Boundary Value Problems, Fredholm Integral equations, SOR and KSOR Methods

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Abstract: The main interest is the numerical treatment of boundary value problems of the second and fourth order with their equivalent Fredholm integral equation forms. Comparison of the performance of the SOR and the KSOR methods on the systems arise from the differential form and those arise from the equivalent Fredholm form by using discretization techniques of the same accuracy are considered. It is found that the SOR and the KSOR use the same number of iterations with the same system but with different relaxation factors. The number of iterations in case of the integral representations is approximately less than quarter the number of iterations in case of the differential representations in the same time the computational work per iteration in the differential form (sparse systems) is less than that of the integral form. We discussed the advantages of using the integral representation over the use of the differential representation especially when we have a good approximation of the relaxation parameters. All calculations are done with the help of computer algebra system (MATHEMATICA 8.0).
[I.K.Youssef and R.A.Ibrahim. **Boundary Value Problems, Fredholm Integral equations, SOR and KSOR Methods.** *Life Sci J* 2013;10(2):304-312]. (ISSN: 1097-8135). http://www.lifesciencesite.com. 48

Keywords: boundary value problems; Fredholm integral equation; SOR and KSOR

1. Introduction:

Many problems in science and engineering can be formulated as mathematical models in a form of differential equations (involve local interactions) or integral equations (involve local and global interactions). Most ordinary differential equations can be expressed as integral equations, but the reverse is not true in general, [1, 2]. Usually, the analytical solutions of real models are excluded for different reasons concerning the structure of the model, the domain under consideration or the properties of the required solution. There are many interesting numerical treatments of such models each has its benefits and limitations, the final stage in the numerical treatment is the solution of an algebraic system. We concerned in this work with linear models which can be formulated equivalently in both differential and integral equations and consequently give rise to large linear algebraic systems and the problem is reduced to that of the efficient use of iterative techniques for solving large linear systems. Youssef, [3] introduced the KSOR method as a new variant of the SOR method in which the domain of the relaxation parameter is extended and the sensitivity around the optimum value of the relaxation parameter is decreased. Because, differential equations involve local interactions, they thus give rise to systems of large linear equations with sparse coefficient matrices, these sparse systems in many cases tend to be badly conditioned. Integral equations by contrast in many cases give rise to dense large well-conditioned coefficient matrices, [4]. The most advantages of using integral equations can be summarized in the following three points, the integral equation representation usually involve fewer dimensions than the differential equation representation as illustrated in the fourth order case below, the integral equation (Fredholm) representation include the boundary conditions which the problem must satisfy in addition to the convenient theory of existence and uniqueness available [5].

We consider the relation between the numerical treatment of two well- known two point boundary value problems and the numerical treatment of the equivalent integral equations:

The first:

$$y''(x)=f(x,y(x)), \quad (0\le x\le 1)$$
$$y(0)=\alpha, \quad y(1)=\beta \tag{1}$$

And its equivalent second kind Fredholm integral equation is

$$y(x)=\lambda\int_0^1 k(x,t)y(t)dt$$
$$-\int_0^1 k(x,t)f(t,y(t))dt+(\beta-\alpha)x+\alpha \tag{2}$$

The second is [5]:

$$y^{(4)}(x)=\lambda y(x), \quad (0\le x\le 1)$$
$$y(0)=y''(0)=\sigma,\ y(1)=y''(1)=\tau \tag{3}$$

And its equivalent second kind Fredholm integral equation is:

$$y(x)=\lambda\int_0^1\left\{\int_0^1 k(x,s)k(s,t)ds\right\}y(t)dt + +\sigma\left(1+\frac{x^2}{2}\right)+\left(\frac{5\tau-8\sigma}{6}\right)x+\left(\frac{\tau-\sigma}{6}\right)x^3 \quad (4)$$

Where λ, σ and τ are real parameters.

1.1 The finite Difference approximations [6]:

The basic idea of the finite difference approximation is the replacement of derivatives or integrals by difference approximations; accordingly we obtain relations between functional values. The region [0, 1]or in general [a, b] of the differential or integral equations is super imposed with a uniform mesh with mesh size $h > 0$, and the grid points are defined by

$$x_j = a + jh;\ j = 0,1,\ldots n;\quad h = \frac{b-a}{n}$$

1.1.1 The central difference approximations:

Let y_j denote the functional value at the point x_j, the common central difference approximations for the second and fourth order derivatives are:

$$y''(i) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \quad (5)$$

$$y^{(4)}(i) = \frac{y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}}{h^4} \quad (6)$$

1.1.2.The trapezoidal rule:

It is well known that the value of a definite integral can be approximated by a combination of functional values of the integrand with different methods depending on the required accuracy and the grid points used. The trapezoidal rule uses only the end points of the interval of integration and gives second order accuracy and it takes the form:

$$\int_a^b f(x)dx = \frac{b-a}{2}\{f(a)+f(b)\} - \frac{(b-a)^3}{12}f''$$

The composite form of the trapezoidal rule takes the form

$$\int_a^b f(x)dx = \frac{h}{2}\left\{f(x_0)+2\sum_{j=1}^{n-1}f(x_j)+f(x_n)\right\} - \frac{(b-a)}{12}h^2 f'' \quad (7)$$

Where $\frac{b-a}{n}h^2 f''(\zeta)$ is the error term, and $\zeta \in (a,b)$

1.2. Iterative methods:

The general form of a linear algebraic system $AX = b$ can be written in component form as:

$$\sum_{j=1}^{m} a_{ij}x_j = b_i;\ i = 1,2,\cdots,m$$

For the use of standard iterative methods, the equations are arranged such that $a_{ii} \neq 0$.

