Some two-step simultaneous methods for determining all the roots of a non-linear equation

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Abstract: We construct some two-step simultaneous methods for finding all the real and complex roots of a non-linear equation. The convergence analysis of these methods is also discussed. The methods are then compared numerically. It was found that the methods are very effective, efficient and provide good numerical results.
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1 Introduction

Consider the non-linear equation

$$f(x)=0 \quad (1)$$

The methods for finding simultaneously all the zeros of non-linear equations are very attractive as compared to finding the single root at a time. These methods are more stable, have wider region of stability and can be implemented for the parallel computing see [1-3,5-14].

2 Two-step Simultaneous Methods for Finding Distinct Roots

In this section, we develop two-step iterative methods for the simultaneous approximation of all the zeros of a non-linear equation **Error! Reference source not found.**.

Let us consider the methods which are derived from integral inequalities by Mir and Naila [1]

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.\frac{f(x_n)}{f(x_n)-f(y_n)}, \quad (2)$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.\frac{f(x_n)+f(y_n)}{f(x_n)}, \quad (3)$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f(x_n)}{f'(x_n)}.\frac{f(y_n)(f(x_n)+f(y_n))}{f^2(x_n)-f(x_n)f(y_n)-3f^2(y_n)} \quad (3)$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} * \frac{f(x_n)+f(y_n)}{f(x_n)-f(y_n)}, \quad (5)$$

for determining zeros of single variable non-linear equation **Error! Reference source not found.**.

Let

$$W_i(x_i) = \frac{f(x_i)}{\prod_{\substack{j\neq i \\ j=1}}^{n}(x_i - x_j)} \quad \text{(Weierstrass'Correction)}. \quad (5)$$

Then, replacing $f(x)/f'(x)$ by $W_i(x_i)$ in the methods (1), (2), (3) and (4), we get the following(1) two-step methods for determining simultaneously all real and complex zeros of a non-linear equation **Error! Reference source not found.**:

$$\begin{cases} y_i = x_i - W_i(x_i), \\ z_i = x_i - W_i(x_i)\dfrac{f(x_i)}{f(x_i)-f(y_i)}, \end{cases} \quad (6)$$

(2)

$$\begin{cases} y_i = x_i - W_i(x_i), \\ z_i = x_i - W_i(x_i)\dfrac{f(x_i)+f(y_i)}{f(x_i)}, \end{cases} \quad (7)$$

$$\begin{cases} y_i = x_i - W_i(x_i), \\ z_i = y_i - W_i(x_i)\dfrac{f(y_i)(f(x_i)+f(y_i))}{f^2(x_i)-f(x_i)f(y_i)-3f^2(y_i)}, \end{cases} \tag{9}$$

and

$$\begin{cases} y_i = x_i - W_i(x_i), \\ z_i = y_i - W_i(x_i)\dfrac{f(y_i)(f(x_i)+f(y_i))}{f(x_i)(f(x_i)-f(y_i))}, \end{cases} \tag{10}$$

where $W_i(x_i)$ is given by (5).

2.1 Convergence Analysis

In this section, we prove that the two-step simultaneous methods described by the equations (6), (7), **Error! Reference source not found.** and **Error! Reference source not found.** have cubic convergence.

Theorem 1.

Let n be the number of distinct roots $\xi_1, \xi_2, \ldots, \xi_n$ of a non linear equation **Error! Reference source not found.** . *If $x_1, x_2, \ldots, x_n$ are the initial approximations of the roots respectively, then, for sufficiently close initial approximations, the order of convergence of (6) equals three.*

We denote: $\varepsilon_i = x_i - \xi_i$, $\varepsilon_i' = y_i - \xi_i$ and $\hat{\varepsilon}_i = z_i - \xi_i$. Considering the first equation of (7) ,we have

$$y_i = x_i - W_i(x_i),$$

$$\varepsilon_i' = \varepsilon_i - W_i(x_i),$$

$$= \varepsilon_i(1 - A_i), \tag{11}$$

where

$$A_i = \frac{W_i(x_i)}{\varepsilon_i} = \prod_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{x_i - \xi_j}{x_i - x_j}\right).$$

