The Symmetric (2, 4)-nets

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Abstract : We give a vector space presentation of the unique symmetric (2, 4)-net con- structing it from sets of subspaces of V (5, 2) satisfying a certain condition. All such sets of subspaces in V (5, 2) are determined.
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1 Introduction

A t-(v, k, λ) design Π is an incidence structure with v points, k points on a block and any subset of t points is contained in exactly λ blocks, where $v > k$, $\lambda > 0$. The number of blocks is b and the number of blocks on a point is r.

The design Π is resolvable if its blocks can be partitioned into r parallel classes, such that each parallel class partitions the point set of Π. Blocks in the same parallel class are parallel. Clearly each parallel class has $m = v/k$ blocks. Π is affine resolvable, or simply affine, if it can be resolved so that any two non-parallel blocks meet in μ points, where $\mu = k/m = k^2/v$ is constant. Affine 1-designs are also called nets. The dual design of a design Π is denoted by Π^*. If Π and Π^* are both affine, we call Π a symmetric net. We use the terminology of Jungnickel [2] see also [3] In this case, if $r > 1$, then $v = b = \mu m^2$ and $k = r = \mu m$. That is, Π is an affine $1-(\mu m^2, \mu m, \mu m)$ design whose dual Π^* is also affine with the same parameters. For short we call such a symmetric net *a (μ, m)-net* (see [1]).

If Π is a symmetric net we shall refer to the parallel classes of Π as block classes of Π and to the parallel classes of Π^* as point classes of Π.

The incidence matrix of the symmetric (2, 4)-net is as shown below. In 2007, this symmetric net has been shown by *V. D*. Tonchev [4] to be unique using an exhaustive computer search.

$$M = \begin{pmatrix}
1&1&1&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&1&0&1\\
0&0&0&0&1&0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&1&0&0&0&0&1&0&0&1&1&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&1&0&0&0&1&0&0&0&0&0&1\\
0&0&0&1&1&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0\\
0&0&1&0&0&1&0&0&0&0&0&1&0&1&1&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&1&0&1&1&0&0&1&0&1&0&0&0\\
1&0&0&1&0&0&0&1&0&0&0&1&0&1&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&1&0&0&0&0&1&1&0&0&0&0\\
0&1&0&0&0&1&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&1&1\\
1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0\\
0&0&1&0&0&0&1&0&1&0&0&0&0&1&0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&1&0&0&0&1&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&1&0&0&0&0&1\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&1&1&0&1&0\\
0&0&0&1&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0\\
0&1&0&0&1&0&0&1&1&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&1&1&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&1&0&0&0&1&0&0&0&0&0&1&1\\
0&0&1&1&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0\\
0&1&0&0&1&0&0&0&0&0&1&0&1&1&1&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&1&0&1&1&0&0&1&0&1&0&0&0&0\\
1&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&1&0&0&1&0&0\\
0&0&1&0&1&0&0&0&1&1&0&1&1&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&1\\
0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0&0&1&1&1&0&1&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&0&1&1&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&1&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&1&0&0&0&0&1&0&0\\
0&0&0&0&1&0&1&0&0&0&0&1&0&3&1&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0&0&1
\end{pmatrix}$$

In this paper we shall give algebraic and geometric presentations of this symmetric (2, 4)-net using sets of subspaces of appropriate vector spaces and projective geometries, respectively. of V (5, 2) and determine all such subsets.

2 Various setups

Let V (n, q) denote the vector space of dimension n over the field of q elements and let $V = V$ *(5, 2)*. Hypothesis A: Let $U_0, U_1, . . . , U_8$ be subspaces of V so that

- dim $U_0 = 2$ and dim $U_i = 3$ for $1 \le i \le 8$,
- $U_0 \cap U_i = 0$ for $1 \le i \le 8$,
- $\dim(U_i \cap U_j) = 1$ for $1 \le i < j \le 8$.

Theorem 2.1 Suppose that $U_0, U_1, \ldots, U_8$ are subspaces of V satisfying Hypothesis A. The incidence structure whose points are the elements of V and whose blocks are thecosets of U_i, $1 \le i \le 8$, in V is a net with $m = 4$ and $\mu = 2$. The point parallel classes are the cosets of U_0 in V.

Proof. This is almost immediate. We need only observe that if $i = j$ then the intersection of a coset of U_j with U_i is non-empty and is a coset in U_i of $U_i \cap U_j$, and every coset of U_0 meets every coset of U_i in a single point.

Hypothesis B: Let $V_0, V_1, \ldots, V_8$ be ubspaces of V so that

- dim $V_0 = 3$ and dim $V_i = 2$ for $1 \le i \le 8$,
- $V_i \cap V_j = 0$ for $0 \le i < j \le 8$.

For any subspace U of V, let $U^{\perp} = \{ut : (u, ut) = 0$ for all $u \in U\}$, where $(\cdot, \cdot)$ denotes the standard inner product on V.

Theorem 2.2 V has a set of subspaces satisfying Hypothesis A if, and only if, V has a set of subspaces satisfying Hypothesis B.

Proof. Assume that $U_0, U_1, \ldots, U_8$ are subspaces of V satisfying Hypothesis A. Let $V_i = \cup\, U_i^{\perp}$ for $i = 0, \ldots, 8$.

Then dim $V_i = 5 - \dim U_i$ for $0 \le i \le 8$. For $1 \le i \le 8$, $\dim(U_0 + U_i) = \dim U_0 + \dim U_i - \dim(U_0 \cap U_i) = 2 + 3 - 0 = 5$. Also, for $1 \le i < j \le 8$, $\dim(U_i + U_j) = \dim U_i + \dim U_j - \dim(U_i \cap U_j) = 3 + 3 - 1 = 5$. Hence, $U_i + U_j = V$ if $0 \le i < j \le 8$. So, if $i = j$, $V_i \cap V_j = U_i^{\perp} \cap U_j^{\perp} = (U_i + U_j)^{\perp} = V^{\perp} = 0$.

