On The Exact Solution Of High Even-Order Differential Equation

Abdon Atangana [1], Suares Clovis Oukouomi Noutchie [2]

[1] Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences
University of the Free State, 9300 Bloemfontein, South Africa
[2] Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa
23238917@nwu.ac.za

Abstract: In this work, the homotopy decomposition method (HDM) was applied to derive exact solutions of High even-order differential equation. The reliability of HDM and the reduction in computations give HDM a wider applicability. In all examples, in the limit of infinitely many terms the HDM yields the exact solution. A comparison with the exact solution reveals that HDM is simple, efficient, reliable, and converge very rapidly. In addition, the calculations involved in HDM are very simple and straightforward. It is demonstrated that HDM is a powerful and efficient tool for FPDEs. It was also demonstrated that HDM is more efficient than the Bernstein Galerkin approximation, Bernstein Petrov-Galerkin approximation, Non polynomial spline method, ADM (Adomian decomposition method), VIM (Variational iteration method), HAM (Homotopy analysis method) and HPM (Homotopy perturbation method).

[Atangana A, Oukouomi Noutchie SC. **On the Exact Solution of High Even-Order Differential Equation**. *Life Sci J* 2013;10(3):246-252] (ISSN:1097-8135). http://www.lifesciencesite.com. 39

Keywords: High even-order differential equation; homotopy decomposition method; complexity; advantages.

1. Introduction

Higher order even differential equations arise in many fields. When instability sets in an ordinary convection, it is modelled by tenth-order boundary value problem [1]. Scott and Watts [2] described several computer codes that were developed using the superposition and orthonormalization technique and invariant imbedding. Twizell et al [3] developed numerical methods for 8^{th}-, 10^{th}-, and 12^{th}-order eigen-value problems arising in thermal instability. Siddiqi and Twizell [4] presented the solution of 10th-order boundary value problem using 10^{th} degree spline. Siddiqi and Akram [5] developed the solution of 10th-order boundary value problems using non-polynomial spline. Siddiqi and Akram [6] presented the solution of 10th-order boundary value problem by using 11^{th} degree spline. Rashidinia et al [7] developed numerical methods for 8th-order boundary value problem using non-polynomial spline. Dijidejeli and Twizell [8] derived numerical method for special nonlinear boundary-value problems of 2^{nd} order. Abdellah Lamnii et al [9] developed and analyzed numerical method for approximating solutions of some general linear boundary value problems. Ramadan et al [10] have been applied non-polynomial spline function for approximating solutions of $2\mu^{th}$ order two point BVPs.

The paper is structured as follows: In Section 2, we present the basic ideal of the homotopy decomposition method for solving high even orders partial differential equations. We present the application of the HDM for Higher order even differential equations and numerical results in Section 3. In section 4 we present the complexity of the method for solving high even order partial differential equations. The conclusions are then given in the final Section 5.

1. **Homotopy decomposition method** [12,25]

To illustrate the basic idea of this method we consider a general nonlinear non-homogeneous differential equation with initial conditions of the following form

$$\frac{\partial^m U(x)}{\partial x^m} = L(U(x)) + N(U(x)) + f(x), \quad m = 1,2,3\ldots \tag{2.1}$$

Subject to the initial condition

$$\frac{\partial^i U(0)}{\partial x^i} = y_i, \qquad \frac{\partial^{m-1} U(0)}{\partial x^{m-1}} = 0\ , i = 0,1,2\ldots m-2$$

Where, f is a known function, N is the general nonlinear differential operator and L represents a linear differential operator. The method first step here is to apply the inverse operator of $\frac{\partial^m}{\partial x^m}$ on both sides of equation (2.1) to obtain

$$U(x) = \sum_{k=0}^{m-1} \frac{x^i}{i!} y_i + \int_0^x \int_0^{x_1} .. \int_0^{x_{m-1}} L(U(\tau)) + N(U(\tau)) + f(\tau) d\tau .. dx \tag{2.2}$$

The multi-integral in Eq (2.2) can be transformed to

$$\int_0^x\int_0^{x_1}..\int_0^{x_{m-1}} L(U(\tau))+N(U(\tau))+f(\tau)d\tau..dx =$$
$$\frac{1}{(m-1)!}\int_0^x (x-\tau)^{m-1}L(U(\tau))+N(U(\tau))+f(\tau)d\tau.$$