Now, if ξ_i is a simple root, then for small enough ε, $|x_i - x_j|$ is bounded away from zero, and so

$$\left(\frac{x_i - \xi_j}{x_i - x_j}\right) = 1 + \left(\frac{x_j - \xi_j}{x_i - x_j}\right) = 1 + O(\varepsilon)$$

and

$$\prod_{\substack{j \neq i \\ j=1}}^{n} \frac{(x_i - \xi_j)}{(x_i - x_j)} = (1 + O(\varepsilon))^{n-1}$$

$$= 1 + (n-1)O(\varepsilon) + \ldots = 1 + O(\varepsilon).$$

This implies,

$$A_i = 1 + O(\varepsilon).$$

Hence

$$A_i - 1 = O(\varepsilon). \tag{12}$$

Thus, (8) gives

$$\varepsilon_i' = O(\varepsilon^2). \tag{13}$$

Now, considering second equation of (6), we have

$$z_i = x_i - W_i(x_i)\left(1 + \frac{\dfrac{f(y_i)}{f(x_i)}}{1 - \dfrac{f(y_i)}{f(x_i)}}\right)$$

$$\hat{\varepsilon}_i = \varepsilon_i - \varepsilon_i A_i\left(1 + \frac{\dfrac{f(y_i)}{f(x_i)}}{1 - \dfrac{f(y_i)}{f(x_i)}}\right) \tag{14}$$

Now, $f(y_i)$ can be written as

$$f(y_i) = \prod_{\substack{j \neq i \\ j=1}}^{n} (y_i - \xi_j) = (y_i - \xi_i)_{\substack{j \neq i \\ j=1}}^{n} (y_i - \xi_j)$$

$$= (x_i - W_i - \xi_i)\prod_{\substack{j \neq i \\ j=1}}^{n} (x_i - W_i - \xi_j)$$

$$= (\varepsilon_i - W_i)\prod_{\substack{j \neq i \\ j=1}}^{n} (x_i - W_i - \xi_j) \tag{15}$$

$$= \varepsilon_i(1 - A_i)_{\substack{j \neq i \\ j=1}}^{n} (x_i - W_i - \xi_j). \tag{8}$$

This implies

$$\frac{f(y_i)}{f(x_i)} = \frac{\varepsilon_i(1 - A_i)\prod_{\substack{j \neq i \\ j=1}}^{n} (x_i - W_i - \xi_j)}{\varepsilon_i \prod_{\substack{j \neq i \\ j=1}}^{n} (x_i - \xi_j)}$$

$$= (1 - A_i)\prod_{\substack{j \neq i \\ j=1}}^{n} \frac{(x_i - W_i - \xi_j)}{(x_i - \xi_j)}$$

$$= (1 - A_i)G_i, \tag{16}$$

where

$$G_i = \prod_{\substack{j\neq i\\ j=1}}^{n} \frac{(x_i - W_i - \xi_j)}{(x_i - \xi_j)}$$

Using (11) in (9), we get

$$\hat{\varepsilon}_i = \varepsilon_i - \varepsilon_i A_i \left(1 + \frac{(1-A_i)G_i}{1-(1-A_i)G_i}\right)$$
$$= \varepsilon_i - \varepsilon_i A_i - \frac{\varepsilon_i A_i (1-A_i)G_i}{1-(1-A_i)G_i}$$
$$= \varepsilon_i (1-A_i) - \varepsilon_i (1-A_i) \frac{A_i G_i}{1-(1-A_i)G_i}$$
$$= \varepsilon_i (1-A_i)\left[1 - \frac{A_i G_i}{1-(1-A_i)G_i}\right]$$
$$= \frac{\varepsilon_i (1-A_i)(1-G_i)}{1-(1-A_i)G_i}. \quad (17)$$

It is easy to verify that

$$\left[1 - A_i G_i\right] = 1 - \prod_{\substack{j\neq i\\ j=1}}^{n} \left(\frac{x_i - \xi_j}{x_i - x_j}\right) \prod_{\substack{j\neq i\\ j=1}}^{n} \frac{(x_i - W_i - \xi_j)}{(x_i - \xi_j)}$$
$$= 1 - \prod_{\substack{j\neq i\\ j=1}}^{n} \frac{(x_i - W_i - \xi_j)}{(x_i - x_j)}$$
$$= 1 - G_i = O(\varepsilon). \quad (18)$$

Hence, using **Error! Reference source not found.** and (13) in (12), gives

$$\hat{\varepsilon}_i = \frac{\varepsilon_i O(\varepsilon) O(\varepsilon)}{1 - O(\varepsilon) G_i} = O(\varepsilon^3),$$

which proves the theorem.

