Upper and Lower Solutions Method for G-BSDEs in the Reverse Order

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Abstract: The main intention of this paper is to develop the upper and lower solutions method in the reverse order for backward stochastic differential equations under G-Brownian motion (G-BSDEs) of the form $X_t = E[\xi + \int_t^T b(s,\ X_s)ds + \int_t^T \theta(s,\ X_s)d\langle B\rangle_s \mid \Omega_t],\qquad t \in [0,\ T]$. The existence of solutions for G-BSDEs via the method of upper and lower solutions in the reverse order is established. Very general results are studied by considering b or θ or both coefficients of the G-BSDEs as discontinuous functions with the stated technique in the reverse order.

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1. Introduction

In the year of 1972, very important equations known as backward stochastic differential equations (BSDEs) were introduced by Bismut (Bismut, 1973). While in 1990, they were generalized by E. Pardoux and S. Peng (Pardoux and Peng, 1990). These equations have immense applications in mathematical finance and are appropriately connected with stochastic controls, nonlinear semigroups and nonlinear partial differential equations. Under the G-Brownian motion the above mentioned equations with the existence and uniqueness theorem were launched by Peng, see the appendix of this paper or (Peng, 2010) chapter IV page 83. Later X. Bai and Y. Lin developed the existence and uniqueness of solutions for backward stochastic differential equations under G-Brownian motion (G-BSDEs) with the integral Lipschitz coefficients (Bai and Lin, 2010). Also, see (Xu, 2010) for the BSDEs under super linear G-expectation characterizing a class of stochastic control problems. Recently, Faizullah and Rahman established the upper and lower solutions method in the usual order for G-BSDEs and entrenched the existence theory for G-BSDEs with a discontinuous drift coefficient (Faizullah and Rahman 2012). Also see (Faizullah and Piao 2012) for the upper and lower solutions in the usual order. Now in contrast to the above here we introduce the method of upper and lower solutions in the reverse order to study the existence theory for G-BSDEs. We discuss a very general case by considering b or θ or both coefficients of the G-BSDEs as discontinuous functions with upper and lower solutions in the reverse order.

We consider the following backward stochastic differential equation under G-Brownian motion

$$X_t = E[\xi + \int_t^T b(s,\ X_s)ds + \int_t^T \theta(s,\ X_s)d\langle B\rangle_s \mid \Omega_t],\qquad t \in [0,\ T], \tag{1.1}$$

where $\xi \in L_G^1(\Omega_T)$ is given, $b(t,\ x)$, $\theta(t,\ x)$ are $M_G^1(0,\ T;\ R^n)$ measurable functions and $\{\langle B\rangle_t\}_{t\ge 0}$ is the quadratic variation process of one dimensional (only for simplicity) G-Brownian motion $\{B_t\}_{t\ge 0}$. A process $X_t \in M_G^1(0,\ T;\ R^n)$ satisfying the G-BSDE (1.1) is said to be its solution. We assume that both coefficients are monotonically increasing functions. But they do not need to be continuous.

This paper is organized as follows. Section 2 presents some basic notions and definitions. In section 3 we develop the method of upper and lower solutions. The comparison theorem is proved in section 4 while the existence of solutions for G-BSDEs with discontinuous functions is shown in section 5. Appendix is given in section 6.

2. Preliminaries

We remind the following notions and definitions (Denis et al., 2010; Faizullah and Piao, 2012; Faizullah et. al., 2012; Gao, 2009; Li and Peng, 2011; Peng, 2010).

Let Ω be a (non-empty) basic space and $\mathbf{H}$ be a linear space of real valued functions defined on Ω such that the constant $c \in \mathbf{H}$ and if $X_1,\ X_2,\ldots,\ X_n \in \mathbf{H}$ then $\varphi(X_1,\ X_2,\ldots,\ X_n) \in \mathbf{H}$ for each $\varphi \in C_{l.Lip}(R^n)$, where $C_{l.Lip}(R^n)$ is the space of linear functions φ defined as the following

$C_{l.Lip}(R^n) = \{\varphi : R^n \to R \mid \exists C \in R^+, m \in N$

$s.t. |\varphi(x) - \varphi(y)| \le c(1 + |x|^m + |y|^m)|x - y|\}$,

for $x,\ \ y \in R^n$. We consider that $\mathbf{H}$ is the space of random variables.