Equation (2.2) can then be reformulated as
$$U(x)=\sum_{i=0}^{m-1}\frac{x^i y_i}{i!}+\frac{1}{(m-1)!}\int_0^x (x-\tau)^{m-1}L(U(\tau))+N(U(\tau))+f(\tau)d\tau$$
(2.3)

Using the Homotopy scheme the solution of the above integral equation is given in series form as:
$$U(x,p)=\sum_{n=0}^{\infty}p^n U_n(x) \qquad (2.4)$$
$$U(x)=\lim_{p\to 1}U(x,p)$$
and the nonlinear term can be decomposed as
$$NU(x)=\sum_{n=1}^{\infty}p^n\mathcal{H}_n(U)$$
Where $p\epsilon\ (0,\ 1]$ is an embedding parameter. $\mathcal{H}_n(U)$ is the He's polynomials [11] that can be generated by
$$\mathcal{H}_n(U_0,\cdots\cdots,U_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{j=0}^{n}p^jU_j(x)\right)\right],$$
$n=0,1,2\cdots\cdots$

The homotopy decomposition method is obtained by the beautiful coupling of decomposition method with He's polynomials and is given by
$$\sum_{n=0}^{\infty}p^nU_n(x)=T(x)+$$
$$p\frac{1}{(m-1)!}\int_0^x(x-\tau)^{m-1}\left[\begin{array}{c}f(\tau)+\\ L\left(\sum_{n=0}^{\infty}p^nU_n(\tau)\right)+\sum_{n=0}^{\infty}p^n\mathcal{H}_n(U)\end{array}\right]d\tau$$
with
$$T(x)=\sum_{i=0}^{m-1}\frac{x^i}{i!} \qquad (2.5)$$
Comparing the terms of same powers of p gives solutions of various orders. The initial guess of the approximation is $T(x)$. The convergence of the method can be found in [12].

2.1. Advantages of the method

Numerical method for solving ODEs can be viewed as insurance companies. It is important to know what advantages are offered by the insurance company, before becoming a member. Therefore in this section we present the advantages offered by HDM. The homotopy decomposition method is chosen to solve this nonlinear problem because of the following advantages the method has over the existing methods.

1- Method does not require the linearization or assumptions of weak nonlinearity
2- The solutions are not generated in the form of general solution as in Adomian decomposition method. With ADM the recursive formula allows repetition of terms in the case of nonhomgeneous differential equation, and this lead to the noisy solution [13]
3- The solution obtained is noise free compare to the variational iteration method [14]
4- No correctional function is required as in the case of the variational homotopy decomposition method
5- No Lagrange multiplier is required in the case of the variational iteration method [14]
6- it is more realistic compared to the method of simplifying the physical problems
7- If the exact solution of the differential equation exists, the approximated solution via the method converge to the exact solution [12]
8- A construction of a homotopy $v(r,p):\Omega\times(0,1]$ is not needed as in the case of the homotopy perturbation method [11]
9- the calculations involved in HDM are very simple and straightforward
10- HDM provides us with a convenient way to control the convergence of approximation series without adapting h, as in the case of [15-16] which is a fundamental qualitative difference in analysis between HDM and other methods.

2.3 Complexity of HDM

It is very important to test the computational complexity of the method or algorithm. Complexity of an algorithm is the study of how long a program will take to run, depending on the size of its input and long of loops made inside the code [12]. We compute a numerical example which is solved by the homotopy decomposition method. The code has been presented with Mathematica 8 according to the following code.

Step 1: Set $m\leftarrow 0$

Step 2: Calculated the recursive relation after the comparison of the terms of the same power is done.

Step 3: If $\|U_{n+1}(x)-U_n(x)\|<r$ with r the ratio of the neighbourhood of the exact solution [At] then go to step 4, else $m\leftarrow m+1$ and go to step 2

Step 4: Print out:

$$U(x) = \sum_{n=0}^{\infty} U_n(x)$$

as the approximate of the exact solution [12].

Lemma 1: *If the exact solution of the partial differential equation (2.1) exists, then* $\|U_{n+1}(x) - U_n(x)\| < r$ *for all* $x \in X$

Proof: Let $x \in T$, then since the exact solution exists, then we have that following

$$\|U_{n+1}(x) - U_n(x)\| = \|U_{n+1}(x) - U(x) + U(x) - U_n(x)\|$$

$$\leq \|U_{n+1}(x) - U(x)\| + \|U_n(x) - U(x)\| \leq \frac{r}{2} + \frac{r}{2} = r$$

The last inequality follows from [12]

Lemma 2: *The complexity of the homotopy decomposition method is of order* $O(n)$

Proof: The number of computations including product, addition, subtraction and division are

In step 2

U_0: 0 because, obtains directly from the initial guess [12]

U_1: 3

U_n:3

Now in step 4 the total number of computations is equal to $\sum_{j=0}^{n} U_j(x,t) = 3n = O(n)$

2. Application

In learning science examples are useful than rules'' (Isaac Newton). In this section we apply this method for solving High even-order differential.