Theorem 2.

Let n be the number of distinct roots $\xi_1, \xi_2, \ldots, \xi_n$ of a non linear equation **Error! Reference source not found.** . *If $x_1, x_2, \ldots, x_n$ are the initial approximations of the roots respectively, then, for sufficiently close initial approximations, the order of convergence of (7) equals three.*

Proof.

We denote: $\varepsilon_i = x_i - \xi_i$, $\varepsilon_i' = y_i - \xi_i$ and $\hat{\varepsilon}_i = z_i - \xi_i$. Now, second equation of (7) can be written as,

$$z_i = y_i - W_i(x_i)\left(1 + \frac{f(y_i)}{f(x_i)}\right)$$
$$z_i = x_i - W_i(x_i) - W_i(x_i)\left(1 + \frac{f(y_i)}{f(x_i)}\right). \quad (19)$$

Using (8) and (11) in (14), we get

$$z_i = x_i - W_i(x_i) - W_i(x_i)\left(1 + (1-A_i)G_i\right)$$
$$\hat{\varepsilon}_i = \varepsilon_i - \varepsilon_i A_i \left(1 + (1-A_i)G_i\right)$$
$$= \varepsilon_i \left(1-A_i\right)\left[1 - A_i G_i\right] \quad (20)$$

Using **Error! Reference source not found.** and (13) in **Error! Reference source not found.**, we have:

$$\hat{\varepsilon}_i = \varepsilon_i O(\varepsilon) O(\varepsilon) = O(\varepsilon^3), \quad (12)$$

which proves the theorem.

Theorem 3.

Let n be the number of distinct roots $\xi_1, \xi_2, \ldots, \xi_n$ of a non linear equation **Error! Reference source not found.** . *If $x_1, x_2, \ldots, x_n$ are the initial approximations of the roots respectively, then for sufficiently close initial approximations, the order of convergence of* ***Error! Reference source not found.*** *equals three.*

Proof.

We denote: $\varepsilon_i = x_i - \xi_i$, $\varepsilon_i' = y_i - \xi_i$ and (13) $\hat{\varepsilon}_i = z_i - \xi_i$. Now, second equation of **Error! Reference source not found.** can be written as,

$$z_i = y_i - W_i(x_i) \frac{\frac{f(y_i)}{f(x_i)} + \left(\frac{f(y_i)}{f(x_i)}\right)^2}{1 - \frac{f(y_i)}{f(x_i)} - 3\left(\frac{f(y_i)}{f(x_i)}\right)^2}. \quad (15)$$

Using (8) and (11) in (15), we get

$$z_i = x_i - W_i - W_i \frac{(1-A_i)G_i + (1-A_i)^2 G_i^2}{1-(1-A_i)G_i - 3(1-A_i)^2 G_i^2}$$
$$\hat{\varepsilon}_i = \varepsilon_i - \varepsilon_i A_i - \varepsilon_i A_i \frac{(1-A_i)G_i + (1-A_i)^2 G_i^2}{1-(1-A_i)G_i - 3(1-A_i)^2 G_i^2}$$
$$= \varepsilon_i (1-A_i) - \varepsilon_i (1-A_i) \frac{A_i G_i + A_i (1-A_i) G_i^2}{1-(1-A_i)G_i - 3(1-A_i)^2 G_i^2}$$

$$= \varepsilon_i (1-A_i)\left[1 - \frac{A_i G_i + A_i(1-A_i)G_i^2}{1-(1-A_i)G_i - 3(1-A_i)^2 G_i^2}\right]$$

$$= \varepsilon_i (1-A_i)\left[\frac{1-(1-A_i)G_i - 3(1-A_i)^2 G_i^2 - A_i G_i - A_i(1-A_i)G_i^2}{1-(1-A_i)G_i - 3(1-A_i)^2 G_i^2}\right]$$

$$= \varepsilon_i (1-A_i)\left[\frac{(1-G_i) - 3(1-A_i)^2 G_i^2 - A_i(1-A_i)G_i^2}{1-(1-A_i)G_i - 3(1-A_i)^2 G_i^2}\right]. \tag{16}$$

Using **Error! Reference source not found.** and (13) in (16), we have:

$$\hat{\varepsilon}_i = \varepsilon_i O(\varepsilon) * \left[\frac{O(\varepsilon) - 3(O(\varepsilon))^2 G_i^2 - A_i O(\varepsilon)}{1-O(\varepsilon) - 3(O(\varepsilon))^2}\right] G_i^2$$

$$= \varepsilon_i * O(\varepsilon) * O(\varepsilon)\left[\frac{1 - 3O(\varepsilon)G_i^2 - A_i G_i^2}{1-O(\varepsilon)G_i - 3(O(\varepsilon))^2 G_i^2}\right]$$

$$= O(\varepsilon^3),$$

which proves the theorem.