Definition 1

A functional $\mathsf{E} : \mathbf{H} \to \mathsf{R}$ is said to be a sublinear expectation, if for all $X, Y \in \mathbf{H}$, $c \in \mathsf{R}$ and $\lambda \ge 0$ it satisfies the following properties

1. (Monotonicity). If $X \ge Y$ then $\mathsf{E}[X] \ge \mathsf{E}[Y]$.
2. (Constant preserving). $\mathsf{E}[c] = c$.
3. (Sub-additivity). $E[X+Y] \le E[X] + E[Y] or E[X] - E[Y] \le E[X-Y]$.
4. (Positive homogeneity). $E[\lambda X] = \lambda E[X]$.

The triple $(\Omega, \mathbf{H}, \mathsf{E})$ is called a sublinear expectation space.

Let Ω be the space of all R-valued continuous paths $(w_t)_{t \in \mathsf{R}^+}$ with $w_0 = 0$ equipped with the distance

$$\rho(w^1, w^2) = \sum_{k=1}^{\infty} \frac{1}{2^k} (\max_{t \in [0,k]} |w_t^1 - w_t^2| \wedge 1),$$

and consider the canonical process $B_t(w) = w_t$ for $t \in [0, \infty),\ \ w \in \Omega$ then for each fixed $T \in [0, \infty)$ we have

$$L_{ip}(\Omega_T) = \{\varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_n}) :\ t_1, \ldots, t_n \in [0, T],\ \varphi \in \mathsf{C}_{l.Lip}(\mathsf{R}^n), n \in \mathsf{N}\},$$

where $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T)$ for $t \le T$ and $L_{ip}(\Omega) = \cup_{m=1}^{\infty} L_{ip}(\Omega_m)$.

For $0 = t_0 < t_1 < \ldots < t_n < \infty,\ \ \varphi \in \mathsf{C}_{l.Lip}(\mathsf{R}^n)$ and each

$X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}) \in L_{ip}(\Omega)$,

$$\mathsf{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] = \mathsf{E}[\varphi(\sqrt{t_1 - t_0}\,\xi_1, \ldots, \sqrt{t_n - t_{n-1}}\,\xi_n)].$$

The conditional sublinear expectation of $X \in L_{ip}(\Omega_t)$ is defined by

$$\mathsf{E}[X \mid \Omega_t] = \mathsf{E}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}) \mid \Omega_t] = \psi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_j} - B_{t_{j-1}}),$$

where

$$\psi(x_1, \ldots, x_j) = \mathsf{E}[\varphi(x_1, \ldots, x_j, \sqrt{t_{j+1} - t_j}\,\xi_{j+1}, \ldots, \sqrt{t_n - t_{n-1}}\,\xi_n)],$$

such that ξ_i is G-normally distributed and ξ_{i+1} is independent of $(\xi_1, \xi_2, \ldots, \xi_i)$ for each $i = 1, 2, \ldots, n-1$.

Definition 2

The sublinear expectation $\mathsf{E} : L_{ip}(\Omega) \to \mathsf{R}$ is called a G-expectation if the corresponding canonical process $\{B_t\}_{t \ge 0}$ on the sub-linear expectation space $(\Omega, L_{ip}(\Omega), \mathsf{E})$ is a G-Brownian motion, that is, for $0 \le s < t,$ it satisfies the following conditions

1. $B_0(w) = 0$.
2. The increment $B_{t+s} - B_t$ is $\mathbf{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$-distributed and independent of $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$, for each

$n \in Z^{+}$ and $0 \le t_1 \le \ldots \le t_n \le t.$

The completion of $L_{ip}(\Omega)$ under the norm $\| X \|_p = (E[| X |^p])^{1/p}$ for $p \ge 1$ is denoted by $L_G^p(\Omega)$ and $L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$ for $0 \le t \le T < \infty.$