Problem 1: Consider the boundary value problem [17] (3.1)

$$u_{x,x}(x) - u(x) = (4 - 2x^2)sin(x) + 4xcos(x), x \in [0,1]$$

Subject to the boundary conditions $u(0) = u(1) = 0$

Following the discussion presented in section 2 we arrive at the following (3.2)

$$\sum_{n=0}^{\infty} p^n u_n(x) = u(0) + \int_0^x \left(\sum_{n=0}^{\infty} p^n u_n(t) + (4 - 2t^2)sin(t) + 4tcos(t)\right)(x - t)dt$$

Comparing the terms of the same power of p yields: (3.3)

$$p^0: u_0(x) = 0, u_0(0) = -x$$

$$p^1: u_1(x) = \int_0^x (x - t)\left(u_0 + (4 - 2t^2)sin(t) + 4tcos(t)\right)dt,$$

$$p^n: u_n(x) = \int_0^x (x - t)u_{n-1}(t)dt, u_n(0) = 0$$

and the following solutions are read: (3.4)

$$u_0(x) = -x$$

$$u_1(x) = 4x - \frac{x^3}{6} + 4xcos(x) - 8\,sin(x) + 2x^2\,sin(x)\ )$$

$$u_2(x) = -16x + \frac{2x^3}{3} - \frac{x^5}{120} - 12xcos(x) + 28\sin(x) - 2x^2\,sin(x)$$

$$u_3(x) = 44x - \frac{8x^3}{3} + \frac{x^5}{30} - \frac{x^7}{5040} + 20xcos(x) - 64\sin(x) + 2x^2\,sin(x)$$

$$u_4(x) = -88x + \frac{22x^3}{3} - \frac{2x^5}{15} + \frac{x^7}{1260} - \frac{x^9}{362880} - 28xcos(x) + 116\sin(x) - 2x^2\,sin(x)$$

$$u_5(x) = 148x - \frac{44x^3}{3} + \frac{11x^5}{30} - \frac{x^7}{315} + \frac{x^9}{90720} - \frac{x^{11}}{39916800} + 36xcos(x) - 184\sin(x) + 2x^2\,sin(x)$$

$$u_6(x) = -224x + \frac{74x^3}{3} - \frac{11x^5}{15} + \frac{11x^7}{1260} - \frac{x^9}{22680} + \frac{x^{11}}{9979200} - \frac{x^{13}}{6227020800} - 44xcos(x) + 268\sin(x) - 2x^2\,sin(x)$$

$$u_7(x) = 316x - \frac{112x^3}{3} + \frac{37x^5}{30} - \frac{11x^7}{630} + \frac{11x^9}{90720} - \frac{x^{11}}{2494800} + \frac{x^{13}}{1556735200} - \frac{x^{15}}{1307674368000} + 52xcos(x) - 368\sin(x) + 2x^2\,sin(x)$$

Using the package program of Mathematica, in the same manner one can obtain the rest of the components. But, here, 8 terms were computed and the asymptotic solution is given by:

$$u_{N=7}(x) = \sum_{n=0}^{7} u_n(x)$$

Notice that the Taylor series of order 15 of the above solution gives: (3.5)

$$-x + \frac{7x^3}{6} - \frac{7x^5}{40} + \frac{43x^7}{5040} - \frac{73x^9}{356400} + \frac{37x^{11}}{13305600} - \frac{211x^{15}}{1307674368000} + o(x^{16})$$

Thus, the approximated solution via HDM is given below as (3.6)

$$u(x) = \sum_{n=0}^{\infty} (x^2 - 1)\frac{(-1)^n x^{2n+1}}{(2n+1)} = (x^2 - 1)\sin(x)$$

This is the exact solution of problem 1.

