The p-Metric Space of χ^2 Defined by Musielak

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Abstract: In the present paper we introduce the p-metric space of χ^2 multiplier defined by a Musielak modulus function. We study some topological properties and prove some inclusion relations between these spaces. Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to define the sequence space l_M which is called an Orlicz sequence space. Another generalization of Orlicz sequence spaces is due to Woo [31].
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1. Introduction:

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequence, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [2]. Later on, they were investigated by Hardy [3], Moricz [7], Moricz and Rhoades [8], Basarir and Solankan [1], Tripathy [11], Turkmenoglu [12], and many others.
We procure the following sets of double sequences:

$M_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty\right\}$,

$C_p(t) := \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{m,n} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C}\right\}$,

$C_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\}$,

$L_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\}$,

$C_{bp}(t) := C_p(t) \cap M_u(t) \text{ and } C_{0bp}(t) = C_{0p}(t) \cap M_u(t)$;

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Where $t = (t_{mn})$ is the sequence of strictly positive real's t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringshein's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $M_u(t)$, $C_p(t)$, $C_{0p}(t)$, $L_u(t)$, $C_{bp}(t)$ and $C_{0bp}(t)$ reduce to the sets Mu, C_p, C_{0p}, L_u, C_{bp} and C_{0bp}, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gokhan and Colak [14,15] have proved that $M_u(t)$ and $C_p(t)$, $C_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α–, β–, γ– duals of the spaces $M_u(t)$ and $C_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [16] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [17] and Tripathy [11] have independently introduced that statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Altay and Basar [20] have defined the spaces BS, $BS(t)$, CS_p, CS_{bp}, CS_r and BV of double sequences consisting of all double series whose sequence of partial sums are in the spaces M_u, $M_u(t)$, C_p, C_{bp}, C_r and L_u, respectively, and also examined some properties of those sequence spaces and determined the α– duals of the spaces BS, BV, CS_{bp} and the $\beta(v)$ - duals of the spaces CS_{bp} and CS_r of double series. Basar and Server [21] have introduced the Banach space L_q of double sequences corresponding to the well-known space l_q of single sequences and examined some properties of the space L_q. Quite recently Subramanian and Misra [22] have studied the space $\chi^2_M(p,q,u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesaro summable with respect to a modulus was introduced by Maddox [6] as an extension of the definition of strongly Cesaro summable sequences. Connor [23] further extended this definition to a definition of strong A– summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A– summability, strong A– summability with respect to a modulus, and A– statistical convergence. In [24] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [25]-[26], and [27] the four dimensional matrix transformation $(Ax)_{k,l} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{kl}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \qquad (1.1)$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if any only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \left(m, n \in \mathbb{N}\right)$.

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $\left((m+n)!|x_{mn}|^{1/m+n} \to 0\right)a$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let ϕ = {all finite sequences}.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i, j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})$ $(m, n \in \mathbb{N})$ are also continuous.

Let M and Φ are mutually complementary modulus functions. Then, we have:

(i) For all u, y ≥ 0,
$uy \leq M(u) + \Phi(y)$, (Young's ineuqality) [See[13]] (1.2)

(ii) For all u ≥ 0,
$$u\eta(u) = M(u) + \Phi(\eta(u)). \qquad (1.3)$$

(iii) For all $u \geq 0$, and o $< \lambda < 1$
$$M(\lambda u) \leq \lambda M(u) \qquad (1.4)$$

Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to construct Orlicz sequence space

$$l_M = \left\{x \in w: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, for\, some\, \rho > 0\right\},$$

The space l_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p \left(1 \leq p < \infty\right)$, the spaces l_M coincide with the classical sequence space l_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$g_{mn}(v) = \sup\left\{|v|u - \left(f_{mn}\right)(u) : u \leq 0\right\}, m, n = 1, 2, \ldots$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f and its sub-space h_f are defined as follows

$$t_f = \left\{x \in w^2 : I_f\left(\left((m+n)!|x_{mn}|\right)\right)^{1/m+n} \to 0\, as\, m, n \to \infty\right\},$$

$$h_f = \left\{x \in w^2 : I_f\left(\left((m+n)!|x_{mn}|\right)\right)^{1/m+n} \to 0\, as\, m, n \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m \in I_\eta} \sum_{n \in I_\mu} f_{mn}\left((m+n)!|x_{mn}|\right)^{1/m+n}, x = (x_{mn}) \in t_f a \cdot$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sup_{mn}\left\{\inf\left(\frac{1}{\lambda_{\eta\mu}}\right)\sum_{m \in I_\mu}\sum_{n \in I_\eta} u_{mn} f_{mn}\left(\left((m+n)!|x_{mn}|\right)^{1/m+n}\right) \leq 1\right\}$$