Itô's integral of G-Brownian motion. Consider the following simple process: Let $p \ge 1$ be fixed. For a given partition $\pi_T = \{t_0, t_1, \ldots, t_N\}$ of $[0, T]$,

$$\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w) I_{[t_i, t_{i+1})}(t),$$

where $\xi_i \in L_G^p(\Omega_t)$, $i = 0,\ 1, \ldots,\ N-1$. The collection containing the above type of processes, that is, containing $\eta_t(w)$ is denoted by $M_G^{p,0}(0,T)$. The completion of $M_G^{p,0}(0,T)$ under the norm $\| \eta \| = \{\int_0^T E[| \eta_v |^p] dv\}^{1/p}$ is denoted by $M_G^p(0,T)$ and for $1 \le p \le q$, $M_G^p(0,\ T) \supset M_G^q(0,\ T).$

Definition 3

For each $\eta_t \in M_G^{2,0}(0,T)$, the Itô's integral of G-Brownian motion is defined as

$$I(\eta) = \int_0^T \eta_v dB_v = \sum_{i=0}^{N-1} \xi_i (B_{t_{i+1}} - B_{t_i}).$$

Definition 4

An increasing continuous process $\{\langle B \rangle_t\}_{t \ge 0}$ with $\langle B \rangle_0 = 0$, defined by

$$\langle B \rangle_t = \lim_{N \to \infty} \sum_{i=0}^{N-1} (B_{t_{i+1}^N} - B_{t_i^N})^2 = B_t^2 - 2\int_0^t B_v dB_v,$$

is called the quadratic variation process of G-Brownian motion.

For the details of the following two definitions see (Denis et al., 2010).

Definition 5

Let $\mathbf{B}(\Omega)$ be the Borel σ -algebra of Ω and $\mathbf{P}$ be a (weakly compact) collection of probability measures P defined on $(\Omega, \mathbf{B}(\Omega))$ then the capacity $\hat{c}(.)$ associated to $\mathbf{P}$ is defined by

$$\hat{c}(A) = \sup_{P \in \mathbf{P}} P(A), \quad A \in \mathbf{B}(\Omega).$$

Definition 6

A set A is said to be polar if its capacity is zero, that is, $\hat{c}(A) = 0$ and a property holds "quasi-surely" $(q.s.)$ if it holds outside a polar set.

The following proposition can be found in (Peng, 2006; 2008).

Proposition 1

For each $X,\ Y \in L_{ip}(\Omega)$ the following properties of $[.\,|\Omega_t]$ hold.

1. If $X \ge Y$ then $E[X | \Omega_t] \ge E[Y | \Omega_t].$
2. $E[\eta | \Omega_t] = \eta$ for each $t \in [0, \infty)$ and $\eta \in L_{ip}(\Omega_t)$.
3. $E[X | \Omega_t] - E[Y | \Omega_t] \le E[X - Y | \Omega_t].$

Also, from (2) and (3) we have $E[X + \eta | \Omega_t] = E[X | \Omega_t] + \eta.$

The following theorem can be found in (Faizullah, 2011). For the proof see appendix.

Theorem 1

Let $X_t,\ Y_t \in M_G^1([0,\ T];\ R^n)$. If $X_t \le Y_t$ for $t \in [0,T]$ and any $w \in \Omega$. Then

$$\int_0^T X_t d\langle B \rangle_t \le \int_0^T Y_t d\langle B \rangle_t.$$

Through out the paper for $X = (x_1,\ x_2, \ldots,\ x_n)$, $Y = (y_1,\ y_2, \ldots,\ y_n)$, $X \le Y$ means $x_i \le y_i$,

$i = 1,\ 2, \ldots,\ n.$

3. The Method of Upper and Lower Solutions

We recall that the concept of upper and lower solutions for the classical SDEs was established in (Assing and Manthey, 1995; Halidias and Kloeden, 2006; Halidias and Michta, 2008) and for G-SDEs in (Faizullah and Piao, 2012; Faizullah and Rahman 2012; Faizullah et. al., 2012).

Definition 7 (Lower and upper solutions)

A process $\alpha_t \in M_G^1(0,\ T;\ R^n)$ is said to be a lower solution of the G-BSDE on the interval $[0,\ T]$ if for any fixed S the inequality (interpreted component wise)

$$\alpha_t \geq E[\alpha_S + \int_t^S b(s,\ \alpha_s)ds + \int_t^S \theta(s,\ \alpha_s)d\langle B\rangle_s \,|\, \Omega_t], \qquad 0 \leq t \leq S \leq T, \quad (3.1)$$

holds q.s.