To assess the accuracy of the HDM for solving the High even-order differential, we establish in the table 1, the error committed by choosing the 7 firsts terms of the series decomposition, which is given below

Table 1: Comparison of numerical values with HDM, the exact solution and absolute error

x	HDM	Exact	error
0.1	-0.0988351	-0.0988351	$2.55351\text{x}10^{-15}$
0.2	-0.190723	-0.190723	$1.44329\text{x}10^{-14}$
0.3	-0.268923	-0.268923	$5.66214\text{x}10^{-15}$
0.5	-0.359569	-0.359569	$8.21565\text{x}10^{-15}$
0.7	-0.328551	-0.328551	$5.77316\text{x}10^{-15}$
0.9	-0.148832	-0.148832	$2.44249\text{x}10^{-14}$
0.98	-0.0328877	-0.0328877	$-1.53141\text{x}10^{-14}$

An Eton proverb says "one *image is equivalent to one thousand words*", the following figure shows the comparison between the approximated solution and the exact for $n = 7$.

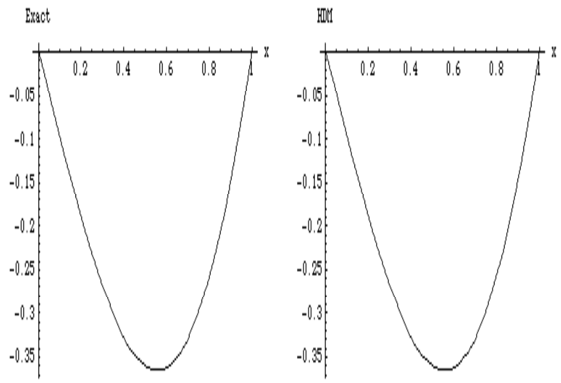

Figure 1: Comparison for n = 7

Remark 1: The same problem was solved by several Authors see for example [18], the obtained an error of order $10x^{-15}$ for N = 18, but here we do obtain error for N= 7 only, indication that the HDM is very accurate, that the method in [18]. Note that, the Petrov-Galerkin method generates a sequence of approximate solutions that satisfy a weak form of the original differential equation as tested against polynomials in a dual space, but the HDM does not need to do it.

Problem 3: Consider the sixth-order BVP [19, 20, 21] (3.7)

$$u^{(6)}(x) - u(x) = -6e^{x}, \quad x \in [0,1]$$

Subject to the boundary conditions

$$u(0) = 1, u'(0) = 0 \; and \; u''(0) = -1$$

$$u(1) = 0, u'(1) = -e \; and \; u''(1) = -2e$$

Following the HDM steps we arrive at the following integral equations: (3.8)

$$p^{0}: u_0(x) = T(x), u_0(0) = 1$$

$$p^{1}: u_1(x) = \frac{1}{5!}\int_0^x (x-t)^5 \left(u_0(t) - 6e^{t}\right)dt, u_1(0) = 0$$

$$p^{n}: u_n(x) = \frac{1}{5!}\int_0^x (x-t)^5 u_{n-1}(t)dt, u_n(x) = 0, n \geq 2$$

The following solutions are obtained: (3.9)

$$u_0(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} - \frac{x^5}{30}$$

$$u_1(x) = \frac{1}{120}\left(720 - 720e^{x} + 720x + 360x^2 + 120x^3 + 30x^4 + 6x^5 + \frac{x^6}{6} - \frac{x^8}{336} - \frac{x^9}{1512} - \frac{x^{10}}{10080} - \frac{x^{11}}{83160}\right)$$

$$u_2(x) = \frac{1}{120}\left(720 - 720e^{x} + 720x + 360x^2 + 120x^3 + 30x^4 + 6x^5 + x^6 + \frac{x^7}{7} + \frac{x^8}{56} + \frac{x^9}{504} + \frac{x^{10}}{5040} + \frac{x^{11}}{55440} + \frac{x^{12}}{3991680} - \frac{x^{14}}{726485760} - \frac{x^{15}}{5448643200} - \frac{x^{16}}{58118860800} - \frac{x^{17}}{741015475200}\right)$$

One can obtain the rest of the components. But, here, 3 terms were computed and the asymptotic solution is given by: (3.10)

$$u_{N=3}(x) = \sum_{n=0}^{2} u_n(x)$$

$$= e^{x} - \sum_{n=0}^{17} \frac{x^{n+1}}{n!}$$

Therefore the approximated solution for infinite N gives

$$u(x) = e^{x} - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = (1-x)e^{x}$$

This is the exact solution of problem 2.