Let $\lambda = \lambda_{\eta\mu}$ be a non-decreasing sequence of positive real's tending to infinity with $\lambda_1 = 1$ and $\lambda_{\eta+1,\mu+1} \leq \lambda_{\eta\mu} + 1$. The generalized de la Vallee-Poussin means is defined by

$$t_{\eta\mu}(x) = \frac{1}{\lambda_{\eta\mu}} \sum_{m \in I_\eta} \sum_{n \in I_\mu} \left((m+n)! x_{mn}\right)^{1/m+n},$$

where $I_{\eta\mu}\left[(\eta\mu) - \lambda_{\eta\mu} + 1, \eta\mu\right]$. A sequence $x = (x_{mn})$ is said to be (V, λ) – summable to a number $t_{\eta\mu} \to 0$ as $\eta\mu \to \infty$.

If X is a sequence space, we give the following definitions:

(i) X' = the continuous dual of X;

(ii) $X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, for\, each\, x \in X\right\}$;

(iii) $X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn}\, x \in X\right\}$;

(iv) $X^\gamma = \left\{\begin{array}{l} a = (a_{mn}) : \sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn} x_{mn}\right| < \infty, \\ for\, each\, x \in X \end{array}\right\}$;

(v) let X bean FK – space $\supset \phi$; then $X^{f}=\left\{f\left(\Im_{mn}\right):f\in X'\right\}$;

(vi) $X^{\delta}=\left\{\begin{array}{l} a=\left(a_{mn}\right):\sup_{mn}\left|a_{mn}x_{mn}\right|^{1/m+n}<\infty, \\ for\,each\,x\in X \end{array}\right\}$;

$X^{\alpha}.X^{\beta}$, X^{γ} are called α – (or Kothe – Toeplitz) dual of X, β – (or generalized – Kothe – Toeplitz) dual of X, γ – dual of X, δ – dual of X respectively. X^{α} is defined by Gupta and Kamptan [13]. It is clear that $X^{\alpha}\subset X^{\beta}$ and $X^{\alpha}\subset X^{\gamma}$, but $X^{\beta}\subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z\left(\Delta\right)=\left\{x=\left(x_k\right)\in w:\left(\Delta x_k\right)\in Z\right\}$$

for $Z=c$, c_0 and l_∞, where $\Delta x_k=x_k-x_{k+1}$ for all $k\in\mathbb{N}$.

Here c, c_0 and l_∞ denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space l_p is introduced and studied in the case $1\le p\le\infty$ by Basar and Altay and in the case $0<p<1$ by Altay and Basar in [20]. The spaces $c(\Delta)$, $c_0(\Delta)$, $l_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\|=|x_1|+\sup k\ge 1\left|\Delta x_k\right| and \|x\|bv_p=\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p},(1\le p<\infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence space defined by

$$Z\left(\Delta\right)=\left\{x=\left(x_{mn}\right)\in w^2:\left(\Delta x_{mn}\right)\in Z\right\}$$

where $Z=\Lambda^2,\chi^2$ and $\Delta x_{mn}=(x_{mn}-x_{mn+1})-(x_{m+1n}-x_{m+1n+1})=x_{mn}-x_{mn+1}-x_{m+1n}+x_{m+1n+1}$ for all $m,n\in\mathbb{N}$.