An upper solution $\beta_t \in M_G^1(0,\ T;\ R^n)$ can be defined in a similar way as above by replacing $\geq$ with $\leq$ and α_t with β_t in (3.1).

Suppose that α_t and β_t are the respective lower and upper solutions of the G-BSDE

$$X_t = E[\xi + \int_t^T b(s,w)ds + \int_t^T \theta(s,w)d\langle B\rangle_s \,|\, \Omega_t]. \qquad (3.4)$$

Define two functions $p,\ q : [0,\ T] \times R^n \times \Omega \to R^n$ by

$$\begin{aligned} p(t,\ x,\ w) &= \min\{\alpha_t(w),\ \max\{\beta_t(w),\ x\}\}, \\ q(t,\ x,\ w) &= p(t,\ x,\ w) - x. \end{aligned} \qquad (3.5)$$

Consider the backward stochastic differential equation

$$X_t = E[\xi + \int_t^T \tilde{b}(s,X_s)ds + \int_t^T \tilde{\theta}(s,X_s)d\langle B\rangle_s \,|\, \Omega_t] \qquad (3.6)$$

where

$$\begin{aligned} \tilde{b}(t,\ x,\ w) &= b(t,\ w) + q(t,\ x,\ w), \\ \tilde{\theta}(t,\ x,\ w) &= \theta(t,w) + q(t,\ x,\ w), \end{aligned} \qquad (3.7)$$

are Lipschitz continuous in x and $\xi \in L_G^1(\Omega_T,\ R^n)$ is given. It is known that the G-BSDE (3.6) has a unique solution $X_t \in M_G^1(0,\ T;\ R^n)$ see the appendix or chapter III page 84 of (Peng, 2010). Also see (Bai and Lin, 2010).

4. Comparison Theorem for G-BSDEs

First we prove an important lemma which will be used in the next comparison theorem.

Lemma 1

Suppose that the respective lower and upper solutions α_t and β_t of the G-BSDE (3.4) satisfy the condition $\alpha_t \geq \beta_t$ for $t \in [0,\ T]$. Then α_t and β_t are lower and upper solutions of the G-BSDE (3.6) respectively.

Proof

Using the given condition $\alpha_t \geq \beta_t$ we have $p(t,\ \alpha_t) = \alpha_t$ and $q(t,\ \alpha_t) = 0$, thus

$$\begin{aligned} &E[\alpha_S + \int_t^S \tilde{b}(s,\ \alpha_s)ds + \int_t^S \tilde{\theta}(s,\ \alpha_s)d\langle B\rangle_s \,|\, \Omega_t] \\ &= E[\alpha_S + \int_t^S [b(s,\ w) + q(s,\ \alpha_s)]ds + \int_t^S [\theta(s,\ w) + q(s,\ \alpha_s)]d\langle B\rangle_s \,|\, \Omega_t] \\ &= E[\alpha_S + \int_t^S b(s,\ w)ds + \int_t^S \theta(s,\ w)d\langle B\rangle_s \,|\, \Omega_t] \leq \alpha_t. \end{aligned}$$

Hence α_t is a lower solution of (3.6). Similarly, we can show that if β_t is an upper solution of the G-BSDE (3.4) then it is an upper solution of the G-BSDE (3.6).

Theorem 2

Suppose that the mappings $b,\ \theta$ are measurable with $\int_t^T E[|J(t,.)|]dt < \infty$ where $J = b\ and\ \theta$ respectively, the respective lower and upper solutions α_t, β_t of the G-BSDE (3.4) with $E[|\alpha_t|] < \infty$, $E[|\beta_t|] < \infty$ satisfy $\alpha_t \ge \beta_t$ for $t \in [0,\ T]$ and $X_T = \xi \in L_G^1(\Omega_T,\ R^n)$ is a given terminal value with $E[|X_T|] < \infty$ such that $\beta_T \le X_T \le \alpha_T$. Then there exists a unique solution $X_t \in M_G^1(0,\ T;\ R^n)$ of the G-BSDE (3.4) such that $\beta_t \le X_t \le \alpha_t$ for $t \in [0,\ T]$ q.s.