Usually, we write $A = D - L - U$ [3, 6], where D is the diagonal part of A , $-L$, $-U$ are the lower and the upper parts of A .

1.2.1 Jacobi method:

The Jacobi method is one of the simplest iterative methods it is a direct application of the fixed point theorem. The eigenvalues of the iteration matrix of the Jacobi method play a central part in the selection of the appropriate relaxation parameters.

$$x_i^{[n+1]} = \frac{1}{a_{ii}}\sum_{\substack{j=1\\ j\neq i}}^{m} a_{ij}x_j^{[n]},\ i = 1,2,\ldots,m$$

$$X^{[n+1]} = D^{-1}(L+U)X^{[n]} + D^{-1}b \quad (8)$$

Where the iteration matrix of Jacobi method is

$$T_j = D^{-1}(L+U)$$

1.2.2 Successive over relaxation method (SOR):

Gauss Seidel method is known as a modification of Jacobi method in the sense of using the most recent calculated values. The successive over relaxation SOR method generalizes the Gauss Seidel method in the sense of using a relaxation parameter $\omega \in (0,2)$. It is well known that $\omega = 1$ gives the Gauss Seidel method, moreover suitable choices of ω increases the convergence.

$$x_i^{[n+1]} = x_i^{[n]} + \frac{\omega}{a_{ii}}\left\{b_i - \sum_{j=1}^{i-1}a_{ij}x_j^{[n+1]} - \sum_{j=i}^{m}a_{ij}x_j^{[n]}\right\} \quad (9)$$

$$,i = 1,2,\ldots,m$$

Or, in matrix notations

$$X^{[n+1]} = (D-\omega L)^{-1}\{(1-\omega)D + \omega U\}X^{[n]} + (D-\omega L)^{-1}b \quad (10)$$

With iteration matrix

$$T_{SOR} = (D-\omega L)^{-1}\{(1-\omega)D+\omega U\};\ \omega \in (0,2)$$

1.2.3 KSOR method:

The KSOR method introduced in [3], in which it is assumed that it is possible to use the current component in addition to the most recent calculated components used in the SOR

$$x_i^{[n+1]} = x_i^{[n]} + \frac{\omega^*}{a_{ii}}\left\{b_i - \sum_{j=1}^{i-1}a_{ij}x_j^{[n+1]} - \sum_{j=i+1}^{m}a_{ij}x_j^{[n]} - a_{ii}x_j^{[n+1]}\right\}$$

$$,i = 1,2,\ldots,m$$

$$X^{[n+1]} = ((1+\omega^*)D - \omega^* L)^{-1}(D+\omega^* U)X^{[n]} + ((1+\omega^*)D - \omega^* L)^{-1}b \quad (11)$$

With iteration matrix

$$T_{KSOR} = ((1+\omega^*)D - \omega^* L)^{-1}(D+\omega^* U);\ \omega^* \in \Re - [-2,0]$$

The rate of convergence of the iterative method depends on the spectral radius of the iteration matrix. It is well known that the smaller spectral radius of the iteration matrix the faster rate of convergence of the corresponding iterative method.

2. BVP and Fredholm integral equations

It is well known that boundary value problems (of the second and fourth orders) can be formulated as Fredholm integral equations with some tricks of integrations. The integral equation representation involves fewer dimensions than the differential equation representation. Moreover, the integral equation (Fredholm) representation includes the boundary conditions which the problem must satisfy [7].

Problem (1):

We consider a linear second order boundary value problem of the form

$$y''(x)+\lambda y(x)=f(x);(0\le x\le 1),\quad y(0)=\alpha,\ y(1)=\beta \tag{12}$$

Integrating twice with respect to x and using the replacement lemma, we obtain the following second kind non-homogeneous Fredholm integral equation

$$y(x)=\lambda\int_0^1 k(x,t)y(t)dt-\int_0^1 k(x,t)f(t)dt+(\beta-\alpha)x+\alpha \tag{13}$$

Problem (2):

We consider the fourth order differential equation:

$$y^{(4)}(x)=\lambda y(x);0\le x\le 1;\quad y(0)=y''(0)=\sigma,\ y(1)=y''(1)=\tau \tag{14}$$

This equation is equivalent to

$$\psi''(x)=-\lambda y(x),\ y''(x)=-\psi(x);(0\le x\le 1),\quad \psi(0)=-\sigma,\ \psi(1)=-\tau;\ y(0)=\sigma,\ y(1)=\tau \tag{15}$$

Employing the same philosophy as in the second order case, integrating with respect to x and using the replacement lemma, we obtain the following second kind Fredholm non-homogenous integral equation

$$y(x)=\lambda\int_0^1\{\int_0^1 k(x,s)k(s,t)ds\}y(t)dt+\sigma(1+\frac{x^2}{2})+(\frac{5\tau-8\sigma}{6})x+\frac{(\tau-\sigma)}{6}x^3 \tag{16}$$

The integral equation are called homogenous if the part

$$\sigma(1+\frac{x^2}{2})+(\frac{5\tau-8\sigma}{6})x+\frac{(\tau-\sigma)}{6}x^3=0$$

Where the kernel $k(x,t)$ is defined as

$$K(x,t)=\begin{cases}t(1-x); & t\le x\\ x(1-t); & x\le t\end{cases}$$

Theorem1: there exits ω and ω^* that make the algebraic system corresponding to integral equation (13), converges faster than that obtained from the equivalent system obtained from the differential equation (12) provided we use methods of approximations of the same accuracy.

