Some Families of Higher Order Three-Step Iterative Techniques

[1]Nazir Ahmad Mir, [1]Saher Akmal Khan, [2]Naila Rafiq, [2]Nusrut Yasmin.

[1]Department of Basic Sciences, Riphah International University, Islamabad, Pakistan,
[2]Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, Pakistan.
nazirahmad.mir@gmail.com

Abstract: We suggest and analyze four new families of higher order three-step iterative techniques for solving non-linear equations. Each of these families has seventh order convergence. Per iteration, these families of methods require three function and one of its derivative evaluations. Thus, each of these families has 1.627 computational efficiency. Several examples are given to illustrate the performance and efficiency of these methods. Comparison with other similar methods is also given

[Nazir Ahmad Mir, Saher Akmal Khan, Naila Rafiq, Nusrut Yasmin. **Some Families of Higher Order Three-Step Iterative Techniques.** *Life Sci J* 2013;10(2s):43-53] (ISSN:1097-8135). http://www.lifesciencesite.com. 8

Keywords: Iterative methods; Three-step methods; Predictor-Corrector type methods; Convergence order.

Introduction

Solving non-linear equation is one of the most important problems in numerical analysis. Newton's method is the most significant iterative method for solving non-linear equation.

$$f(x)=0 \qquad (1)$$

A non-linear equation cannot be solved in general analytically. Therefore, a number of approximate techniques have been developed to compute the zeros of non-linear equations. These techniques have been developed using error analysis, Adomian Decomposition methods, Quadrature formulas, etc.

In recent years, mathematicians have developed many two-step and three-step iterative methods, see [1-8].

In this paper, four new families of higher order three-step methods are developed using variants of Changbum Chun [5] and Li. Tai fang [7] techniques by applying binomial approximation on them and using the strategy of Weihong Bi et al [6].

Convergence analysis of these families of methods is discussed. Each of these families is of seventh order convergence. Per iteration,. these families of methods require three function and one of its derivative evaluations. Thus, each of these families has 1.627 computational efficiency.

1 Construction of families of three-step iterative methods

Consider Newton's Method:

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)} \qquad (2)$$

A family of iterative method by Li-Fang et.al [7] is given by:

$$z_n=y_n-\frac{f(y_n)}{f'(y_n)-\alpha f(y_n)}, \qquad (3)$$

where α is a real number and y_n is defined by (2).

Consider the approximation, by Chun [5]: (1)

$$f'(y_n)=\frac{f'(x_n)[f(x_n)-f(y_n)]}{f(x_n)+f(y_n)} \qquad (4)$$

Using (4) in (3), we obtain a new family of Iterative Methods,

$$z_n=y_n-\frac{f(y_n)(f(x_n)+f(y_n))}{f'(x_n)(f(x_n)-f(y_n))-\alpha f(y_n)(f(x_n)+f(y_n))}, \qquad (5)$$

where $\alpha \in \mathrm{R}$ and y_n is defined by (2).

Second family of methods is obtained by applying binomial expansion upto first order on (5), namely

$$z_n=y_n-\frac{f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\left[1+\frac{\alpha f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\right], \qquad (6)$$

where $\alpha \in \mathrm{R}$ and y_n is given by (2).

Similarly, third family of methods is obtained using second approximation on (5), that is

$$z_n = y_n - \frac{f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\left[1+\frac{\alpha f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]} + \left(\frac{\alpha f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\right)^2\right], \tag{7}$$

where $\alpha \; \varepsilon \; \mathrm{R}$ and y_n is given by (2).

Fourth family of methods is given by using family of C.Chun and Y.Ham [8], and applying first binomial approximation on it, that is

$$z_n = y_n - [2f(x_n)+(2\beta-1)f(y_n)] \frac{[(2f(x_n)-(2\beta-5)f(y_n)]}{4f^2(x_n)} \frac{f(y_n)}{f'(x_n)} \tag{8}$$

Further, considering the strategy of Weihong et.al [6], we propose three- step family of methods combining equations (2) and (5):

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - \frac{f(y_n)(f(x_n)+f(y_n))}{f'(x_n)(f(x_n)-f(y_n))-\alpha f(y_n)(f(x_n)+f(y_n))} \tag{9}$$

and

$$x_{n+1} = z_n - G(\mu_n)\frac{f(z_n)}{f'(z_n)},$$

where $\mu_n = \dfrac{f(z_n)}{f(x_n)}$ and $G(t)$ represents a real-valued function.