Proof

We define the functions $p,\ q : [0,\ T] \times R^n \times \Omega \to R^n$ by (3.5) and consider the backward stochastic differential equation (3.6).

Now the G-BSDE (3.6) has a unique solution and by lemma 1 if α_t and β_t are the lower and upper solutions of the G-BSDE (3.4) respectively then they are for the G-BSDE (3.6). We also note that any solution X_t of the modified G-BSDE (3.6) such that

$$\beta_t \le X_t \le \alpha_t, \quad t \in [0,\ T], \qquad (4.1)$$

q.s. is also a solution of the G-BSDE (3.4). Thus we only need to show that any solution X_t of the modified problem (3.6) does satisfy the inequality (4.1).

Suppose that there exists an arbitrary interval $(t_1,\ t_2) \subset [0,\ T]$ such that $X_{t_2} = \alpha_{t_2} = \zeta$ and $X_t > \alpha_t$ for $t \in (t_1, t_2)$. Then

$$\begin{aligned}
X_t - \alpha_t &= E[\zeta + \int_t^{t_2} \tilde{b}(s,\ X_s)ds + \int_t^{t_2} \tilde{\theta}(s,\ X_s)d\langle B\rangle_s \,|\, \Omega_t] \\
&\quad - E[\zeta + \int_t^{t_2} \tilde{b}(s,\ \alpha_s)ds + \int_t^{t_2} \tilde{\theta}(s,\ \alpha_s)d\langle B\rangle_s \,|\, \Omega_t] \\
&\le E[\zeta + \int_t^{t_2} \tilde{b}(s,\ X_s)ds + \int_t^{t_2} \tilde{\theta}(s,\ X_s)d\langle B\rangle_s \\
&\quad - \zeta - \int_t^{t_2} \tilde{b}(s,\ \alpha_s)ds - \int_t^{t_2} \tilde{\theta}(s,\ \alpha_s)d\langle B\rangle_s \,|\, \Omega_t] \\
&= E[\int_t^{t_2} [b(s,\ w) + q(s,\ X_s)]ds + \int_t^{t_2} [\theta(s,\ w) + q(s,\ X_s)]d\langle B\rangle_s \\
&\quad - \int_t^{t_2} [b(s,\ w) + q(s,\ \alpha_s)]ds - \int_t^{t_2} [\theta(s,\ w) + q(s,\ \alpha_s)]d\langle B\rangle_s \,|\, \Omega_t].
\end{aligned}$$

Since $\alpha_t \ge \beta_t$ gives $p(t,\ \alpha_t) = \alpha_t$ and $q(t,\ \alpha_t) = 0$. Also by supposition $X_t > \alpha_t$ implies $X_t > \beta_t$ so $p(t,\ X_t) = \alpha_t$. Therefore we have $p(t,\ X_t) = \alpha_t$.

$$X_t - \alpha_t \le E[\int_t^{t_2} q(s,\ X_s)ds + \int_t^{t_2} q(s,\ X_s)d\langle B\rangle_s \,|\, \Omega_t] \le 0,$$

because $X_t > \alpha_t$ gives $q(t,\ X_t) = \alpha_t - X_t < 0$ in $(t_1,\ t_2)$. This is a contradiction. Thus $X_t \le \alpha_t$ for $t \in [0,\ T]$. Using similar arguments as above we can show that $X_t \ge \beta_t$ for $t \in [0,\ T]$.

5. G-BSDEs with Discontinuous Coefficients

We now take the following G-BSDE

$$X_t = E[\xi + \int_t^T b(s,\ X_s)ds + \int_t^T \theta(s,\ X_s)d\langle B\rangle_s \,|\, \Omega_t], \qquad t \in [0,\ T]. \qquad (5.1)$$

Here both coefficients $b(t,\ x)$ and $\theta(t,x)$ do not need to be continuous but suppose that they are increasing, that is, if $x \ge y$ then $J(t,\ x) \ge J(t,\ y)$ for $J = b$ and θ respectively (where the inequalities are interpreted component wise).