Proof:

Consider the equation $y''(x)+\lambda y(x)=f(x)$; in this equation using finite difference scheme (5) and putting $x_i=ih$, h is the step size, $i=0,1,\dots n$

$$y_{i-1}+(h^2\lambda-2)y_i+y_{i+1}=h^2f(x_i) \tag{17}$$

Notice that for $i=1$ and $i=n$ the equation will involve y_0 and y_{n+1} which are known quantities. Thus from (17) we get a linear system of the form

$AY=b$

where A is the coefficient matrix:

$$A=\begin{pmatrix}\lambda h^2-2 & 1 & 0 & 0 & \dots & 0 & 0 & 0\\ 1 & \lambda h^2-2 & 1 & 0 & \dots & 0 & 0 & 0\\ 0 & 1 & \lambda h^2-2 & 1 & \dots & 0 & 0 & 0\\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots\\ 0 & 0 & 0 & 1 & \dots & 1 & \lambda h^2-2 & 1\\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & \lambda h^2-2\end{pmatrix} \tag{18}$$

Y is the vector of unknown variables and b is the right hand side

The matrix A is tridiagonal, it is strongly diagonally dominant for values of λ which makes the term $\lambda y(x)$ has opposite sign to that of $y''(x)$ as in example (1) below.

The above mentioned iterative methods converge; the rate of convergence depends on the relaxation parameter chosen.

But according to the equation

$$y(x)=\lambda\int_0^1 k(x,t)y(t)dt-\int_0^1 k(x,t)f(t)dt+(\beta-\alpha)x+\alpha \tag{19}$$

Naming $F(x)=(\beta-\alpha)x+\alpha-\int_0^1 k(x,t)f(t)dt$

So we have

$$y(x)=\lambda\int_0^1 k(x,t)y(t)dt+F(x) \tag{20}$$

Using the trapezoid rule (7) to approximate the integral in (13), we obtain a functional relation [8] which is satisfied at each point x_j of the interval of integration. Accordingly we find the linear system of algebraic equations, $A_1Y_1=B$ where the matrix of the coefficients A_1 has the form

$$A_1=\begin{pmatrix}1-\lambda h^2(1-h) & -\lambda h^2(1-2h) & -\lambda h^2(1-3h) & \cdots & -\lambda h^2(1-(n-1)h)\\ -\lambda h^2(1-2h) & 1-2\lambda h^2(1-2h) & -2\lambda h^2(1-3h) & \cdots & -2\lambda h^2(1-(n-1)h)\\ -\lambda h^2(1-3h) & -2\lambda h^2(1-3h) & 1-3\lambda h^2(1-3h) & \cdots & -3\lambda h^2(1-(n-1)h)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -\lambda h^2(1-(n-1)h) & -2\lambda h^2(1-(n-1)h) & -3\lambda h^2(1-(n-1)h) & \cdots & 1-(n-1)\lambda h^2(1-(n-1)h)\end{pmatrix} \tag{21}$$

It is clear that for small values of h and λ we can prove with the help of Gerschgorin theorem that the matrix A_1 is strictly diagonally dominant, positive definite and symmetric.

The quantities

$$\lambda h^2(1-h), 2\lambda h^2(1-2h), \ldots, (n-1)\lambda h^2(1-(n-1)h)$$

are very small comparing with unity.

Theorem2: the algebraic system corresponding to equation (14) is divergent for every ω and ω^* while that corresponding to equivalent system (15) and (16) are converges.

Proof:

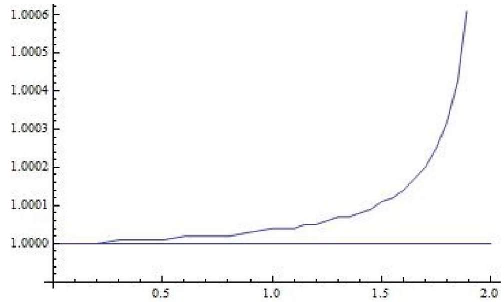

Figure 1: The behavior of the spectral radius of the iteration matrix T_{SOR} as a function in ω for a general fourth order differential equation (14) and (30).

From the previous figure we see that there is no value of ω and ω^* gives a value of the spectral radius of the iteration matrix of Jacobi less than one, consequently there is no value of ω and ω^* make the resultant linear system of the differential equation convergent.

But according to the system of differential equations

$$\psi''(x) = -\lambda y(x), \quad y''(x) = -\psi(x); (0 \le x \le 1)$$

$$\psi(0) = -\sigma, \quad \psi(1) = -\tau; \quad y(0) = \sigma, \quad y(1) = \tau$$

Using finite difference method for this system with the notation introduced in section (1.1) we get

$$\begin{aligned} &\psi_{i-1} - 2\psi_i + \psi_{i+1} + \lambda h^2 y_i = 0; \\ &y_{i-1} - 2y_i + y_{i+1} + h^2\psi_i = 0 \end{aligned} \qquad (22)$$

$$\psi(0) = -\sigma, \psi(1) = -\tau; y(0) = \sigma, y(1) = \tau$$

So equation (22) gives a linear system of algebraic equations of dimension $2n-2$, of the form

$$A_2 Y = b \qquad (23)$$

Where A_2, is the coefficients matrix Y is the corresponding vector of unknown variables and b is the right hand side matrix.