Using the Taylor expansion, $f(z_n)$ and $f'(z_n)$ can be approximated as:

$$f(z_n) = f(z_n - y_n + y_n) \approx f(y_n) + f'(y_n)(z_n - y_n) + \frac{1}{2}f''(y_n)(z_n-y_n)^2, \tag{10}$$

$$f'(z_n); \; f'(y_n) + f''(y_n)(z_n - y_n), \tag{11}$$

this implies

$$f'(z_n) \approx \frac{f(z_n)-f(y_n)}{z_n - y_n} + \frac{1}{2}f''(y_n)(z_n - y_n)$$
$$= f[z_n, y_n] + \frac{1}{2}f''(y_n)(z_n - y_n) \tag{12}$$

In order to avoid the computation of the second derivative, we approximate $f''(y_n)$ as follows:

$$f''(y_n) \approx 2f[z_n, x_x, x_n] = \frac{2(f[z_n, x_x] - f'(x_n))}{z_n - x_n}, \tag{13}$$

where z_n and x_n are sufficiently close to y_n when n is sufficiently big integer.

Substituting (13) into (12) and replacing $f'(z_n)$ with approximation in (12), we propose the following three-step family of methods from(9):

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(y_n)(f(x_n)+f(y_n))}{f'(x_n)(f(x_n)-f(y_n))-\alpha f(y_n)(f(x_n)+f(y_n))}, \tag{14}$$

and

$$x_{n+1} = z_n - G(\mu_n) \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)},$$

where $\mu_n = \dfrac{f(z_n)}{f(x_n)}$ and $G(t)$ represents a real-valued function.

Similarly from $(2), (6)$ and the third equation of (14), we have second family of three-step methods:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\left[1+\frac{\alpha f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\right] \tag{15}$$

and

$$x_{n+1} = z_n - G(\mu_n)\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)},$$

where $\mu_n = \frac{f(z_n)}{f(x_n)}$ and $G(t)$ represents a real-valued function.

Also from $(2),(7)$ and third equation of (14), third family of methods is given by:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\left[1+\frac{\alpha f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}+\left(\frac{\alpha f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\right)^2\right] \quad (16)$$

and

$$x_{n+1} = z_n - G(\mu_n)\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)},$$

where $\mu_n = \frac{f(z_n)}{f(x_n)}$ and $G(t)$ represents a real-valued function.

Fourth family is obtained by combining $(2),(8)$ and third equation of (14):

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{[2f(x_n)+(2\beta-1)f(y_n)][(2f(x_n)-(2\beta-5)f(y_n)]}{4f^2(x_n)}\frac{f(y_n)}{f'(x_n)} \quad (17)$$

and

$$x_{n+1} = z_n - G(\mu_n)\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)},$$

where $\mu_n = \frac{f(z_n)}{f(x_n)}$ and $G(t)$ represents a real-valued function.

We now discuss convergence for families of three step methods $(14-17)$.

2 Convergence Analysis

Theorem 1

Assume that $f: D \subset R \to R$. is a scalar function on some open interval D. Suppose $x^* \in D$, $f'(x^*) \neq 0$. If the initial point x_0 is sufficiently close to x^* and G is any function with $G(0)=1$, $G'(0)=1$, $|G''(0)| < \infty$, then the family of three-step methods (14) is of convergence order seven for any $\alpha \in R$.

Proof.

Let $e_n = x_n - x^*$, $\tilde{e}_n = y_n - x^*$ and $d_n = z_n - x^*$. Denote $c_k = \frac{f^{(k)}(x^*)}{k!f'(x^*)}$, $k = 2,3,....$ Using the Taylor expansion and taking into account $f(x) = 0$, we have:

$$f(x_n) \approx f'(x^*)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O(e_n^6)] \quad (18)$$

$$f'(x_n) \approx f'(x^*)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + O(e_n^5)] \quad (19)$$

From((18) and (19), we get:

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + [2c_2^2 - 2c_3]e_n^3 + [7c_2c_3 - 4c_2^3 - 3c_4]e_n^4 + [8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5]e_n^5 + O(e_n^6) \quad (20)$$

Thus, from (2), we have:

$$e_n^{:} = c_2e_n^2 + [2c_3 - 2c_2^2]e_n^3 + [4c_2^3 - 7c_2c_3 + 3c_4]e_n^4 + [-8c_2^4 + 20c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5]e_n^5 + O(e_n^6) \quad (21)$$

Again expanding $f(y_n)$ about x^*, we have:

$$f(y_n) = f'(x^*)[e_n^{:} + c_2e_n^{:2} + O(e_n^{:3})]$$

$$f(y_n)=f'(x^*)[c_2e_n^2+(2c_3-2c_2^2)e_n^3+ (3c_4+5c_2^3-7c_2c_3)e_n^4+ (-12c_2^4+24c_2^2c_3-10c_2c_4-6c_3^2+4c_5)e_n^5 +O(e_n^6)]. \quad (22)$$

From $(18),(19)$ and (22), we have

$$\frac{f(y_n)(f(x_n)+f(y_n))}{f'(x_n)[f(x_n)-f(y_n)]-\alpha f(y_n)(f(x_n)+f(y_n)]} =c_2e_n^2+(2c_3-2c_2^2)e_n^3+(3c_4-6c_2c_3+\alpha c_2^2+c_2^3)e_n^4 +(4\alpha c_2c_3+10c_2^4-4\alpha c_2^3+4c_5-4c_3^2-8c_2c_4)e_n^5 +O(e_n^6) \quad (23)$$