Theorem 3

Suppose that the mappings $b(t,\ x)$, $\theta(t,\ x)$ are increasing in x, α_t and β_t are the respective lower and upper solutions of the G-BSDE (5.1) with $\int_t^T E[|J(\alpha_t)|]dt < \infty$, $\int_t^T E[|J(\beta_t)|]dt < \infty$ for $J = b$ and θ respectively and $\alpha_t \geq \beta_t$ for $t \in [0,\ T]$.

Then there exists at least one solution $X_t \in M_G^1(0,\ T;\ R^n)$ of the G-BSDE (5.1) such that $\beta_t \leq X_t \leq \alpha_t$ for $t \in [0,\ T]$ q.s.

Proof

We define the space of all d-dimensional stochastic processes by $\mathbf{H}$, that is, $\mathbf{H} = \{X = \{X_t,\ t \in [0,\ T]\} : E[|X_t|] < \infty\}$ with the norm $\|X_t\| = \int_t^T E[|X_t|]dt$ for all $t \in [0,\ T]$, which is a Banach space, see chapter 4 page 45 of (Peng, 2010) or see (Peng, 2006; 2008).

Now we denote the order interval $[\beta,\ \alpha]$ in $\mathbf{H}$ by $\mathbf{K}$, that is, $\mathbf{K} = \{X : X \in \mathbf{H}\ \ and\ \ \beta_t \leq X_t \leq \alpha_t\}$ for $t \in [0,\ T]$, which is closed and bounded by the above norm. By using the monotone convergence theorem one can prove the convergence of a monotone sequence that belongs to $\mathbf{K}$ in $\mathbf{H}$ (Denis et al., 2010). Thus $\mathbf{K}$ is a regularly ordered metric space with the above norm. It is clear that for any process $V \in \mathbf{K}$, α and β are lower and upper solutions for the G-BSDE

$$X_t = E[\xi + \int_t^T b(s,\ V_s)ds + \int_t^T \theta(s,\ V_s)d\langle B\rangle_s \,|\, \Omega_t], \qquad t \in [0,\ T]. \qquad (5.2)$$

Thus by theorem 2, for any given $X_T \in L_G^1(\Omega_T,\ R^n)$ with $E[|X_T|] < \infty$ and $\beta_T \leq X_T \leq \alpha_T$, the G-BSDE (5.2) has a unique solution $X_t \in M_G^1(0,\ T;\ R^n)$ such that $\beta_t \leq X_t \leq \alpha_t$ for $t \in [0,\ T]$ q.s.

We define an operator $N : \mathbf{K} \to \mathbf{K}$ by $N(V) = X$, where X is the unique solution of the G-BSDE (5.2). We will use theorem 6 to show that N has a fixed point, which is then the required solution. If we show that N is an increasing mapping then it has a fixed point.

We have to prove that if $V^{(1)}$ and $V^{(2)}$ are stochastic processes in $\mathbf{K}$ such that $V_t^{(1)} \leq V_t^{(2)}$ then $X_t^{(1)} \leq X_t^{(2)}$ for all $t \in [0,\ T]$, where $X^{(1)} = N(V^{(1)})$ and $X^{(2)} = N(V^{(2)})$.

Let $V_t^{(1)} \leq V_t^{(2)}$ for all $t \in [0,\ T]$ and define $X^{(1)} = N(V^{(1)})$, $X^{(2)} = N(V^{(2)})$ where $V^{(1)},\ V^{(2)} \in \mathbf{K}$. Since it is given that the coefficient b and θ are increasing functions therefore $X_t^{(1)}$ is an upper solution of the G-BSDE

$$X_t = E[\xi + \int_t^T b(s,\ V_s^{(2)})ds + \int_t^T \theta(s,\ V_s^{(2)})d\langle B\rangle_s \,|\, \Omega_t], \qquad t \in [0,\ T]. \qquad (5.3)$$

But this problem has a lower solution α_t. Hence by theorem 2, the G-BSDE (5.3) has a solution $X_t^{(2)}$ such that $X_t^{(1)} \leq X_t^{(2)} \leq \alpha_t$. Thus N is an increasing mapping and by theorem 6, it has a fixed point $X^{(*)} = N(X^{(*)}) \in \mathbf{K}$ such that $\beta_t \leq X_t^{(*)} \leq \alpha_t$ q.s. where

$$X_t^{(*)} = E[\xi + \int_t^T b(s,\ X_s^{(*)})ds + \int_t^T \theta(s,\ X_s^{(*)})d\langle B\rangle_s \,|\, \Omega_t], \qquad t \in [0,\ T].$$