Theorem 4.

Let n be the number of distinct roots $\xi_1, \xi_2, \ldots, \xi_n$ of a non linear equation **Error! Reference source not found.** . *If $x_1, x_2, \ldots, x_n$ are the initial approximations of the roots respectively, then, for sufficiently close initial approximations, the order of convergence of* ***Error! Reference source not found.*** *equals three.*

Proof.

We denote: $\varepsilon_i = x_i - \xi_i$, $\varepsilon_i' = y_i - \xi_i$ and $\hat{\varepsilon}_i = z_i - \xi_i$. From equation **Error! Reference source not found.**, we have

$$\varepsilon_i' = O(\varepsilon^2).$$

Now, second equation of **Error! Reference source not found.** can be written as,

$$= y_i - W_i(x_i)\frac{f(y_i)}{f(x_i)}\frac{1+\frac{f(y_i)}{f(x_i)}}{1-\frac{f(y_i)}{f(x_i)}}. \tag{17}$$

Using (8), **Error! Reference source not found.** in (17), we get

$$z_i = x_i - W_i - W_i(1-A_i)G_i\frac{1+(1-A_i)G_i}{1-(1-A_i)G_i}$$

$$\hat{\varepsilon}_i = \varepsilon_i(1-A_i) - \varepsilon_i(1-A_i) \times A_i G_i \frac{1+(1-A_i)G_i}{1-(1-A_i)G_i}$$

$$= \varepsilon_i(1-A_i)\left[1 - \frac{A_i G_i + A_i(1-A_i)G_i^2}{1-(1-A_i)G_i}\right]$$

$$= \varepsilon_i(1-A_i)\left[\frac{1-(1-A_i)G_i - A_i G_i - A_i(1-A_i)G_i^2}{1-(1-A_i)G_i}\right]$$

$$= \varepsilon_i(1-A_i)\left[\frac{1-(1-A_i)G_i(1+A_i G_i) - A_i G_i}{1-(1-A_i)G_i}\right].$$

$$= \varepsilon_i(1-A_i)\left[\frac{(1-A_i G_i) - (1-A_i)G_i(1+A_i G_i)}{1-(1-A_i)G_i}\right] \tag{18}$$

From (13), we have the following form of (18),

$$\hat{\varepsilon}_i = \varepsilon_i O(\varepsilon)\left[\frac{O(\varepsilon) - O(\varepsilon)G_i O(\varepsilon)(1+A_i G_i)}{1-O(\varepsilon)G_i}\right]$$

$$\hat{\varepsilon}_i = O(\varepsilon^3),$$

which proves the theorem

2.2 Numerical Results

We give here some numerical results in order to present the performance of our third order two-step methods, (6), (7), **Error! Reference source not found.** and **Error! Reference source not found.** namely $NM1$, $NM2$, $NM3$, and $NM4$ respectively. We compare our methods with Zhang et al. method [13] (abbreviated as ZPH method) of order five. All the computations are performed using Maple 10.0, with 64 digits floating point arithmetic. We take $\varepsilon_i = 10^{-20}$ as tolerance and use the following stopping criteria for estimating the zeros:

$$e_i = \left|z_i^{(n+1)} - z_i^{(n)}\right| < \varepsilon_i \quad \text{foreach} i,$$

where e_i represents the absolute error and it , number of iterations.

Numerical examples are also taken from [13].