To assess the accuracy of the HDM for solving the High even-order differential, we establish in the table 2, the error committed by choosing the 3 firsts terms of the series decomposition which is given below

Table 2: Comparison of numerical values with HDM, the exact solution and absolute error.

x	HDM	Exact	error
0.1	0.9954654	0.9954654	$9.07513x10^{-16}$
0.2	0.977122	0.977122	$2.13296x10^{-16}$
0.3	0.944901	0.944901	$1.04212x10^{-15}$
0.5	0.824361	0.824361	$1.77337x10^{-15}$
0.7	0.604126	0.604126	$3.47412x10^{-15}$
0.9	0.24596	0.24596	$1.83266x10^{-17}$
0.98	0.0532891	0.0532891	$3.65965x10^{-15}$

Remark 2: The same problem was solved by several Authors see for example [18], the obtained an error of order 10^{-16} for N = 18, but here we do obtain error of order 10^{-17} for N= 7 only, indication that the HDM is very accurate, that the method in [18].

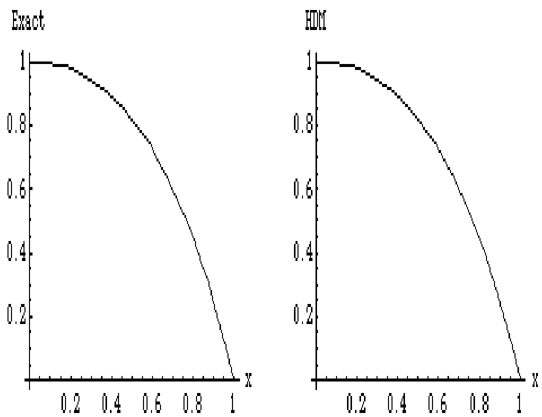

Figure 2: Comparison of the exact with approximated for n = 3

Problem 2: Consider the fourth-order two point boundary value problem [21] (3.12)

$$u^{(4)}(x) - 3u(x) = -2e^x, \quad x \in [0,1]$$

Subject to the boundary conditions

$$u(0) = 1, u(1) = e, u'(0) = 1 \; and \; u'(1) = e$$

Following the HDM steps we arrive at the following integral equations: (3.13)

$$p^0: u_0(x) = T(x), u_0(0) = 1$$

$$p^1: u_1(x) = \frac{1}{3!}\int_0^x (x-t)^3\,(3u_0(t) - 2e^t)dt, u_1(0) = 0$$

$$p^n: u_n(x) = \frac{1}{3!}\int_0^x (x-t)^3 3u_{n-1}(t)dt\,, u_n(x) = 0, n \geq 2$$

The following solutions are obtained: (3.14)

$$u_0(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$u_1(x) = \frac{1}{6}\left(12 - 12e^x + 12x + 6x^2 + 2x^3 + \frac{3x^4}{4} + \frac{3x^5}{20} + \frac{x^6}{40} + \frac{x^7}{280}\right)$$

$$\frac{1}{6}\left(36 - 36e^x + 36x + 18x^2 + 6x^3 + \frac{3x^4}{2} + \frac{3x^5}{10} + \frac{x^6}{20} + \frac{x^7}{140} + \frac{3x^8}{2240} + \frac{x^9}{6720} + \frac{x^{10}}{67200} + \frac{x^{11}}{739200}\right)$$

Using the package Mathematica, in the same manner one can obtain the rest of the components. But, here, 3 terms were computed and the asymptotic solution is given by:

$$u_{N=3}(x) = 9 + 9x + \frac{9x^2}{2} + \frac{3x^3}{2} + \frac{3x^4}{8} + \frac{3x^5}{40} + \frac{x^6}{80} + \frac{x^7}{560} + \frac{x^8}{4460} + \frac{x^9}{40320} + \frac{x^{10}}{403200} + \frac{x^{11}}{4435200} - 8e^x$$

$$u_{N=3}(x) = 9\sum_{n=0}^{11}\frac{x^n}{n!} - 8e^x$$

Now taking the limit as N tends to infinity yields to:

$$u(x) = 9e^x - 8e^x = e^x$$

This is the exact solution to problem 1

Problem 4: We consider the following boundary-value problem (3.17)

$$u^{(10)}(x) - xu(x) = -89(21x + x^2 - x^3)e^x, \quad -1 \leq x \leq 1$$

$$u(x) = (1 - x^2)e^x$$

Following the HDM steps we arrive at the following integral equations: (3.18)

$$p^0: u_0(x) = 1 + x - \frac{x^2}{2} - \frac{5x^3}{6} - \frac{11x^4}{24} - \frac{19x^5}{120} - \frac{29x^6}{720} - \frac{41x^7}{5040} - \frac{11x^8}{8064} - \frac{71x^9}{362880}, u_0(0) = 1$$

$$p^1: u_1(x) = \frac{1}{9!}\int_0^x (x-t)^9\,(xu_0(t) - 89(21t + t^2 - t^3)e^t)dt, u_1(0) = 0$$

$$p^n: u_n(x) = \frac{1}{9!}\int_0^x (x-t)^9 3u_{n-1}(t)dt\,, u_n(x) = 0, n \geq 2$$