Example 1

Consider the following scalar G-BSDE

$$X_t = E[\xi + \int_t^T u(X_s)ds + \int_t^T u(X_s)d\langle B\rangle_s \,|\, \Omega_t], \qquad t \in [0,\ T], \qquad (3.3)$$

where $u : R \to R$ is the unit step function defined by $u(x) = \begin{cases} 0, \text{ if } x < 0\,; \\ 1, \text{ if } x \geq 0. \end{cases}$

Then $\alpha_t = E[\xi + \int_t^T ds + \int_t^T d\langle B\rangle_s \,|\, \Omega_t]$ and $\beta_t = E[\xi \,|\, \Omega_t] = \xi$ for $t \in [0,\ T]$ are the respective lower and upper solutions for the G-BSDE (3.3) which are shown below. Since $\alpha_t = E[\xi + \int_t^T ds + \int_t^T d\langle B\rangle_s \,|\, \Omega_t] = E[\alpha_S + \int_t^S ds + \int_t^S d\langle B\rangle_s \,|\, \Omega_t]$ where $\alpha_S = E[\xi + \int_S^T ds + \int_S^T d\langle B\rangle_s \,|\, \Omega_t]$ for any fixed S such that $0 \le t \le S \le T$. And

$$E[\alpha_S + \int_t^S ds + \int_t^S d\langle B\rangle_s \,|\, \Omega_t] \ge E[\alpha_S + \int_t^S u(\alpha_s) ds + \int_t^S u(\alpha_s) d\langle B\rangle_s \,|\, \Omega_t],\ 0 \le t \le S \le T,$$

which shows that $\alpha_t = E[\xi + \int_t^T ds + \int_t^T d\langle B\rangle_s \,|\, \Omega_t]$ is the lower solution of (3.3).

Similarly, one can show that $\beta_t = E[\xi \,|\, \Omega_t] = \xi$ for $t \in [0,\ T]$ is the upper solution of (3.3). Thus by the above theorem 3 there exists at least one solution $X_t^{(*)}$ of the G-BSDE (3.3) such that $\xi \le X_t^{(*)} \le E[\alpha_S + \int_t^S ds + \int_t^S d\langle B\rangle_s \,|\, \Omega_t]$.

Remark 1

The above results (i.e. theorem 2 and theorem 3) are more general. Therefore the existence theory for G-BSDEs with upper and lower solution in the reversed order where the first or second coefficient is a discontinuous function can be obtained in a similar manner.

6. Appendix

For the following definition and theorem see (Heikkila and Hu, 1993).

Definition 9

An ordered metric space M is called regularly (resp. fully regularly) ordered, if each monotone and order (resp. metrically) bounded ordinary sequence of M converges.

Theorem 6

If $[a,\ b]$ is a nonempty order interval in a regularly ordered metric space, then each increasing mapping $N:[a,\ b] \to [a,\ b]$ has the least and the greatest fixed point.

Proof of theorem 1

Since $\{\langle B\rangle_t : t \ge 0\}$ is an increasing continuous process with $\langle B\rangle_0 = 0$. Therefore for any $w \in \Omega$ and $t_{i+1} \ge t_i$, $\langle B\rangle_{t_{i+1}} - \langle B\rangle_{t_i} \ge 0,\ \ i = 0,\ 1, \ldots,\ N-1$. Also for $X_t,\ Y_t \in M_G^1([0,\ T];\ R^n)$, $X_t = \sum_{i=0}^{N-1} \xi_i I_{[t_i,\ t_{i+1})}$ and $Y_t = \sum_{i=0}^{N-1} \tilde{\xi}_i I_{[t_i,\ t_{i+1})}$ where $\xi_i,\ \tilde{\xi}_i \in L_G^2(\Omega_i)$, $i = 0,\ 1, \ldots,\ N-1$. Then $X_t \le Y_t$ implies that

$$\sum_{i=0}^{N-1} \xi_i I_{[t_i,\ t_{i+1})} \le \sum_{i=0}^{N-1} \tilde{\xi}_i I_{[t_i,\ t_{i+1})},$$

this yields

$$\sum_{i=0}^{N-1} \xi_i [\langle B\rangle_{t_{i+1}} - \langle B\rangle_{t_i}] \le \sum_{i=0}^{N-1} \tilde{\xi}_i [\langle B\rangle_{t_{i+1}} - \langle B\rangle_{t_i}].$$