From (5), we have:

$$d_n=e_n-\frac{f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]-\alpha f(y_n)[f(x_n)+f(y_n)]}$$

Using (21) and (23) in the above result, we obtain:

$$d_n=(3c_2^3-c_2c_3-\alpha c_2^2)e_n^4+(20c_2^2c_3-4\alpha c_2c_3-18c_2^4 +4\alpha c_2^3-2c_3^2-2c_2c_4)e_n^5+O(e_n^6) \quad (24)$$

Expanding $f(z_n)$ about x^*, we have:

$$f(z_n)=f'(x^*)[d_n+c_2d_n^2+c_3d_n^3+c_4d_n^4+O(d_n^5)] \quad (25)$$

Thus by first equation of (22) and (25), we have:

$$f[z_n,y_n]=\frac{f(z_n)-f(y_n)}{z_n-y_n}=f'(x^*)\frac{[d_n+c_2d_n^2+c_3d_2^3+O(d_n^4)-(e_n+c_2e_n^2+c_3e_n^3+O(e_n^4))]}{d_n-e_n}$$

This implies

$$f[z_n,y_n]=f'(x^*)[1+c_2d_n+c_2e_n+c_3e_n^2+O(e_n^6)]$$

Using(21) and (24), we have:

$$f[z_n,y_n]=f'(x^*)[1+c_2^2e_n^2+2c_2(c_3-c_2^2)e_n^3 +(7c_2^4-7c_2^2c_3+3c_2c_4-\alpha c_2^3)e_n^4+O(e_n^5)] \quad (26)$$

Moreover, by (18) and (25), we have:

$$f[z_n,x_n]=\frac{f(z_n)-f(x_n)}{z_n-x_n}=\frac{f(z_n)-f(x_n)}{d_n-e_n} =f'(x^*)\frac{[d_n+c_2d_n^2+c_3d_2^3+O(d_n^4)-(e_n+c_2e_n^2+c_3e_n^3+ c_4e_n^4+O(e_n^5))]}{d_n-e_n}$$

This implies,

$$f[z_n,x_n]=f'(x^*)[1+c_2(e_n+d_n)+c_3(e_n^2+d_ne_n+d_n^2) +c_4(e_n^3+e_n^2d_n+e_nd_n^2+d_n^3)+O(d_n^4)] \quad (27)$$

Now,

$$f[z_n,x_n,x_n]=\frac{f[z_n,x_n]-f'(x_n)}{z_n-x_n} =\frac{f[z_n,x_n]-f'(x_n)}{d_n-e_n}$$

Then by (19) and (27), we obtain:

$$f[z_n,x_n,x_n]=f'(x^*)[c_2+c_3(e_n+(e_n+d_n)) +c_4(e_n^2+(d_n+e_n)e_n+(d_2^n+e_n^2+d_ne_n)+O(e_n^3)]$$

Using (24), we have:

$$f[z_n,x_n,x_n]=f'(x^*)[c_2+2c_3e_n+3c_4e_n^2+O(e_n^3)] \quad (28)$$

Equations (24) further gives:

$$f[z_n,x_n,x_n](z_n-y_n)=f'(x^*) [c_2+2c_3e_n+3c_4e_n^2](d_n-e_n)$$

Using (24), we obtain:

$$f[z_n,x_n,x_n](z_n-y_n)=f'(x^*) [-c_2^2e_n^2+(-4c_2c_3+2c_2^3)e_n^3 +(-4c_3^2-\alpha c_2^3-6c_2c_4-c_2^4+10c_2^2c_3)e_n^4+O(e_n^5)] \quad (29)$$

Equation (26) and (29), imply:

$$f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)=f'(x^*) [1-2c_2c_3e_n^3+(6c_2^4+3c_2^2c_3-3c_2c_4-2\alpha c_2^3-4c_3^2)e_n^4+O(e_n^5)] \quad (30)$$

Now from (18) and (25), we have:

$$\mu_n=\frac{f(z_n)}{f(x_n)}=\frac{d_n}{e_n}-c_2d_n+O(e_n^5) \quad (31)$$

Using the Taylor expansion and considering $\left|G''(0)\right| < \infty$, we get

$$G(\mu_n) = G(0) + G'(0)\mu_n + O(\mu_n^2)$$

Using (31), we have:

$$G(\mu_n) = G(0) + G'(0)\left(\frac{d_n}{e_n} - c_2 d_n\right) + O(\mu_n^2) \quad (32)$$