The matrix A_2 is sparse and banded; computing the spectral radius of the iteration matrix of Jacobi method of the system (23) we find it is less than unity which means that the system is convergent.

But according to the equation (16)

$$y(x) = \lambda \int_0^1 \{\int_0^1 k(x,s)k(s,t)ds\} y(t)dt + \sigma(1 + \frac{x^2}{2}) + (\frac{5\tau - 8\sigma}{6})x + \frac{(\tau - \sigma)}{6}x^3$$

On using the composed trapezoidal rule (7), the above equation can be written as a linear system of algebraic equations.

For explanation take $n = 10, \lambda = 1$ since $y(0)$ and $y(1)$ are given then we get a system of dimension 9 of the form

$$A_3 Y = b \qquad (24)$$

Where

$$A_3 = \begin{pmatrix} 0.99925 & -0.001296 & -0.001134 & -0.000972 & -0.00081 & -0.000648 & -0.000486 & -0.000324 & -0.000162 \\ -0.00013867 & 0.998667 & -0.00209067 & -0.001792 & -0.0014933 & -0.00119467 & -0.000896 & -0.00059733 & -0.00029867 \\ -0.000252 & -0.000504 & 0.99825 & -0.002352 & -0.00196 & -0.001568 & -0.001176 & -0.000784 & -0.000392 \\ -0.000352 & -0.000704 & -0.001056 & 0.998 & -0.00216 & -0.001728 & -0.00196 & -0.000864 & -0.000432 \\ -0.00041667 & -0.00083333 & -0.00125 & -0.0016667 & 0.997917 & -0.0016667 & -0.00125 & -0.00083333 & -0.00041667 \\ -0.00432 & -0.000864 & -0.00196 & -0.001728 & -0.00216 & 0.998 & -0.001056 & -0.000704 & -0.000352 \\ -0.000392 & -0.000784 & -0.001176 & -0.001568 & -0.00196 & -0.002352 & 0.99825 & -0.000504 & -0.000252 \\ -0.00029867 & -0.00059733 & -0.000896 & -0.00119467 & -0.0014933 & -0.001792 & -0.00209067 & 0.998667 & -0.00013867 \\ -0.000162 & -0.000324 & -0.000486 & -0.000648 & -0.00081 & -0.000972 & -0.001134 & -0.001296 & 0.99925 \end{pmatrix}$$

The matrix A_3 is always strictly diagonally dominant, computing the spectral radius of the iteration matrix of Jacobi method for A_3 we find that it is less than unity, which mean that the system (24) is convergent.

Theorem 3: the SOR and the KSOR are completely consistent in the sense of Young [8] for both algebraic systems obtained from differential and integral equations.

2. Numerical Examples:

Example 1:

Consider the second order B.V.P. **[9]**

$$-y''(x) + \pi^2 y(x) = 2\pi^2 \sin(\pi x); 0 \le x \le 1; y(0) = y(1) = 0 \qquad (25)$$

Whose exact solution is:

$$y(x) = \sin(\pi x) \qquad (26)$$

On using the theoretical methodology described for the general second order boundary value problem (12), we obtain the following Fredholm integral equation

$$y(x) = 2\sin(\pi x) - \pi^2 \int_0^x t(1-x)y(t)dt - \pi^2 \int_x^1 x(1-t)y(t)dt \qquad (27)$$

It is an easy task to see that the Fredholm integral equation (27) satisfies the boundary conditions in (25). Moreover the closed form solution (26) satisfies both the differential equation (25) and the integral equation (27)

Using the finite difference described in equation (17) we obtain a linear system of algebraic equations, corresponding to differential equation (25) with coefficient matrix ($h = 0.1$).

$$A = \begin{pmatrix} 2.098696 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2.098696 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2.098696 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2.098696 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2.098696 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2.098696 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2.098696 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2.098696 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2.098696 \end{pmatrix} \quad (28)$$

Using the trapezoidal rule (7) we obtain a functional relation, which is valid at each point of the grid points $x_j = jh; j = 1(1)9; h = 0.1$

We obtain a linear system of algebraic equations, corresponding to the integral equation (27) with coefficient matrix ($h = 0.1$).

$$B = \begin{pmatrix} 1.0888264 & 0.0789568 & 0.0690872 & 0.0592176 & 0.049348 & 0.0394784 & 0.0296088 & 0.0197392 & 0.0098696 \\ 0.0789568 & 1.157914 & 0.138174 & 0.118435 & 0.098696 & 0.0789568 & 0.0592176 & 0.0394784 & 0.0197392 \\ 0.0690872 & 0.138174 & 1.207262 & 0.177653 & 0.1488044 & 0.118435 & 0.0888264 & 0.0592176 & 0.0296088 \\ 0.0592176 & 0.118435 & 0.177653 & 1.23687 & 0.197392 & 0.157914 & 0.118435 & 0.0789568 & 0.0394784 \\ 0.049348 & 0.098696 & 0.148044 & 0.197392 & 1.24674 & 0.197392 & 0.148044 & 0.098696 & 0.049348 \\ 0.0394784 & 0.0789568 & 0.118435 & 0.157914 & 0.197392 & 1.23687 & 0.177653 & 0.118435 & 0.0592176 \\ 0.0296088 & 0.0592176 & 0.0888264 & 0.118435 & 0.1488044 & 0.177653 & 1.207262 & 0.138174 & 0.0690872 \\ 0.0197392 & 0.0394784 & 0.0592176 & 0.0789568 & 0.098696 & 0.118435 & 0.138174 & 1.157914 & 0.0789568 \\ 0.0098696 & 0.0197392 & 0.0296088 & 0.0394784 & 0.049348 & 0.0592176 & 0.0690872 & 0.0789568 & 1.0888264 \end{pmatrix} \quad (29)$$

In the following, we summarize the results of using the SOR and the KSOR in the strongly diagonally dominant systems arising from the differential form and the system arising from the integral form. Also we illustrate the behavior of the spectral radius of the iteration matrices of the above systems varses the relaxation parameters as in figures [2,3,4,5]. We note that, we shifted the origin slightly to make the graphs readable.