2. DEFINITION AND PRELIMINARIES

Let $n\in\mathbb{N}$ and X be a real vector space of dimension w, where $n\le w$. A real valued function $d_p\left(x_1,\ldots,x_n\right)=\left\|\left(d_1\left(x_1\right),\ldots,d_n\left(x_n\right)\right)\right\|_p$ on X satisfying the following four conditions:

(i) $\left\|d_1\left(x_1\right),\ldots,d_n\left(x_n\right)\right\|_p=0$ if and only if $d_1(x_1),\ldots,d_n(x_n)$ are linearly de-pendent,

(ii) $\left\|\left(d_1\left(x_1\right),\ldots,d_n\left(x_n\right)\right)\right\|_p$ is invariant under permutation,

(iii) $\left\|\left(\alpha d_1\left(x_1\right),\ldots,d_n\left(x_n\right)\right)\right\|_p=|\alpha|\left\|d_1\left(x_1\right),\ldots,d_n\left(x_n\right)\right\|_p,\alpha\in\mathbb{R}$

(iv) $d_p\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\ldots\left(x_n,y_n\right)\right)=\left(d_X\left(x_1,x_2,\ldots,x_n\right)^p+d_Y\left(y_1,y_2,\ldots,y_n\right)\right)^{1/p}$ for $1\le p<\infty$; (or)

(v) $d\left(\left(x_1,y_1\right),\left(x_2,y_2\right),\ldots,\left(x_n,y_n\right)\right):=\sup\left\{\begin{array}{l} d_X\left(x_1,x_2,\ldots,x_n\right), \\ d_Y\left(y_1,y_2,\ldots y_n\right)\end{array}\right\}$

for $x_1,x_2,\ldots,x_n\in X$, $y_1,y_2,\ldots,y_n\in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n sub-spaces.

A trivial example of p product metric of n metric space is the p norm space is $X=\mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\left\|\left(d_1\left(x_1\right),\ldots,d_n\left(x_n\right)\right)\right\|_E=\sup\left(\left|\det\left(d_{mn}\left(x_{mn}\right)\right)\right|\right)=\sup\left(\left|\begin{array}{cccc} d_{11}\left(x_{11}\right) & d_{12}\left(x_{12}\right) & \ldots & d_{1n}\left(x_{1n}\right) \\ d_{21}\left(x_{21}\right) & d_{22}\left(x_{22}\right) & \ldots & d_{2n}\left(x_{2n}\right) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}\left(x_{n1}\right) & d_{n2}\left(x_{n2}\right) & \ldots & d_{nn}\left(x_{nn}\right)\end{array}\right|\right)$$

where $x_i=(x_{i1},\ldots,x_{in})\in\mathbb{R}^n$ for each $i=1,2,\ldots,n$.

If every Cauchy sequence in X converges to some $L\in X$, then X is said to be complete with respect to the p– metric. Any complete p– metric space is said to be p– Banach metric space.

Let X be a linear metric space. A function $w: X\to\mathbb{R}$ is called Para normed, if

1. $w(x)\ge 0$, for all $x\in X$;
2. $w(-x)=w(x)$, for all $x\in X$;
3. $w(x+y)\le w(x)+w(y)$, for all $x,y\in X$;
4. If (σ_{mn}) is a sequence of scalars with $\sigma_{mn}\to\sigma$ as $m,n\to\infty$ and (x_{mn}) is a sequence of vectors with $w(x_{mn}-x)\to 0$ as $m,n\to\infty$, then $w(\sigma_{mn}x_{mn}-\sigma x)\to 0$ as $m,n\to\infty$.

A paranormed w for which $w(x)=0$ implies $x=0$ is called total paranorm and the pair (X,w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (sec [32], Theorem 10.42, p.183).

Let $f=(f_{mn})$ be a Musielak-modulus function, $\left(X,\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right)$ be a p-metric space, $q=(q_{mn})$ be bounded sequence of strictly positive real numbers and $u=(u_{mn})$ be any sequence of strictly positive real numbers. By $S(p-X)$ we denote the space of all sequences defined over $\left(X,\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right)a$. In the present paper we define the following sequence spaces:

$$\left[\chi^{2q}_{f,V,u}\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]=\lim_{\eta\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[u_{mn}f_{mn}\left(\left\|\left(\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n},\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right)\right\|_p\right)\right]^{q_{mn}}=0,$$

$$\left[\Lambda^{2q}_{f,V,u},\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]=\sup_{\eta,\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum n\in I_\mu$$

$$\left[u_{mn}f_{mn}\left(\left\|x_{mn}\right|^{1/m+n},\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}<\infty,$$

If we take $f_{mn}(x) = x$, we get

$$\left[\chi^{2q}_{f,V,u}\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]=\lim_{\eta\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}$$

$$\left[u_{mn}\left(\left\|\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n},\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}=0,$$

$$\left[\Lambda^{2q}_{f,V,u},\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]=\sup_{\eta,\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}$$

$$\left[u_{mn}\left(\left\|\left|x_{mn}\right|^{1/m+n},\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|\right)_p\right]^{q_{mn}}<\infty,$$