From (24), we have

$$\frac{d_n}{e_n} = (3c_2^3 - c_2c_3 - \alpha c_2^2)e_n^3 + (20c_2^2c_3 - 4\alpha c_2c_3 - 18c_2^4 + 4\alpha c_2^3 - 2c_3^2 - 2c_2c_4)e_n^4 + O(e_n^5) \quad (33)$$

Thus, the error equation for family of methods (14) is given by:

$$e_{n+1} = x_{n+1} - x^* = z_n - G(\mu_n)\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)} - x^*$$

$$= d_n - G(\mu_n)\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}$$

$$= d_n - \frac{[G(0) + G'(0)(\frac{d_n}{e_n} - c_2 d_n) + O(\mu_n^2)][d_n + c_2 d_n^2 + O(d_n^3)]}{1 - 2c_2c_3e_n^3 + (6c_2^4 + 3c_2^2c_3 - 3c_2c_4 - 2\alpha c_2^3 - 4c_3^2)e_n^4 + O(e_n^5)} \quad (34)$$

$$= d_n - [G(0)d_n + G'(0)\frac{d_n}{e_n}d_n - G'(0)c_2d_n^2 + G(0)c_2d_n^2 + G'(0)c_2\frac{d_n}{e_n}d_n^2 - G'(0)c_2^2d_n^3 + \ldots]$$
$$[1 + 2c_2c_3e_n^3 - (6c_2^4 + 3c_2^2c_3 - 3c_2c_4 - 2\alpha c_2^3 - 4c_3^2)e_n^4 + O(e_n^5)]$$

On simplification, we have:

$$e_{n+1} = [1 - G(0)]d_n + [-G'(0)(3c_2^3 - c_2c_3 - \alpha c_2^2)^2 - 2c_2c_3G(0)(3c_2^3 - c_2c_3 - \alpha c_2^2)]e_n^7 + O(e_n^8) \quad (35)$$

Thus, the convergence order of the family (14) is seventh order if $G(0) = 1$ and the error equation is given by

$$e_{n+1} = [-G'(0)(3c_2^3 - c_2c_3 - \alpha c_2^2)^2 - 2c_2c_3(3c_2^3 - c_2c_3 - \alpha c_2^2)]e_n^7 + O(e_n^8) \quad (36)$$

With $G'(0) = 1$, error equation is given by

$$e_{n+1} = (c_2^2c_3^2 - 9c_2^6 - \alpha^2c_2^4 + 6\alpha c_2^5)e_n^7 \quad (37)$$

We have the same error equation for family of methods (15) and (16) with seventh order convergence.

3 THE CONCRETE ITERATIVE METHODS

The three-step family of methods (14) suggests some new methods which are stated as under:

3.1 METHOD 1 (Abbreviated as MK1)

For the given function G, defined by

$$G(t) = 1 + \frac{t}{1 + \beta t}, \quad (38)$$

where $\beta \;\varepsilon\; R$, it can easily be seen that the function $G(t)$ of (38) satisfies the conditions of Theorem 1. Hence, we get a new two-parameter seventh-order family of methods from (14):

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - \frac{f(y_n)(f(x_n) + f(y_n))}{f'(x_n)(f(x_n) - f(y_n)) - \alpha f(y_n)(f(x_n) + f(y_n))} \quad (39)$$

$$x_{n+1} = z_n - \frac{f(x_n) + (\beta + 1)f(z_n)}{f(x_n) + \beta f(z_n)} \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}$$

3.2 METHOD 2 (Abbreviated as MK2)

For the function G, defined by

$$G(t) = \frac{1}{(1 - \beta t)^{\frac{1}{\beta}}} \quad (40)$$

where $\beta \neq 0$, it can easily be seen that the function $G(t)$ of (40) satisfies the conditions of Theorem 1. Hence, we get a new two-parameter seventh-order family of methods from(14):

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - \frac{f(y_n)(f(x_n) + f(y_n))}{f'(x_n)(f(x_n) - f(y_n)) - \alpha f(y_n)(f(x_n) + f(y_n))} \quad (41)$$

$$x_{n+1}=z_n-\left(\frac{f(x_n)}{f(x_n)-\beta f(z_n)}\right)^{\frac{1}{\beta}}\frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}$$

3.3 METHOD 3 (Abbreviated as MK3)

For the function G, defined by (38)

$$G(t)=1+\frac{t}{1+\beta t},$$

where $\beta \ \varepsilon \ R$, it can easily be seen that the function $G(t)$ of (38) satisfies the conditions of Theorem 1. Hence, we get a new two-parameter seventh-order family of methods from (15):

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$z_n=y_n-\frac{f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\left[1+\frac{\alpha f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\right] \quad (42)$$

$$x_{n+1}=z_n-\frac{f(x_n)+(\beta+1)f(z_n)}{f(x_n)+\beta f(z_n)}\frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}$$