Table 1: The solution of the algebraic system obtained from both the differential equation (25) and the corresponding Fredholm integral equation (27), with $h = 0.1$, the number of iterations for nearly optimal values of the relaxation parameters was given.

x	Y_{ext}	Differential equation		Integral equation	
		SOR $\omega = 1.402$ (25 iter.)	KSOR $\omega^* = -3.65$ (25 iter.)	SOR $\omega = 0.96$ (8 iter.)	KSOR $\omega^* = -19$ (8 iter.)
0.1	0.309017	0.31031	0.31031	0.307745	0.307745
0.2	0.587785	0.59022	0.59022	0.585366	0.585366
0.3	0.809017	0.812357	0.812357	0.805688	0.805688
0.4	0.951057	0.954978	0.954978	0.947142	0.947142
0.5	1	1.00412	1.00412	0.995884	0.995884
0.6	0.951057	0.954974	0.954974	0.947142	0.947142
0.7	0.809017	0.812348	0.812348	0.805688	0.805688
0.8	0.587785	0.590205	0.590205	0.585366	0.585366
0.9	0.309017	0.310289	0.310289	0.307745	0.307745

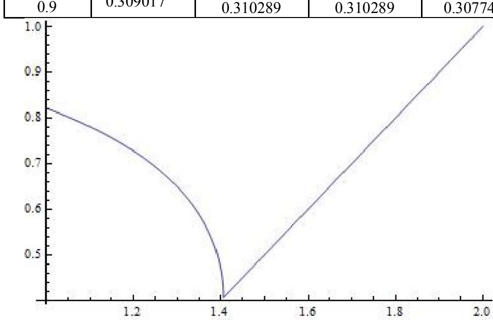

Figure 2: The behavior of the spectral radius of the iteration matrix T_{SOR} as a function in ω for equation (25),h=0.1.

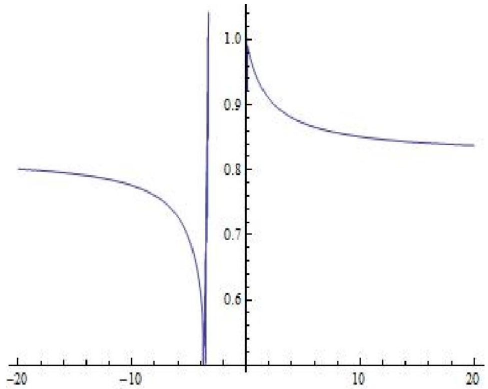

Figure 3: The behavior of the spectral radius of the iteration matrix T_{KSOR} as a function in ω^* for equation (25),h=0.1.

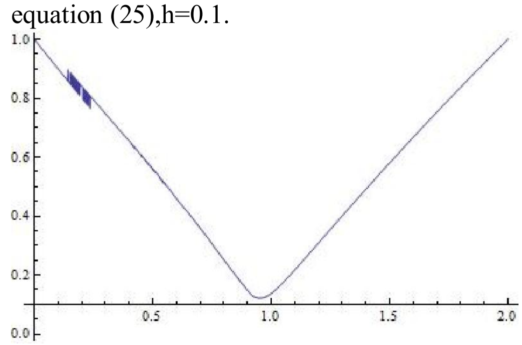

Figure 4: The behavior of the spectral radius of the iteration matrix T_{SOR} as a function in ω for equation (27),h=0.1.

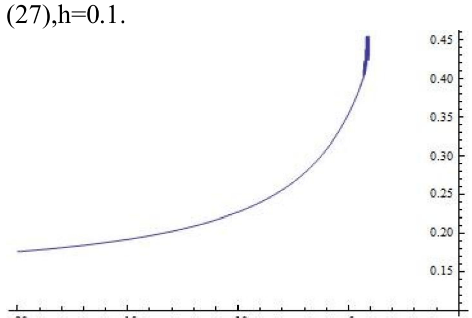

Figure 5: The behavior of the spectral radius of the iteration matrix T_{KSOR} as a function in ω^* for equation (27), h=0.1.