The following solutions are obtained: (3.19)

$$u_0(x) = 1 + x - \frac{x^2}{2} - \frac{5x^3}{6} - \frac{11x^4}{24} - \frac{19x^5}{120} - \frac{29x^6}{720} - \frac{41x^7}{5040} - \frac{11x^8}{8064} - \frac{71x^9}{362880}$$

$$u_1(x) = \frac{1}{9!}\left(x\left(355622400 + \frac{475009920}{x} + 129366720x + 30360960x^2 + 5155920x^3 + 677376x^4 + 73080x^5 + 7056x^6 + 693x^7 + \frac{1}{670442572800x}\left(243290200817664000e^x(-1309 + 329x - 31x^2 + x^3) - x^{11}(-6094932480 - 1015822080x + 117210240x^2 + 55814400x^3 + 10232640x^4 + 1325592x^5 + 138852x^6 + 12464x^7 + 990x^8 + 71x^9)\right)\right)\right)$$

Using the package of program Mathematica, in the same manner one can obtain the rest of the components. But, here, 2 terms were computed and the asymptotic solution is given by:

$$u_{N=2}(x) = u_0(x) + u_1(x)$$

Taking the Taylor series of the above solution we obtained:

$$u(x) = \sum_{n=0}^{\infty}(1 - x^2)\frac{x^n}{n!} = (1 - x^2)e^x$$

This is the exact solution of problem 4.

To assess the accuracy of the HDM for solving the High even-order differential, we establish in the table 3 the error committed by choosing the 2 firsts terms of the series decomposition which is given below

Table 3: Comparison of numerical values with HDM, the exact solution and absolute error

x	HDM	Exact	error
0.1	1.09412	1.09412	2.65472×10^{-13}
0.2	1.17255	1.17255	2.0357×10^{-13}
0.3	1.22837	1.22837	1.86974×10^{-13}
0.5	1.23654	1.23654	1.52119×10^{-13}
0.7	1.02701	1.02701	8.05206×10^{-14}
0.9	0.467325	0.467325	3.73951×10^{-13}
0.98	0.105512	0.105512	1.06939×10^{-12}

Remark 3: The same problem was solved by several Authors see for example [24], the obtained an error of order 10^{-8} for N = 18, but here we do obtain error of order 10^{-14} for N= 2 only, indication that the HDM is very accurate, that the method in [24].
An Eton proverb says "one *image is equivalent to one thousand words*", the following figure shows the comparison between the approximated solution and the exact for $n = 2$.

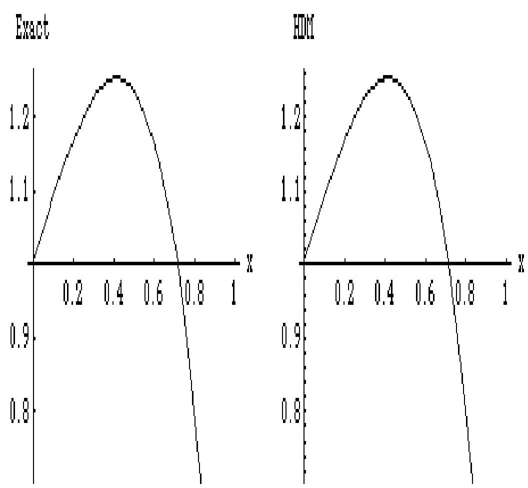

Figure 3: Comparison between exact and approximated for n =2

Problem 5: We Consider the following boundary-value problem [22, 23] (In [22] the boundary conditions are different)

$$y^{(10)}(x) + 5y(x) = 10\cos(x) + 4(-1+x)\sin(x), \qquad 0 \le x \le 1$$

$$y(x) = (x-1)sin(x)$$

Following the HDM steps we arrive at the following integral equations: (3.21)

$$p^0: u_0(x) = T(x), \quad u_0(0) = 0$$

$$p^1: u_1(x) = \frac{1}{9!}\int_0^x (x-t)^9\left(-5u_0(t) + 10\cos(t) + 4(-1+x)\sin(t)\right)dt, u_1(0) = 0$$

$$p^n: u_n(x) = -\frac{1}{9!}\int_0^x (x-t)^9 5u_{n-1}(t)dt, u_n(x) = 0, n \ge 2$$