3.4 METHOD 4 (Abbreviated as MK4)

For the function G, defined by (40)

$$G(t)=\frac{1}{(1-\beta t)^{\frac{1}{\beta}}},$$

where $\beta\neq 0$, it can easily be seen that the function $G(t)$ of (40) satisfies the conditions of Theorem 1. Hence, we get a new two-parameter seventh-order family of methods from(15):

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$z_n=y_n-\frac{f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\left[1+\frac{\alpha f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\right] \quad (43)$$

$$x_{n+1}=z_n-\left(\frac{f(x_n)}{f(x_n)-\beta f(z_n)}\right)^{\frac{1}{\beta}}\frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}$$

3.5 METHOD 5 (Abbreviated as MK5)

For the function G, defined by (38)

$$G(t)=1+\frac{t}{1+\beta t}$$

where $\beta \ \varepsilon \ R$, it can easily be seen that the function $G(t)$ of (38) satisfies the conditions of Theorem 1. Hence, we get a new two-parameter seventh-order family of methods from(16):

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$z_n=y_n-\frac{f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\left[1+\frac{\alpha f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}+\left(\frac{\alpha f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\right)^2\right] \quad (44)$$

$$x_{n+1}=z_n-\frac{f(x_n)+(\beta+1)f(z_n)}{f(x_n)+\beta f(z_n)}\frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}$$

3.6 METHOD 6 (Abbreviated as MK6)

For the function G, defined by (40)

$$G(t)=\frac{1}{(1-\beta t)^{\frac{1}{\beta}}},$$

where $\beta\neq 0$, it can easily be seen that the function $G(t)$ of (40) satisfies the conditions of Theorem 1. Hence, we get a new two-parameter seventh-order family of methods from(16):

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - \frac{f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\left[1+\frac{\alpha f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]} + \left(\frac{\alpha f(y_n)[f(x_n)+f(y_n)]}{f'(x_n)[f(x_n)-f(y_n)]}\right)^2\right] \quad (45)$$

$$x_{n+1} = z_n - \left(\frac{f(x_n)}{f(x_n)-\beta f(z_n)}\right)^{\frac{1}{\beta}} \frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}$$

Theorem

Assume that $f: D \subset R \to R$ is a scalar function on some open interval D. Suppose $x^* \in D$, $f'(x^*) \neq 0$. If the initial point x_0 is sufficiently close to x^* and G is any function with $G(0)=1$, $G'(0)=1$, $|G''(0)| < \infty$, then the family of three-step methods (17) is of convergence order seven for $\beta = \frac{5}{2}$.

Proof. Let $e_n = x_n - x^*$, $\tilde{e}_n = y_n - x^*$ and $d_n = z_n - x^*$. Denote $c_k = \frac{f^{(k)}(x^*)}{k! f'(x^*)}$, $k = 2,3,\ldots$ Using the Taylor expansion and taking into account $f(x)=0$, we have:

From (19)and (22), we have:

$$\frac{f(y_n)}{f'(x_n)} = c_2e_n^2 + (2c_3-4c_2^2)e_n^3 + (13c_2^3 - 14c_2c_3 + 3c_4)e_n^4 + (-38c_2^4 + 64c_2^2c_3 - 12c_3^2 - 20c_2c_4 + 4c_5)e_n^5 + O(e_n^6) \quad (46)$$

Thus, using (19), (22) and (46), we get

$$\frac{2f(x_n)+(2\beta-1)f(y_n)][(2f(x_n)-(2\beta-5)f(y_n)]}{4f^2(x_n)}\frac{f(y_n)}{f'(x_n)}$$
$$= c_2e_n^2 + (2c_3 + (2\beta-7)c_2^2)e_n^3 + (-26c_2c_3 + (\beta^2 - 17\beta + \frac{141}{4})c_2^3 + (8\beta-26)c_2c_3 + 3c_4)e_n^4 + (4c_5 + (-6\beta^2 + 94\beta - \frac{331}{2})c_2^2c_3 + (10\beta^2 - 96\beta + \frac{283}{2})c_2^4 + (18-8\beta)c_3^2 + (12\beta-38)c_2c_4)e_n^5 + O(e_n^6) \quad (47)$$

Now, from equation(17), we get:

$$d_n = \tilde{e}_n - \frac{[2f(x_n)+(2\beta-1)f(y_n)][(2f(x_n)-(2\beta-5)f(y_n)]}{4f^2(x_n)}\frac{f(y_n)}{f'(x_n)} \quad (48)$$

Using equation(46) and (48) and for $\beta = \frac{5}{2}$, we have:

$$d_n = (5c_2^3 - c_2c_3)e_n^4 + (32c_2^2c_3 - 36c_2^4 - 2c_2c_4 - 2c_3^2)e_n^5 + O(e_n^6) \quad (49)$$