Example (2) **[5]:**

The normal modes of free flexural vibration of a thin, uniform rod of unit length are governed approximately by the differential equation

$$y^{(4)}(x) = \lambda y(x); 0 \le x \le 1; \qquad (30)$$
$$y(0) = y''(0) = 1, y(1) = y''(1) = e$$

Whose exact solution when $\lambda = 1$ is

$$y(x) = e^{x} \qquad (31)$$

Where $y(x)$ represents the transverse displacement of the centroid of the cross-section of the rod, at position x, from its equilibrium position, and λ is proportional to σ^2,where σ the frequency of vibration, is not known in advance [5].
On using the theoretical methodology described for the general fourth order boundary value problem (14) and its equivalent system of second order differential equations (15)

$$\psi''(x) = -y(x), y''(x) = -\psi(x); (0 \le x \le 1)$$
$$\psi(0) = -1, \psi(1) = -e; y(0) = 1, y(1) = e \quad (32)$$

We obtain the following Fredholm nonhomogeneous integral equation of the second kind

$$y(x) = \int_0^x t\left(\frac{x^2}{2} - \frac{5x^3}{6} + \frac{x^4}{3}\right)y(t)dt + \int_x^1 (1-t)\left(\frac{x}{6} - \frac{x^3}{2} + \frac{x^4}{3}\right)y(t)dt + f(x) \quad (33)$$

Where $f(x) = 1 + \frac{(5e-8)}{6}x + \frac{x^2}{2} + \frac{(e-1)}{6}x^3$

It is an easy task to see that the Fredholm integral equation (33) satisfies the boundary conditions in (30). Moreover the closed form solution (31) satisfies both the differential equation (30) and the Fredholm integral equation (33).
In this example we solve in three steps:
Firstly: we solve the fourth order boundary value problem (30) Using the finite difference described in equation (6), taking $h = 0.1; y(0) = 1; y(1) = e$

$$y_{i-2} - 4y_{i-1} + 5.9999y_i - 4y_{i+1} + y_{i+2} = 0 \quad (34)$$

Taking $i = 1(1)9$ we obtain a linear system of algebraic equations,
$AY = b$
With coefficient matrix A in the form

$$A = \begin{pmatrix} 2.9999 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 5.9999 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 5.9999 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 5.9999 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 5.9999 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 5.9999 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 5.9999 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -4 & 5.9999 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 2.9999 \end{pmatrix}$$

Table (2) illustrates the solution of this system by the SOR and the KSOR methods.
Also we used different vales for h, taking $h = 0.05$ in equation (32) we obtain on a linear system of algebraic equations its solution by the SOR and KSOR methods will be shown in table (3).
Secondly: we solve the equivalent system of second order differential equation (32) to the basic problem (30), using finite difference method (5) in equation (32) with $h = 0.2$ we obtain

$$y_{i-1} - 2y_i + y_{i+1} + 0.04\psi_i = 0; y(0) = 1, y(1) = e$$
$$\psi_{i-1} - 2\psi_i + \psi_{i+1} + 0.04y_i = 0; \psi(0) = -1, \psi(1) = -e; \quad (35)$$

Taking $i = 1(1)9$ we obtain a linear system of algebraic equations,
$A_1 Y = b_1$
With coefficient matrix A_1

$$A_1 = \begin{pmatrix} -2 & 0.04 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.04 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0.04 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0.04 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0.04 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0.04 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0.04 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.04 & -2 \end{pmatrix}$$

Table (4) illustrates the solution of this system by the SOR and the KSOR methods.

Also we take $h = 0.1$ in equation (35) we get on a linear system of algebraic equations with coefficient matrix of dimension (18×18), the solution of this system by the SOR and the KSOR methods will be shown in table (5).
Thirdly: Using the trapezoidal rule (7) for the equation

$$y(x) = \int_0^x t\left(\frac{x^2}{2} - \frac{5x^3}{6} + \frac{x^4}{3}\right)y(t)dt + \int_x^1 (1-t)\left(\frac{x}{6} - \frac{x^3}{2} + \frac{x^4}{3}\right)y(t)dt + f(x)$$

we obtain a functional relation, which is valid at each point of the grid points
$x_j = jh; j = 1(1)9; h = 0.1$
We obtain a linear system of algebraic equations, corresponding to the Fredholm integral equation (33) with coefficient matrix $A_2; h = 0.1$ in the form

$$A_2 = \begin{pmatrix} -0.99925 & 0.001296 & 0.001134 & 0.000972 & 0.00081 & 0.000648 & 0.000486 & 0.000324 & 0.000162 \\ 0.00013867 & -0.99867 & 0.0020907 & 0.001792 & 0.0014933 & 0.0011947 & 0.000896 & 0.0005973 & 0.00029867 \\ 0.000252 & 0.000504 & -0.99825 & 0.002352 & 0.00196 & 0.001568 & 0.001176 & 0.000784 & 0.000392 \\ 0.000352 & 0.000704 & 0.001056 & -0.998 & 0.00216 & 0.001728 & 0.001296 & 0.000864 & 0.000432 \\ 0.00041667 & 0.0008333 & 0.00125 & 0.0016667 & -0.997917 & 0.0016667 & 0.00125 & 0.0008333 & 0.00041667 \\ 0.000432 & 0.000864 & 0.001296 & 0.001728 & 0.00216 & -0.998 & 0.001056 & 0.000704 & 0.000352 \\ 0.000392 & 0.000784 & 0.001176 & 0.001568 & 0.00196 & 0.002352 & -0.99825 & 0.000504 & 0.000252 \\ 0.00029867 & 0.0005973 & 0.000896 & 0.0011947 & 0.0014933 & 0.001792 & 0.0020907 & -0.99867 & 0.00013867 \\ 0.000162 & 0.000324 & 0.000486 & 0.000648 & 0.00081 & 0.000972 & 0.001134 & 0.001296 & -0.99925 \end{pmatrix}$$

Table (2) illustrates the solution of this system by the SOR and the KSOR methods.

We also use different values for the step size h, we take $h = 0.05$ and the solution of the reduced system by the SOR and the KSOR methods will be shown in table (3).