The following solutions are obtained: (3.22)

$$u_0(x) = -t + t^2 + \frac{t^3}{6} - \frac{t^4}{6} - \frac{t^5}{120} + \frac{t^6}{120} + \frac{t^7}{5040} - \frac{t^8}{5040} - \frac{t^9}{362880}$$

$$u_1(x) = \frac{1}{9!}\Bigg(7257600 - 3262880x - 3265920x^2 + 60480x^3 + 241920x^4 - 3024x^5 - 7056x^6 + 72x^7 + 108x^8 - x^9 + \frac{x^{11}}{22} - \frac{x^{12}}{132} - \frac{x^{13}}{3432} + \frac{x^{14}}{12012} + \frac{x^{15}}{720720} - \frac{x^{16}}{1921920} - \frac{x^{17}}{196035840} + \frac{x^{18}}{441080640} + \frac{x^{19}}{67044257280} - 18144000\cos(x) + 1451520\sin(x) - 1451520x\sin(x)\Bigg)$$

One can obtain the rest of the components. But, here, 2 terms were computed and the asymptotic solution is given by: (3.23)

$$u_{N=2}(x) = u_0(x) + u_1(x)$$

Taking the Taylor series of the above, we obtained

$$u(x) = \sum_{n=0}^{\infty}(x-1)\frac{x^{2n+1}}{(2n+1)!} = (x-1)\sin(x)$$

This is the exact solution to problem 5
To assess the accuracy of the HDM for solving the High even-order differential, we establish in the table 3 the error committed by choosing the 2 firsts terms of the series decomposition which is given below

Table 4: Comparison of numerical values with HDM, the exact solution and absolute error

x	HDM	Exact	error
0.1	-0.0898501	-0.0898501	4.21378×10^{-15}
0.2	-0.158935	-0.158935	2.327×10^{-15}
0.3	-0.206864	-0.206864	1.14278×10^{-14}
0.5	-0.239713	-0.239713	3.6968×10^{-15}
0.7	-0.193265	-0.193265	6.74302×10^{-15}
0.9	-0.0783327	-0.0783327	1.08212×10^{-14}
0.98	-0.0166099	-0.0166099	4.94549×10^{-15}

Remark 4: The same problem was solved by several Authors see for example [24], the obtained an error of order 10^{-6} for N = 22, but here we do obtain error of order 10^{-12} for N= 2 only, indication that the HDM is very accurate, that the method in [24].

3. Conclusion

In this paper, we put into practice a new analytical technique, the Homotopy Decomposition method, for solving Higher even order differential equations arise in many fields. Comparing the methodology of HDM to Bernstein Galerkin approximation (BGA), Bernstein Petrov-Galerkin approximation (BPGA), Non polynomial spline (NPS), HPM, ADM, VIM and HAM, it shows that the HDM has the advantages. In contrast to BPGM, the HDM is free from generating a sequence of approximate solutions that satisfy a weak form of the original differential equation as tested against polynomials in a dual space. We do not need to introduce some basic notation which will be used in the sequel like in the case of BGM. Disparate the ADM, the HDM is free from the need to use Adomian polynomials. In this method we do not need the Lagrange multiplier, correction functional, stationary conditions, or calculating integrals, which eliminate the complications that exist in the VIM. In contrast to the HAM, this method is not required to solve the functional equations in iteration each the efficiency of HAM is very much depended on choosing auxiliary parameter. In contract to HPM, we do not need to continuously deform a difficult problem to another that is easier to solve. We can easily conclude that the Homotopy Decomposition method is an efficient tool to solve approximate solution of Higher even order differential equations arise in many fields.

4. References

[1] S. Chandrasekhar. "Hydrodynamic and Hydromagnetic Stability". *Clarendon press, New York, 1981. Oxford,* 1961.

[2] M. Scott, H. Watts. "A systematized collection of codes for solving two-point BVPs, Numerical Methods for Differential Systems". *Academic press*, 1976.

[3] E. Twizell, A. Boutayeb, K. Djidjeli. "Numerical methods for eighth, tenth and twelfth-order eigenvalue problems arising in thermal instability". *Advances in Computational Mathematics*, no 2: pp 407–436, 1994.

[4] S. Siddiqi, E. Twizell. "Spline solutions of liner tenth-order boundary value problems". *International Journal of Computer International Journal of Computer*, vol 68, pp 345–362, 1998.