By Taylor Series about x^*, we have:

$$f(z_n) = f'(x^*)[d_n + c_2d_n^2 + c_3d_n^3 + c_4d_n^4 + O(d_n^5)]$$

Now,

$$f[z_n,y_n] = \frac{f(z_n)-f(y_n)}{z_n-y_n} = f'(x^*)\frac{[d_n + c_2d_n^2 + c_3d_n^3 + O(d_n^4) - (\dot{e}_n + c_2\dot{e}_n^2 + c_3\dot{e}_n^3 + O(\dot{e}_n^4))]}{d_n - \dot{e}_n}$$

Using (21) and (49), we get:

$$f[z_n,y_n] = f'(x^*)[1 + c_2e_n^2 + (2c_2c_3 - 2c_2^3)e_n^3 + (-7c_2^2c_3 + 3c_2c_4 + 9c_2^4)e_n^4 + O(e_n^5)] \quad (50)$$

Now, Using (18) and (49), we have:

$$f[z_n,x_n] = f'(x^*)[1 + c_2(e_n + d_n) + c_3(e_n^2 + d_ne_n + d_n^2) + c_4(e_n^3 + e_n^2d_n + e_nd_n^2 + d_n^3) + O(d_n^4)] \quad (51)$$

Also by definition,

$$f[z_n,x_n,x_n] = \frac{f[z_n,x_n] - f'(x_n)}{z_n - x_n} = \frac{f[z_n,x_n] - f'(x_n)}{d_n - e_n}$$

Using (19) and (51), we get:

$$f[z_n,x_n,x_n] = f'(x^*)[c_2 + c_3(e_n + (e_n + d_n)) + c_4(e_n^2 + (d_n + e_n)e_n + (d_2^n + e_2^2 + d_ne_n) + O(e_n^3)]$$

Using (49), we obtain:

$$f[z_n,x_n,x_n] = f'(x^*)[c_2 + 2c_2e_n + 3c_4e_n^2 + O(e_n^3)] \quad (52)$$

This implies ,

$$f[z_n, x_n, x_n](z_n - y_n) =$$
$$f'(x^*)[c_2 + 2c_3e_n + 3c_4e_n^2](d_n - e_n)$$

Using (49), we have:

$$f[z_n,x_n,x_n](z_n-y_n)=f'(x^*)[-c_2^2e_n^2+(-4c_2c_3+2c_2^3)e_n^3$$
$$+(10c_2^2c_3+c_2^4-6c_2c_4-4c_3^2)e_n^4+O(e_n^5)] \quad (53)$$

Equations (50) and (53), imply,

$$f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)=$$
$$f'(x^*)[1-2c_2c_3e_n^3+(-4c_3^2-3c_2c_4+10c_2^4+3c_2^2c_3)e_n^4$$
$$+O(e_n^5)] \quad (54)$$

Using the Taylor expansion and considering $|G''(0)| < \infty$, we get:

$$G(\mu_n) = G(0) + G'(0)\mu_n + O(\mu_n^2)$$

Now from (18) and (49), we have:

$$\mu_n = \frac{f(z_n)}{f(x_n)} = \frac{d_n}{e_n} - c_2d_n + O(e_n^5)$$

Using this result, we have:

$$G(\mu_n) = G(0) + G'(0)\left(\frac{d_n}{e_n} - c_2d_n\right) + O(\mu_n^2)$$

From (49), we have

$$\frac{d_n}{e_n} = (5c_2^3 - c_2c_3)e_n^3 + (32c_2^2c_3 - 36c_2^4 - 2c_3^2 - 2c_2c_4)e_n^4 + O(e_n^5) \quad (55)$$

Thus, the error equation for family of methods (17) is given by

$$e_{n+1} = d_n - \frac{[G(0)+G'(0)(\frac{d_n}{e_n}-c_2d_n)+O(\mu_n^2)][d_n+c_2d_n^2+O(d_n^3)]}{1-2c_2c_3e_n^3+(10c_2^4+3c_2^2c_3-3c_2c_4-4c_3^2)e_n^4+O(e_n^5)} \quad (56)$$

$$= [1-G(0)]d_n - [G'(0)\frac{d_n}{e_n} + 2G(0)c_2c_3e_n^3]d_n$$
$$-[G(0)-G'(0)]c_2d_n^2$$
$$-[10c_2^4+3c_2^2c_3-3c_2c_4-4c_3^2)]G(0)e_n^4d_n + O(e_n^8) \quad (57)$$

Thus, the convergence order of family of methods (17) is seventh-order if $G(0) = 1$, and the error equation is given by

$$e_{n+1} = [10G'(0)c_2^4c_3 - (G'(0)-2)c_2^2c_3^2 - 25G'(0)c_2^6 - 10c_2^4c_3]e_n^7$$
$$+O(e_n^8). \quad (58)$$