Table 2: The solution of the algebraic system obtained from both the differential equation (30) and the corresponding Fredholm integral equation (33), with $h = 0.1$, the number of iterations for nearly optimal values of the relaxation parameters are given

x	Y_{ext}	Differential equation		Integral equation	
		SOR	KSOR	SOR $\omega = 1.0006$ (3 iter.)	KSOR $\omega^* = -87$ (3 iter.)
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9	1.10517 1.22140 1.34986 1.49182 1.64872 1.82212 2.01375 2.22554 2.45960	divergent	divergent	1.10848 1.22445 1.35022 1.48904 1.64426 1.81861 2.01365 2.22944 2.46480	1.10848 1.22445 1.35022 1.48904 1.64426 1.81861 2.01365 2.22944 2.46480

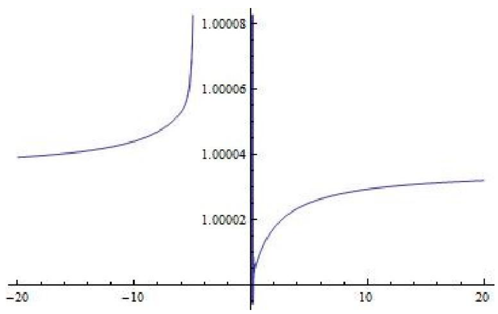

Figure 6: The behavior of the spectral radius of the iteration matrix T_{KSOR} as a function in ω for equation (30),h=0.1.

Table 3:lists the results to the solution of algebraic system reduced from both differential equation (30)and its integral form with comparison between them in the sense of number of iterations and the value of the relaxation parameter when $h = 0.05$.

x	Y_{ext}	Differential equation		Integral equation	
		SOR	KSOR	SOR $\omega = 1.001$ (3 iter.)	KSOR $\omega^* = -34$ (3 iter.)
0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5 0.55 0.6 0.65 0.7 0.75 0.8 0.85 0.9 0.95	1.05127 1.10517 1.16183 1.22140 1.28403 1.34986 1.41907 1.49182 1.56831 1.64872 1.73325 1.82212 1.91554 2.01375 2.117 2.22554 2.33965 2.45960 2.58571	divergent	divergent	1.05341 1.10851 1.16545 1.22448 1.28455 1.35027 1.41783 1.48908 1.56443 1.64427 1.72940 1.81860 1.91348 2.01361 2.11723 2.22938 2.35626 2.46477 2.58931	1.05341 1.10851 1.16545 1.22448 1.28455 1.35027 1.41783 1.48908 1.56443 1.64427 1.72940 1.81860 1.91348 2.01361 2.11723 2.22938 2.35626 2.46477 2.58931

Note: we do not include the solution in the differential form because it is divergent.

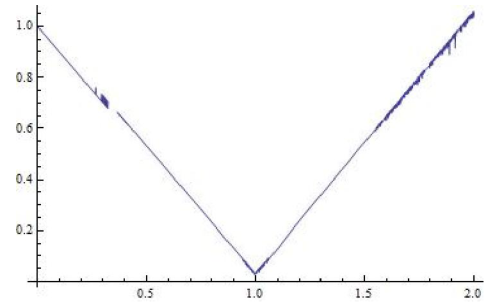

Figure 7: The behavior of the spectral radius of the iteration matrix T_{SOR} as a function in ω for equation (33),h=0.1.

Table 4:lists the results to the solution of algebraic system reduced from the system of second order differential equations(32), the number of iterations and the value of the relaxation parameter when $h = 0.2$ are given.

x	Y_{ext}	SOR ω=1.28 (28 iter.)	KSOR $\omega^* = -4.55$ (28 iter.)
0.2 0.4 0.6 0.8	1.22140 1.49182 1.82212 2.22554	1.22177 1.49241 1.82275 2.22599	1.22177 1.49241 1.82275 2.22599

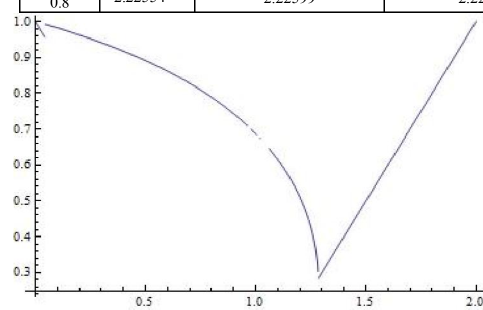

Figure 8: The behavior of the spectral radius of the iteration matrix T_{SOR} as a function in ω for equation (32), h=0.2.

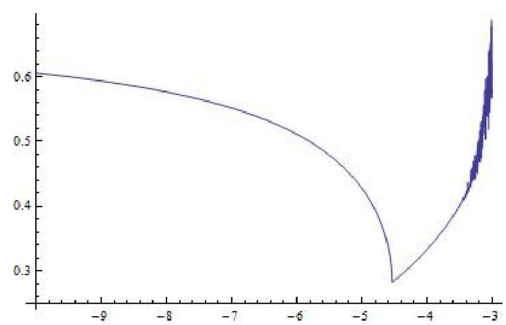

Figure 9: The behavior of the spectral radius of the iteration matrix T_{KSOR} as a function in ω^* for equation (32),h=0.2.