If we take $q = (q_{mn}) = 1$ for all $m, n \in \mathbb{N}$, we get

$$\left[\chi^{2q}_{f,V,u}\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]=\lim_{\eta\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}$$

$$\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n},\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]=0,$$

$$\left[\Lambda^{2q}_{f,V,u},\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]=\sup_{\eta,\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}$$

$$\left[u_{mn}f_{mn}\left(\left\|\left|x_{mn}\right|^{1/m+n},\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]<\infty,$$

If we take $q = (q_{mn}) = 1$ and $u = (u_{mn}) = 1$ for all $m, n \in \mathbb{N}$, we get

$$\left[\chi^{2q}_{f,V,u}\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]=\lim_{\eta\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}$$

$$\left[f_{mn}\left(\left\|\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n},\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]=0,$$

$$\left[\Lambda^{2q}_{f,V,u},\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]=\sup_{\eta,\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}$$

$$\left[f_{mn}\left(\left\|\left|x_{mn}\right|^{1/m+n},\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|\right)_p\right]<\infty,$$

The following inequality will be used throughout the paper. If $0 \le q_{mn} \le \sup q_{mn} = H, K = \max\left(1, 2^{H-1}\right)$ then

$$\left|a_{mn}+b_{mn}\right|^{q_{mn}} \le K\left\{\left|a_{mn}\right|^{q_{mn}}+\left|b_{mn}\right|^{q_{mn}}\right\}$$

for all m, n and a_{mn} , $b_{mn} \in \mathbb{C}$. Also $|a|^{q_{mn}} \le \max\left(1,|a|^H\right)$ for all a $\in \mathbb{C}$.

The main aim of this paper is to introduce some multiplier sequence spaces defined by a Musielak-modulus function over p-metric spaces also study some topological properties and inclusion relation on above defined sequence spaces.

3. MAIN RESULTS

3.1. Theorem

Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be analytic sequence of positive real numbers and $u = (u_{mn})$ be any sequence of strictly positive real numbers. Then the spaces $\left[\chi^{2q}_{f,V,u}\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]$ and $\left[\Lambda^{2q}_{f,V,u},\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]$ are liner spaces.

Proof

It is routine verification. Therefore the proof is omitted.

3.2. Theorem

Let $f = (f_{mn})$ be a Musielak-modulus funtion , $q = (q_{mn})$ be analytic sequence of positive real numbers and $u = (u_{mn})$ be any sequence of strictly positive real numbers. Then spaces $\left[\chi^{2q}_{f,V,u},\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]$ is a paranormed space with respect to the paranormed defined by

$$g(x)=\inf\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n},\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\le 1,$$

where $H = \max\left(1, \sup_{mn} q_{mn} < \infty\right)$.

Proof

Clearly $g(x) \ge 0$ for $x = (x_{mn}) \in \left[\chi^{2q}_{f,V,u}\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]$. Since $f_{mn}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then

$$\inf\left(\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n},\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\le 1\right)=0$$

Suppose that $\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n} \ne 0$ for each $m, n \in \mathbb{N}$.

This implies that $u_{mn}\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n} \ne 0$, for each $m, n \in \mathbb{N}$.

Then $\left\|u_{mn}\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n},\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p \to \infty$.

It follows that

$$\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\right.$$

$$\left.\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n},d(x_1),d(x_2),\ldots,d(x_{n-1})\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\to\infty,$$

which is a contradiction. Therefore $\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}=0$ for each m, n and thus $\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}=0$ for each $m, n \in \mathbb{N}$. Let

$$\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\le 1$$

and

$$\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\right.$$

$$\left.\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|y_{mn}\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\le 1.$$

Then by using Minkowski's inequality, we have

$$\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\right.$$

$$\left.\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|(x_{mn}+y_{mn})\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}$$

$$\le\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}$$

$$+\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\right.$$

$$\left.\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|y_{mn}\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\le 1,$$

so we have $g(x+y)=\inf\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\right.$

$$\left.\left[u_{mn}f_{mn}\left(\left\|\left|(x_{mn}+y_{mn})\right|^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\le 1$$

$$\le\inf\left(\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\right.\right.$$

$$\left.\left.\left[u_{mn}f_{mn}\left(\left\|\left(((m+n)!\left|x_{mn}\right|)\right)^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\le 1\right)+$$

$$\inf\left(\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\right.\right.$$

$$\left.\left.\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|y_{mn}\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\le 1\right)$$