With $G'(0) = 1$, error equation is given by

$$e_{n+1} = (c_2^2c_3^2 - 25c_2^6)e_n^7 + O(e_n^8) \quad (59)$$

4 The CONCRETE FAMILY OF THREE-STEP ITERATIVE METHODS

The three-step family of methods (17) suggest some new methods which are stated as under:

4.1 Method 7 (Abbreviated as MK7)

For the function G, defined by

$$G(t) = 1 + \frac{t}{1+\beta t}, \quad (60)$$

where $\alpha \ \varepsilon \ R$, it can be easily seen that the function $G(t)$ of (60) satisfies the conditions of Theorem 2. Hence, we get a new one-parameter seventh order family of methods from(17):

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = y_n - \frac{[2f(x_n)+f(y_n)]}{2f(x_n)}\frac{f(y_n)}{f'(x_n)} \quad (61)$$

$$x_{n+1} = z_n - \frac{f(x_n)+(\beta+1)f(z_n)}{f(x_n)+\beta f(z_n)}$$
$$\frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}$$

4.2 Method 8 (Abbreviated as MK8)

For the function G, defined by

$$G(t) = \frac{1}{(1-\beta t)^{\frac{1}{\beta}}} \quad (62)$$

where $\alpha \neq 0$, it can easily be seen that the function $G(t)$ of (61) satisfies the conditions of Theorem 2. Hence, we get a new one-parameter seventh-order family of methods

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$z_n = y_n - \frac{[f(x_n)+2f(y_n)]}{f(x_n)}\frac{f(y_n)}{f'(x_n)} \quad (63)$$

$$x_{n+1} = z_n - \left(\frac{f(x_n)}{f(x_n) - \beta f(z_n)} \right)^{\frac{1}{\beta}} \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}$$

Thus, we have constructed four families of three-step methods $MK1$ to $MK8$. Per iteration, each of these families of methods require three evaluations of the function and one evaluation of its first derivative. We consider the definition of efficiency index as $p^{\frac{1}{w}}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method. We observed that each family of methods has the efficiency index $\sqrt[4]{7}$; 1.627.

5 Numerical Examples

Now the Methods $MK1$ to $MK8$ are implemented to solve some non-linear equations and compared with Kou's method of seventh order [2]. Table 1 shows the difference of two consecutive iterates, function value, total number of function evaluatons (TNFE) and total number of iterations. The execution is stopped at the third iteration for the sake of comparison of the methods.

We use the following functions from [2]:

$f_1(x) = x^3 + 4x^2 - 15$,
$x^* = 1.631980805566063518$,

$f_2(x) = \sin(x) - \frac{1}{2}x$,
$x^* = 1.895494267033980947$,

$f_3(x) = e^{-x} + \cos(x), x^* = 1.746139530408012418.$

$f_4(x) = \sin^2(x) - x^2 + 1$,
$x^* = 1.4.4491648215341226$

$f_5(x) = \cos(x) - x$,
$x^* = 0.7390851332151606417$,

$f_6(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$,
$x^* = -1.207647827130919$,

$f_7(x) = 10xe^{-x^2} - 1, x^* = 1.67963061042845.$

Table 1. Comparison of various iterative methods

	$G7$	$MK1$	$MK2$	$MK3$	$MK4$
		$\alpha = \frac{1}{4}, \beta = 2$	$\alpha = \frac{1}{4}, \beta = 2$		$\alpha = 1$
$f_1, x_o = 2$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\lvert x_{n+1} - x_n \rvert$	$.36e-20$	$.10e-37$	$.11e-37$	$.11e-37$	$.10e-37$
$f(x_n)$	$.56e-102$	$-.95e-266$	$-.15e-269$	$-.13e-265$	$-.95e-266$
$f_2, x_o = 2$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\lvert x_{n+1} - x_n \rvert$	$.12e-28$	$.83e-56$	$.84e-56$	$.83e-56$	$.83e-56$
$f(x_n)$	$-.52e-145$	$-.3e-349$	$-.3e-349$	$-.3e-349$	$.3e-349$
$f_3, x_o = 2$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\lvert x_{n+1} - x_n \rvert$	$.169e-25$	$.87e-52$	$.8e-52$	$.89e-52$	$.87e-52$
$f(x_n)$	$-.1e-130$	$-.3e-349$	$.41e-287$	$-.3e-349$	$.3e-349$

$f_4, x_o = 1.6$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\|x_{n+1} - x_n\|$	$.26e-20$	$.27e-38$	$.3e-38$	$.271e-38$	$.27e-38$
$f(x_n)$	$-.21e-102$	$.41e-269$	$.82e-269$	$-.45e-269$	$.41e-269$
$f_5, x_o = 1$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\|x_{n+1} - x_n\|$	$.15e-27$	$.26e-59$	$.25e-59$	$.256e-59$	$.26e-59$
$f(x_n)$	$-.15e-140$	0	0	0	0
$f_6, x_o = -1$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	4	4	4	4	4
$\|x_{n+1} - x_n\|$	$.98e-17$	$.19e-13$	$.15e-14$	$.2e-13$	$.2e-13$
$f(x_n)$	$.21e-83$	$-.23e-92$	$-.34e-100$	$-.24e-92$	$-.23e-92$
$f_7, x_o = 1.8$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\|x_{n+1} - x_n\|$	$.27e-20$	$.35e-33$	$.29e-33$	$.35e-33$	$.5e-33$
$f(x_n)$	$-.57e-102$	$.14e-232$	$.33e-233$	$.15e-232$	$.14e-232$