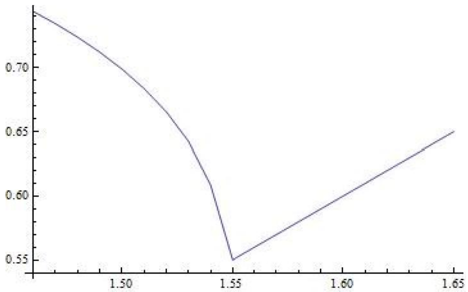

Figure 10: The behavior of the spectral radius of the iteration matrix T_{SOR} as a function in ω for equation (3.3), h=0.1.

Table 5:lists the results to the solution of algebraic system reduced from the system of second order differential equations(37), the number of iterations and the value of the relaxation parameter when $h = 0.1$ are given.

x	Y_{ext}	SOR $\omega=1.55$ (28 iter.)	KSOR $\omega^* = -2.8$ (28 iter.)
0.1	1.10517	1.10522	1.10522
0.2	1.22140	1.22149	1.22149
0.3	1.34986	1.34998	1.34998
0.4	1.49182	1.49197	1.49197
0.5	1.64872	1.64888	1.64888
0.6	1.82212	1.82228	1.82228
0.7	2.01375	2.0139	2.0139
0.8	2.22554	2.22565	2.22565
0.9	2.45960	2.45967	2.45967

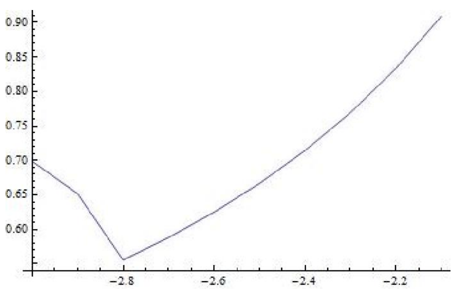

Figure 11: The behavior of the spectral radius of the iteration matrix T_{KSOR} as a function in ω^* for equation (32), h=0.1.

4. Numerical Results:

We performed two groups of numerical experiments:

In the first group we considered the performance of the SOR and the KSOR methods for a second order two point boundary value problem with its Fredholm form:

The algebraic system arising from the discretization of the second order differential equation (25) is solved by SOR and KSOR techniques and we find that the suitable choice of the relaxation parameters is $\omega = 1.402$ and with this value we need 25 iterations to obtain the required solution and $\omega^* = -3.65$ and with this value we need 25 iterations when $h = 0.1$ (Table (1) and Figures(2,3))i.e. both the SOR and the KSOR gives the same solution after the same number of iterations while when we solve the algebraic system arise from the integral representation (27) of the differential equation by the SOR and the KSOR we find that the suitable choice of the relaxation parameters is $\omega = 0.96$ when $h = 0.1$ and with this value we need only 8 iterations to obtain the required solution and $\omega^* = 19$ we need 8 iterations to obtain the same solution obtained with the SOR **(Figures 4,5**) i.e. both the SOR and the KSOR gives the same solution after the same number of iterations.

In the second group we considered the performance of the SOR and the KSOR methods for a fourth order two point boundary value problem [5] with its Fredholm form:

The algebraic system arising from the discretization of the second order differential equation (30) is solved by SOR and KSOR techniques and we find that there are no suitable values for the relaxation parameters make the arising system convergent (Figures 1, 6) i.e. the system is divergent (Tables 2,3)

While when we solve the algebraic system arising from the discretization of the fourth order differential equation as system of two second order differential equations(32) by SOR and KSOR techniques and we find that when $h = 0.2$ the suitable choice of the relaxation parameters is $\omega = 1.28$ with this value we need 28 iterations to obtain the required solution and $\omega^* = -4.55$ with this value we need 28 iterations (Table 4 and Figures 8,9), and when $h = 0.1$ (Figures10,11) $\omega = 1.55$ and with this value we need 28 iterations to obtain the required solution and $\omega^* = -2.8$ with this value we need 28 iterations (Table 5) i.e. both the SOR and the KSOR gives the same solution after the same number of iterations, moreover while when we solve the algebraic system arise from the integral representation (33) of the differential equation by the SOR and the KSOR we find that the suitable choice of the relaxation parameters is $\omega = 1.0006$ and with this value we need only 3 iterations to obtain the required solution and $\omega^* = -87$ we need 3 iterations to obtain the same solution obtained with the SOR when $h = 0.1$ (Table 2 and Figures 7) , also when $h = 0.05$, $\omega = 1.001$ and with this value we need only 3 iterations to obtain the required solution and $\omega^* = -87$ we need only 3 iterations to obtain the same solution obtained with the SOR (Table3) i.e. both the SOR and the KSOR gives the same solution after the same number of iterations.

5. Conclusions:

We have considered the numerical treatment of boundary value problems of the second and fourth orders with integral representation forms. We found that although the algebraic system arising from the differential equation is banded and the optimal value of the relaxation parameters can be accurately approximated the number of iterations used by the SOR with optimal value of the relaxation parameter is the same as that used in the KSOR but the sensitivity of the relaxation parameter in the KSOR is small in comparison with that of the SOR. Also, the number of iteration required to obtain the same accuracy is very small in case of the system obtained from the integral form in both the SOR and the KSOR as shown in table (1) compared with the number of iterations required to obtain the same solution from the algebraic system arising from the differential equations. Moreover, in the fourth order case we found that the algebraic system obtained directly from the differential equation is divergent while that obtained from the integral equation is convergent. Further, we found that the system of algebraic equations obtained from the equivalent system of differential equations is convergent but the number of algebraic equations is doubled but still the algebraic system arising from the integral form is more suitable as shown in tables [2, 3, 4, 5].

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