Hence

$$\int_0^T X_t d\langle B\rangle_t \le \int_0^T Y_t d\langle B\rangle_t.$$

Remark 2

The above theorem shows that the G-Ito's integral w.r.t. the quadratic variation process satisfies the monotonic property. Also if $X_t \le 0$ then $\int_0^T X_t d\langle B\rangle_t \le 0.$

We now consider the G-BSDE (3.6). Define a mapping $\Lambda_t : M_G^1(0,\ T;\ R^n) \to M_G^1(0,\ T;\ R^n)$ on a fixed interval [0, T] by

$$\Lambda_t(X) = E[\xi + \int_t^T \tilde{b}(s,\ X_s) ds + \int_t^T \tilde{\theta}(s,\ X_s) d\langle B\rangle_s \,|\, \Omega_t],\quad t \in [0,\ T].$$

Lemma 4

For each $X,\ \hat{X} \in M_G^1(0,\ T;\ R^n)$ we have the following estimate;

$$E[|\Lambda_t(X) - \Lambda_t(\hat{X})|] \le C\int_0^t E[|X_s - X_s|] ds,\qquad t \in [0,\ T], \qquad (7.2)$$

where C is an arbitrary constant, depends only on the Lipschitz constant K.

Proof

Using lemma 43 (Peng, 2008) or see (Peng, 2006) and the Lipschitz condition on functions b and θ we have

$$\begin{aligned}
E[|\Lambda_t(X)-\Lambda_t(\hat{X})|] &\le E[\int_t^T|\tilde{b}(s,\ X_s)-\tilde{b}(s,\ X_s)|ds+\int_t^T|\tilde{\theta}(s,\ X_s)-\tilde{\theta}(s,\ \hat{X}_s)|d\langle B\rangle_s] \\
&\le E[K\int_t^T|X_s-\hat{X}_s|ds+K\int_t^T|X_s-X_s|d\langle B\rangle_s] \\
&\le K\int_t^T E[|X_s-\hat{X}_s|]ds+K\int_t^T E[|X_s-X_s|]ds \\
&=2K\int_t^T E[|X_s-\hat{X}_s|]ds=C\int_t^T E[|X_s-X_s|]ds,
\end{aligned}$$

where $C=2K$ is an arbitrary constant. Hence it is the required result.

Theorem 7

The backward stochastic differential equation has a unique solution $X_t\in M_G^1(0,\ T;\ R^n)$.

Proof

To show that the G-BSDE (3.6) has a unique solution we prove that $\Lambda(X)$ is a contraction mapping. Multiplying by e^{2Ct} and integrating on $[0,\ T]$ yields

$$\begin{aligned}
\int_0^T E[|\Lambda_t(X)-\Lambda_t(\hat{X})|]e^{2Ct}dt &\le C\int_0^T\int_t^T E[|X_s-X_s|]e^{2Ct}dsdt \\
&=C\int_0^T E[|X_s-\hat{X}_s|]\int_0^v e^{2Ct}dtds \\
&=\frac{1}{2}C\int_0^T(e^{2Cs}-1)E[|X_s-\hat{X}_s|]ds \\
&\le\frac{1}{2}\int_0^T e^{2Cs}E[|X_s-\hat{X}_s|]ds.
\end{aligned}$$

Thus

$$\int_0^T E[|\Lambda_t(X)-\Lambda_t(\hat{X})|]e^{2Ct}dt\le\frac{1}{2}\int_0^T e^{2Ct}E[|X_t-X_t|]dt. \qquad (7.3)$$

One can observe that the following two norms are equivalent in $M_G^1(0,\ T;\ R^n)$, i.e.,

$$\int_0^T E[|X_t|]dt\sim\int_0^T E[|X_t|]e^{2Ct}dt.$$

Hence from (7.3) we get that $\Lambda(X_t)$ is a contraction mapping.

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