Table 2. Comparison of various iterative methods

	$G7$	$MK5$	$MK6$	$MK7$	$MK8$
$f_1, x_o = 2$				$\alpha = 1$	$\alpha = 1$
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\|x_{n+1} - x_n\|$	$.36e-20$	$.105e-37$	$.11e-37$	$.23e-33$	$.25e-33$
$f(x_n)$	$.56e-102$	$-.96e-266$	$-.15e-265$	$-.97e-235$	$-.19e-234$
$f_2, x_o = 2$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\|x_{n+1} - x_n\|$	$.12e-28$	$.83e-56$	$.16e-46$	$.23e-51$	$.24e-51$
$f(x_n)$	$-.52e-145$	$-.3e-349$	$.58e-327$	$-.3e-349$	$-.3e-349$
$f_3, x_o = 2$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\|x_{n+1} - x_n\|$	$.17e-25$	$.87e-52$	$.79e-52$	$.282e-50$	$.269e-50$

$f(x_n)$	$-.102e-130$	$-.3e-349$	$-.3e-297$	$-.3e-349$	$-.3e-349$
$f_4, x_o = 1.6$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\|x_{n+1} - x_n\|$	$.265e-20$	$.268e-38$	$.27e-38$	$.1e-34$	$.11e-34$
$f(x_n)$	$-.21e-102$	$.414e-269$	$.82e-269$	$.15e-243$	$.31e-243$
$f_5, x_o = 1$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\|x_{n+1} - x_n\|$	$.15e-27$	$.26e-59$	$.42e-59$	$.17e-56$	$.18e-56$
$f(x_n)$	$-.15e0140$	0	$.5e-329$	0	0
$f_6, x_o = -1$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\|x_{n+1} - x_n\|$	$.98e-17$	$.2e-13$	$.15e-14$	$.16e-6$	$.9e-9$
$f(x_n)$	$.21e-83$	$-.23e-92$	$-.34e-100$	$-.18e-43$	$-0.26e-59$
$f_7, x_o = 1.8$					
$TNFE$	12	12	12	12	12
$ITERATIONS$	3	3	3	3	3
$\|x_{n+1} - x_n\|$	$.27e-20$	$.35e-33$	$.28e-33$	$.57e-29$	$.45e-29$
$f(x_n)$	$-.57e-102$	$.14e-232$	$.33e-233$	$.1e-202$	$.2e-203$

6 Conclusion

In this work, we presented eight three-step families of seventh order convergent methods. We observe that these iterative methods are comparable with seventh order method (G_7) of Kou, et.al [2] as cited in the Table 1 and Table 2, and in almost all the cases, these families of methods as compared to Kou et.al method of seventh order [2] give better results in terms of absolute error and function value.

Corresponding Author:
Nazir Ahmad Mir
Department of Basic Sciences,
Riphah International University
Islamabad, Pakistan
E-mail: nazirahmad.mir@gmail.com.

References

1. C. Chun, Y. Li, X, Wang, Some sixth-order variants of Ostrowski root-finding methods, Appl. Math. Comput 193(2007) 389-394.
2. J. Kou, Y. Li, X.Wang, Some variants of Ostrowski's method with seventh-order convergence, J. Comput. Appl. Math.190 (2007)111-115.
3. M.A. Noor, New classes of iterative methods for nonlinear equations, Appl. Math. Comput. (2007), doi:10.1016/j.amc.2007.02.2008.
4. M.A. Noor, New classes of iterative methods for non-linear equations, Appl. Math. Comput. (2007), doi:10,1016/j.amc.2007.02.036.
5. C. Chun, A simply constructed third-order modifications of Newton's method, J. Comput. Appl. Math. (2007), doi: 10.1016/j.cam.2007.07.004.
6. Weihong Bi, Hongmin Ren, Qingbiao Wu, Three-step iterative methods with eighth-order convergence for soloving non-lnear equations, J. comput. Appl. Math (2008)
7. Li Tai-fang ^{*}, Li De-sheng, Xu Zhao-di, Fang Ya-ling new iterative methods for non-linear equaitons, Appl. Math. Comput 197 (2008) 755-759.
8. C. Chun, Y. Ham, Some fourth-order modifications of Newton's Method., Appl. Math. Comput. (2007), doi: 10.1016/j.amc.2007.08.003.

1/2/2013