[5] S. Siddiqi, G. Akram. "Solution of 10th-order boundary value problems using non-polynomial spline technique". *Applied Mathematics and Computation*, vol.190, pp 641–651, 2007.

[6] S. Siddiqi, G. Akram. "Solution of tenth-order boundary value problems using eleventh degree spline". *Applied Mathematics and Computation*, vol.165, pp 115–127, 2007.

[7] J. Rashidinia, R. Jalilian, K. Farajeyan. "Spline approximate solution of eighth-order boundary-value problems". *International Journal of Computer Mathematics*, vol.86, pp 1319–1333, 2009.

[8] K. Dijidejeli, E. Twizell. "Numerical methods for special nonlinear boundary value problems of order 2nd". *Journal of Computational and Applied Mathematics*, vol.47,pp 35–45, , 1993.

[9] A. Lamnii, H. Mraoui, et al. "Spline solution of some linear boundary value problems". *Applied Mathematics E-Notes*, vol. 6, pp 171–178, 2008.

[10] M. Ramadan, I. Lashien, W. Zahra. "High order accuracy nonpolynomial spline solutions for 2μ^{th} order two point boundary value problems". *Applied Mathematics and Computation*, vol. 204, 920–927, 2008.

[11] J.H. He, "Homotopy perturbation technique", *Comput. Methods Appl. Mech. Eng*, vol.178, pp 257–262, 1999.

[12] A. Atangana and Aydin Secer. "Time-fractional Coupled-the Korteweg-de Vries Equations" Abstract Applied Analysis, In press (2013)

[13] Kurulay M., Secer A., Akinlar M.A., A "New Approximate Analytical Solution of Kuramoto –Sivashinsky Equation Using Homotopy Analysis Method", *Appl. Math. Inf. Sci*, vol. 7, no. 1, pp 267-271,2013

[14] Z. Atangana Abdon. "New Class of Boundary Value Problems", *Inf. Sci. Lett*. vol.1, no. 2, pp 67-76,2012.

[15] S.J. Liao, "An approximate solution technique not depending on small parameters: a special example", *Int. J. Non-Linear Mech*, vol.30 no.3, pp 371–380, 1995.

[16] S.J. Liao, "Boundary element method for general nonlinear differential operators", *Eng. Anal. Boundary Elem*, vol. 20, no.2, pp 91–99, 1997.

[17] M. El-gamel, "A comparison between the sinc-Galerkin and the modified decomposition methods for solving two-point boundary-value problems," *Journal of Computational Physics*, vol. 223, no. 1, pp. 369–383, 2007.

[18] E. H. Doha, A. H. Bhrawy, andM. A. Saker, **"**On the Derivatives of Bernstein Polynomials: An Application for the Solution of High Even-Order Differential Equations", *Boundary Value Problems,* vol.2011, Article ID 829543, 16 pages

[19] M. A. Ramadan, I. F. Lashien, andW. K. Zahra, "High order accuracy non-polynomial spline solutions for 2μth order two point boundary value problems," *Applied Mathematics and Computation*, vol. 204, no. 2, pp. 920–927, 2008.

[20] S. S. Siddiqi and G. Akram, "Septic spline solutions of sixth-order boundary value problems," Journal of Computational and Applied Mathematics, vol. 215, no. 1, pp. 288–301, 2008.

[21] A. Wazwaz, "The numerical solution of sixth-order boundary value problems by the modified decomposition method," *Applied Mathematics and Computation*, vol. 118, no. 2-3, pp. 311–325, 2001.

[22] S. Siddiqi, G. Akram. "Solution of tenth-order boundary value problems using eleventh degree spline". *Applied Mathematics and Computation*, vol.165, pp 115–127, 2007.

[23] S. Siddiqi, E. Twizell. "Spline solutions of liner tenth-order boundary value problems". *International Journal of Computer International Journal of Computer*, vol. 68 pp 345–362, 1998.

[24] J. Rashidini, R. Jalilian and K. Farajeyan, "Non polynomial spline solutions for special linear tenth-order boundary value problems", *England, UK World Journal of Modelling and Simulation,* vol. 7, no. 1, pp. 40-51, 2011

[25] Atangana A., Botha J.F. "Analytical solution of groundwater flow equation via Homotopy Decomposition Method, *J Earth Sci Climate Change,* vol. 3, pp115. doi:10.4172/2157-7617.1000115 (2012).

7/2/2013