Therefore,

$$g(x+y)\le g(x)+g(y).$$

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x)=\inf\left(\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\right.\right.$$

$$\left.\left.\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\le 1\right)$$

where $t=\frac{1}{|\lambda|}$. Since $|y|^{q_{mn}}\le\max\left(1,|\lambda|^{\sup p_{mn}}\right)$, we have

$$g(\lambda x)\le\max\left(1,|\lambda|^{\sup p_{mn}}\right)\inf\left(t^{q_{mn}/H}:\left(\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\right.\right.$$

$$\left.\left.\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\le 1\right)$$

This completes the proof.

3.3. Theorem

Let $f = (f_{mn})$ be a Musielak-modulus function. Then the following statements are equivalent

i) $\left[\Lambda^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\Lambda^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$

ii) $\left[\chi^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\chi^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$

iii) $\sup_{\eta,\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[u_{mn}f_{mn}\left(\left\|\left|x_{m,n}\right|^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|\right)_p\right]^{q_{mn}}<\infty$

Proof

(i) $\Rightarrow$ (ii) is obvious, since

$$\left[\chi^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\Lambda^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right].$$

(ii) $\Rightarrow$ (iii) Suppose

$$\left[\chi^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\Lambda^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right].$$

and let (iii) does not hold. Then

$$\sup_{\eta,\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[u_{mn}f_{mn}\left(\left\|\left|x_{m,n}\right|^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}=\infty.$$

and therefore there is a sequence $(\eta_i\mu_j)$ of positive integers such that (3.1)

$$\frac{1}{\lambda_{\eta_i\mu_j}}\sum_{m\in I_{\eta_i}}\sum_{n\in I_{\mu_j}}\left[f_{mn}u_{mn}\left(\left\|\frac{(ij)^{-(m+n)}}{(m+n)!},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}>\frac{(ij)^{(m+n)}}{(m+n)!},$$

$i, j = 1, 2, \ldots$

Define $x = (x_{mn})$ by

$$x=(x_{mn})=\begin{cases}\frac{(ij)^{-(m+n)}}{(m+n)!},1\le m\le I_{\eta_i};1\le n\le I_{\mu_j}, & \text{if } i,j=1,2,3,\ldots;\\ 0, & \text{if } m\ge I_{\eta_i},n\ge I_{\mu_j}.\end{cases}$$

Then $x=(x_{mn})\in\left[\chi^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$ but $x=(x_{mn})\notin\left[\Lambda^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$ which contradicts (ii). Hence

(iii) must hold.

(iii) $\Rightarrow$ (i). Suppose $x=(x_{mn})\in\left[\Lambda^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$ and $x=(x_{mn})\notin\left[\Lambda^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$. Then (3.2)

$$\sup_{\eta,\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[u_{mn}f_{mn}\left(\left\|\left|x_{m,n}\right|^{1/m+n},\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}=\infty.$$

which contradicts (iii). Hence (i) must hold.

3.4. Theorem

Let $1\le q_{mn}\le \sup_{mn}q_{mn}<\infty$. Then the following statements are equivalent.

i) $\left[\chi^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\chi^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$,

ii) $\left[\chi^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\Lambda^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$,

iii)

$$\inf_{\eta,\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[u_{mn}f_{mn}\left(\left\|\left|x_{m,n}\right|^{1/m+n},\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}>0,$$

t > 0.

Proof

(i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) Suppose $\left[\chi^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\Lambda^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$ and let (iii) does not hold. Then (3.3)

$$\inf_{\eta,\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[u_{mn}f_{mn}\left(\left\|\left|x_{m,n}\right|^{1/m+n},\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}=0, t>0.$$

We can choose an index sequence $(\eta_i\mu_j)$ such that

$$\frac{1}{\lambda_{\eta_i\mu_j}}\sum_{m\in I_{\eta_i}}\sum_{n\in I_{\mu_j}}\left[u_{mn}f_{mn}\left(\left\|\frac{(ij)^{-(m+n)}}{(m+n)!},\left(d(x_1),d(x_2)\dots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}>\frac{(ij)^{(m+n)}}{(m+n)!}, i,j=1,2,\dots$$

