.

(ii) If $b = 0, A \neq 0$, then we have $x_{4n} = x_{4n+2} = 0$ and $y_{4n-1} = y_{4n+1} = A$.

(iii) If $A = 0, b \neq 0$, then we have $y_{4n-1} = y_{4n+1} = 0$ and $x_{4n} = b, x_{4n+2} = -b$.

(iv) If $B = 0, a \neq 0$, then we have $y_{4n} = y_{4n+2} = 0$ and $x_{4n-1} = a, x_{4n+1} = -a$.

6. On the Solution of System:

$$x_{n+1} = \frac{x_{n-1}}{-1 - x_{n-1}y_n}, \quad y_{n+1} = \frac{y_{n-1}}{-1 + y_{n-1}x_n}.$$

In this section, we study the solutions of the system of two difference equations

$$x_{n+1} = \frac{x_{n-1}}{-1 - x_{n-1}y_n}, \quad y_{n+1} = \frac{y_{n-1}}{-1 + y_{n-1}x_n}, \quad (5)$$

with a nonzero real numbers initial conditions.

Theorem 6 Suppose that $\{x_n, y_n\}$ are solutions of system (5). Also, assume that x_{-1}, x_0, y_{-1} and y_0 are arbitrary nonzero real numbers and let $x_{-1} = a, x_0 = b, y_{-1} = A, y_0 = B$. Then

$$x_{2n-1} = (-1)^n a \prod_{i=0}^{n-1} \frac{1 + (2i)aB}{1 + (2i+1)aB},$$

$$x_{2n} = (-1)^n b \prod_{i=0}^{n-1} \frac{(2i+1)bA - 1}{(2i+2)bA - 1},$$

and

$$y_{2n-1} = (-1)^n A \prod_{i=0}^{n-1} \frac{(2i)bA - 1}{(2i+1)bA - 1},$$

$$y_{2n} = (-1)^n B \prod_{i=0}^{n-1} \frac{1 + (2i+1)aB}{1 + (2i+2)aB}.$$

7. Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to nonlinear difference equations.

Example 1. Consider the difference system equation (1) with the initial conditions $x_{-1} = 0.8, x_0 = 0.5, y_{-1} = 0.24$ and $y_0 = 0.1$. (See Fig. 1).

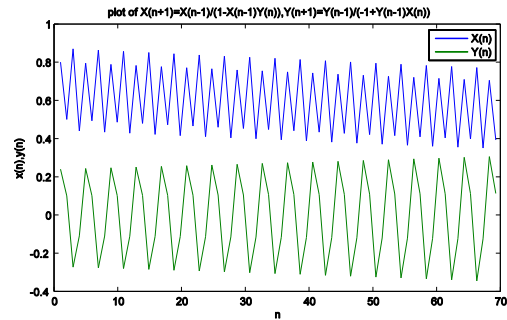


Figure (1)

Example 2. For the the initial conditions $x_{-1} = 0.2, x_0 = 0.5, y_{-1} = 0.4$ and $y_0 = 0.7$ when we take the system (2). (See Fig. 2).

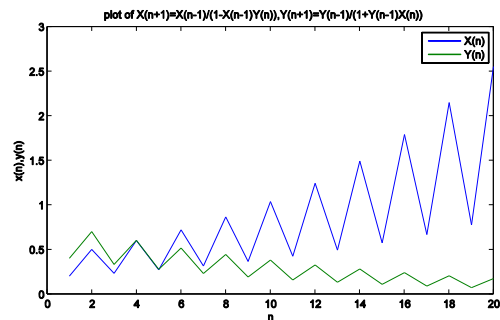


Figure (2)

Example 3. If we consider the difference equation system (3) with the initial conditions $x_{-1} = 0.2, x_0 = 0.5, y_{-1} = 0.4$ and $y_0 = -0.7$. (See Fig. 3).

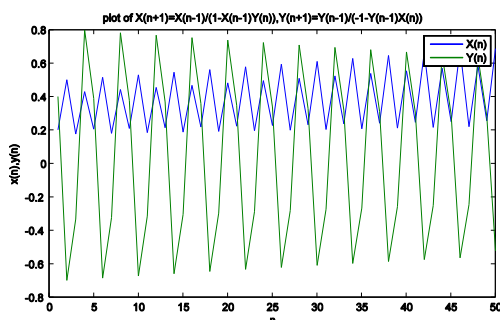


Figure (3)

Example 4. See Figure 4, since we take the difference system equation (4) with the initial conditions

$$x_{-1} = 0.9, x_0 = 0.5, y_{-1} = -0.4 \text{ and } y_0 = 0.3.$$

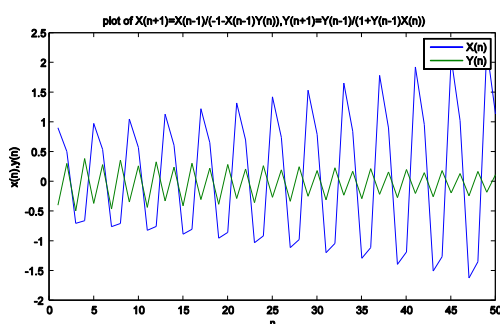


Figure (4)

Example 5. See Figure 5, since we take the difference system equation (5) with the initial conditions

$$x_{-1} = 0.9, x_0 = -0.3, y_{-1} = 0.5 \text{ and } y_0 = 0.1.$$

See Fig. 5.

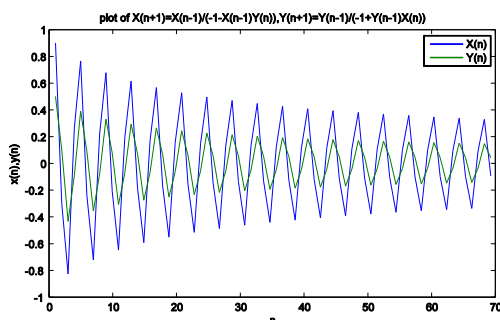


Figure (5)

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