Example 1:

Consider

$$f(z)=z^4-1,$$

with the exact zeros

$$\xi_1=1,\xi_2=-1,\xi_3=i,\xi_4=-i.$$

Take initial approximations as:

$$z_1^{(0)}=0.5+0.5i,\ z_2^{(0)}=-1.36+0.42i,$$

$$z_3^{(0)}=-0.25+1.28i,z_4^{(0)}=0.46-1.37i.$$

The numerical comparisons is given in the table 1

Table 1. Present and Literature Results of Example 1

Methods	*it*	e_1	e_2	e_3	e_4
NM 1	5	$0.105061e^{-19}$	$0.235849e^{-19}$	$770448e^{-21}$	$0.803483e^{-19}$
NM 2	6	$0146934e^{-28}$	$0.610141e^{-36}$	$0.777496e^{-36}$	$0.475622e^{-29}$
ZPH	4	$0.100000e^{-17}$	0.0	$0.154240e^{-18}$	$0.100000e^{-18}$
NM 3	6	$0.343138e^{-28}$	$0.852370e^{-29}$	$0.237760e^{-28}$	$0.188811e^{-28}$
NM 4	13	$0.136148e^{-32}$	$0.17239e^{-35}$	$0.111343e^{-33}$	$0.108202e^{-32}$

Example 2:

Consider

$$f(z)=z^7+z^5-10z^4-z^3-z+10,$$

with the exact zeros

$$\xi_1=2,\xi_2=1,\xi_3=-1,$$

$$\xi_4=i,\xi_5=-i,\xi_6=-1+2i,\xi_7=-1-2i.$$

Take initial approximations as:

$$z_1^{(0)}=1.66+0.23i,z_2^{(0)}=1.36-0.31i,z_3^{(0)}=-0.76+0.18i,$$

$$z_4^{(0)}=-0.35+1.17i,z_5^{(0)}=0.29-1.37i,z_6^{(0)}=-0.75+2.36i,$$

$$z_7^{(0)}=-1.27-1.62i.$$

The numerical comparison is shown in the table 2.

Table 2 Present and Literature Results of Example 2

Methods	*NM* 1	*NM* 2	*ZPH*	*NM* 3	*NM* 4
Iteration	6	6	3	8	7
e_1	$0.103036e^{-39}$	$0.593378e^{-25}$	$0.163337e^{-5}$	$0.262624e^{-25}$	$0.212406e^{-24}$
e_2	$0.272431e^{-40}$	$0.190635e^{-36}$	$0.232441e^{-5}$	$0.150380e^{-23}$	$0.608994e^{-26}$
e_3	$0.421318e^{-45}$	$0.540219e^{-36}$	0.0	$0.879347e^{-22}$	$0.205031e^{-25}$
e_4	$0.376408e^{-45}$	$0.82025e^{-39}$	$0.509990e^{-10}$	$0.642865e^{-25}$	$0.326272e^{-27}$
e_5	$0.838318e^{-43}$	$0.675336e^{-34}$	$0.454933e^{-9}$	$0.329247e^{-21}$	$0.267003e^{-25}$
e_6	$0.119473e^{-44}$	$0.313808e^{-32}$	0.0	$0.101475e^{-26}$	$0.322004e^{-27}$
e_7	$0.889241e^{-43}$	$0.216807e^{-25}$	0.0	$0.922604e^{-21}$	$0.533404e^{-24}$

Example 3:

Consider

$$f(z)=z^3+5z^2-4z-20$$

$$+\cos(z^3+5z^2-4z-20)-1,$$

with the exact zeros

$$\xi_1=-5,\xi_2=-2,\xi_3=2.$$

Take initial approximations as:

$$z_1^{(0)}=-5.2,\ z_2^{(0)}=-1.4,z_3^{(0)}=2.4.$$

The numerical comparison is shown in the table 3.

Table 3 Present and Literature Results of Example 3				
Methods	iterations	e_1	e_2	e_3
$NM\,1$	5	$0.176358e^{-34}$	$0.703298e^{-31}$	$0.344328e^{-24}$
$NM\,2$	5	$0.125647e^{-43}$	$0.243345e^{-27}$	$0.151340e^{-19}$
ZPH	**8**	**0.0**	$\mathbf{0.1e^{-8}}$	$\mathbf{0.9e^{-8}}$
$NM\,3$	5	$0.384662e^{-26}$	0.0	$0.105770e^{-20}$
$NM\,4$	5	0.414932e-29	$0.136890e^{-19}$	0.428373e-26

3 Conclusions

We have developed and extended here three iterative methods for determining single root at a time of a single variable non-linear equations to three simultaneous iterative methods for finding all the roots of a non-linear equation, each of convergence order three. From the tables 1 to 3, we observe that although our methods are of convergence order three but are very effective, efficient and more accurate in terms of accuracy as compared to fifth order simultaneous method of X. Zhang, et al. method [13].

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