Define $x=(x_{mn})$ by

$$x=(x_{mn})=\begin{cases}\frac{(ij)^{(m+n)}}{(m+n)!}, 1\le m\le I_{\eta_i}; 1\le n\le I_{\mu_j}, & if\ i,j=1,2,3,\dots;\\ 0, & if\ m\ge I_{\eta_i}, n\ge I_{\mu_j}.\end{cases}$$

Thus by (3.3) we have $x=(x_{mn})\in\left[\chi^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$ but $x=(x_{mn})\notin\left[\Lambda^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i). Let $x=(x_{mn})\in\left[\chi^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$ That is, (3.4)

$$\inf_{\eta,\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{m,n}\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}=0,$$

Suppose (iii) hold and $x=(x_{mn})\notin\left[\chi^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$. Then for some number $\epsilon>0$ and index $\eta_0\mu_0$, we have

$$\left[f_{mn}(\epsilon_0)\right]^{q_{mn}}\le\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{m,n}\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}$$

and consequently (3.4)

$$\lim_{\eta\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[f_{mn}(\epsilon_0)\right]^{q_{mn}}=0,$$

which contradicts (iii). Hence $\left[\chi^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\chi^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$.

This completes the proof.

3.5. Theorem

Let $f=(f_{mn})$ be a Musielak-modulus function. Let $1\le q_{mn}\le\sup_{mn}q_{mn}<\infty$. Then $\left[\Lambda^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\chi^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$ hold if and only if

(3.5)

$$\lim_{\eta,\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{m,n}\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}=\infty$$

Proof

Suppose $\left[\Lambda^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\chi^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$ and let (3.5) does not hold. There is a number $t_0>0$ and an index sequence $(\eta_i\mu_j)$ such that (3.6)

$$\frac{1}{\lambda_{\eta_i\mu_j}}\sum_{m\in I_{\eta_i}}\sum_{n\in I_{\mu_j}}\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{m,n}\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\le N<\infty, i=1,2,\dots$$

Define $x=(x_{mn})$ by

$$x=(x_{mn})=\begin{cases}(t_0)^{(m+n)}, 1\le m\le I_{\eta_i}; 1\le n\le I_{\mu_j}, & if\ i,j=1,2,3,\dots;\\ 0, & if\ m\ge I_{\eta_i}, n\ge I_{\mu_j}.\end{cases}$$

Therefore, $x=(x_{mn})\in\left[\Lambda^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$ but $x=(x_{mn})\notin\left[\chi^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$. Hence (3.5) must hold.

Conversely, if $x=(x_{mn})\in\left[\Lambda^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$, then for each s, η and μ

(3.7)

$$\frac{1}{\lambda_{\eta_i\mu_j}}\sum_{m\in I_{\eta_i}}\sum_{n\in I_{\mu_j}}\left[u_{mn}f_{mn}\left(\left\|\left|x_{m,n}\right|^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\leq N<\infty$$

Suppose that $x=(x_{mn})\notin\left[\chi^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$. Then for some number $\epsilon_0>0$ we have

$$\left[f_{mn}(\epsilon_0)\right]^{q_{mn}}\leq\left[u_{mn}f_{mn}\left(\left\|\left|x_{m,n}\right|^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}$$

and hence for m, n we get

$$\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[f_{mn}(\epsilon_0)\right]^{q_{mn}}\leq N<\infty,$$

for some N > 0, which contradicts (3.5). Hence $\left[\Lambda^{2q}_{f,V,u},\left\|\left(d(x_1)d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\chi^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$. This completes the proof.

3.6. Theorem

Let $f=(f_{mn})$ be a Musielak-modulus function. Let $1\leq q_{mn}\leq \sup_{mn}q_{mn}<\infty$. Then $\left[\Lambda^{2q}_{V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\chi^{2q}_{f,V,u},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$ hold if and only if

(3.8)

$$\lim_{\eta,\mu}\frac{1}{\lambda_{\eta\mu}}\sum_{m\in I_\eta}\sum_{n\in I_\mu}\left[u_{mn}f_{mn}\left(\left\|\left((m+n)!\left|x_{m,n}\right|\right)^{1/m+n},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}=0$$

Proof: It is similar to above. Therefore we omit the proof.

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