Thus, $V_0, V_1, \ldots, V_8$ satisfy Hypothesis B.

The converse statement is established by a similar argument. Let Π be $PG(5, 2)$ and let π be one of its hyperplanes.

Hypothesis C: Let Q be a point of Π not on π, and let $\pi_0, \pi_1, \ldots, \pi_8$ be subspaces of Π containing Q so that

- dim $\pi_0 = 2$ and dim $\pi_i = 3$ for $1 \le i \le 8$,
- $\pi_0 \cap \pi_i = \{Q\}$ for $1 \le i \le 8$,
- $\dim(\pi_i \cap \pi_j) = 1$ for $1 \le i < j \le 8$.

(In this hypothesis, dimension means geometric dimension.)

Theorem 2.3 V has a set of subspaces satisfying Hypothesis B if, and only if, Π has a set of subspaces satisfying Hypothesis C.

Proof. Assume that $V_0, V_1, \ldots, V_8$ are subspaces of V satisfying Hypothesis B. Consider V to have its natural affine geometry structure $AG(5, 2)$. Complete this to Π, the projective geometry $PG(5, 2)$, by adding a hyperplane π. Let πi be the completion of Vi in Π. Then Hypothesis C is seen to hold with Q being the 0 of V.

For the converse, let Q be a point of Π not on π, and let $\pi_0, \pi_1, \ldots, \pi_8$ be subspaces of Π containing Q and satisfying Hypothesis C. Look at the affine geometry structure on the complement of π, where Q is taken as the 0 of V. We find a set of subspaces of V satisfying Hypothesis B immediately.

3 Justifying the hypotheses

Here we show that it is possible to find sets of subspaces satisfying the hypotheses. In view of the equivalence of the hypotheses, we shall deal with Hypothesis A.

Let $V = V(5, 2)$. Let W be a 3-dimensional subspace of V and let A and B be two

2-dimensional subspaces such that $A \cap W = B \cap W = A \cap B = 0$. Since $\dim(A + B) = 4$, $X = (A + B) \cap W$ is a 2-dimensional subspace of W. (As $A + W = V$, $\dim X = \dim(A + B) + \dim W - \dim(A + B + W) = 4 + 3 - 5 = 2$.)

There are two 2-dimensional subspaces Y and Z of $A + B$ which intersect A, B and X trivially, and intersect one another trivially also. This can be seen be a counting argument (look at all 2-dimensional spaces of $(A + B)$, or by listing all elements of $A + B$ and examining the 2-dimensional spaces in detail, or by considering the following argument.

There are six elements in $A + B - (A \cup B \cup X)$. Label them $u1, \ldots, u6$. Suppose $u_i + u_k$, $u_j + u_k \in A$, where i, j and k are distinct. Then $A = \{0, u_i + u_k, u_j + u_k, u_i + u_j\}$ So, $A + u_k = \{u_i, u_j, u_k, u_i + u_j + u_k\}$ must contain a non-zero element of B and a non-zero element of X. A similar argument applies to B and X, since $A + B = A + X = B + X$.

This contradiction shows that, for a given k, at most three of the elements $u_i + u_k$, with $i = k$, are in $A \cup B \cup X$. Hence, for a given k we can find i so that (u_i, u_k) meets $A \cup B \cup X$ in 0.

Let $W \setminus X = \{t_1, t_2, t_3, t_4\}$. $\dim(Y + (t_i)) = \dim(Z + (t_i)) = 3$ for $1 \le i \le 4$. Since $Y \cap Z = 0$, $\dim((Y + (t_i)) \cap (Z + (t_j))) = 1$, for $1 \le i, j \le 4$. Let $(x_{i,j}) = (Y + (t_i)) \cap (Z + (t_j))$. If $i = j$ then $x_{i,j} = t_i$. The sixteen elements $x_{i,j}$ are distinct since any pair belong to different cosets of Y or different cosets of Z.

We now show how to pick six 2-dimensional subspaces, meeting pairwise in 0, and meeting $Y \cup Z \setminus \{0\}$ in distinct points. Note that $Y + (t_i) = Y \cup \{x_{i,j} : 1 \le j \le 4\}$ and $Z + (t_j) = Y \cup \{x_{i,j} : 1 \le i \le 4\}$.

Pick an arbitrary 2-dimensional subspace in some $Y + (t_i)$ meeting Y in a 1-dimensional subspace. Rearranging labels, we may suppose that it is $(x_{1,3}, x_{1,4})$ in $Y + (t_1)$.

Suppose that $Z + (t_3)$ contains one of our six 2-dimensional subspaces. Then, since it does not contain $x_{1,3}$, it is ($x_{2,3}$, $x_{4,3}$). If Y + (t_4) contains one of our six 2-dimensional subspaces, it must be ($x_{4,1}$, $x_{4,2}$). It is impossible to complete the selection of six 2-dimensional subspaces. The alternative is that Y + (t_2) and Y + (t_3) contain one each of our six 2-dimensional subspaces, the in the former it must be ($x_{2,1}$, $x_{2,4}$). So, ($x_{3,1}$, $x_{4,1}$) must also be one. As must ($x_{3,2}$, $x_{3,4}$) and ($x_{1,2}$, $x_{4,2}$).

We must verify that these six 2-dimensional subspaces meet pairwise in 0; they clearly meet A and B in0. Suppose that w ∈ ($x_{i,j}$, $x_{i,4}$) ∩ ($x_{i',j'}$, $x_{i',4}$) and w = 0, with $i = I'$.

Since $x_{i,j}$ and $x_{i,4}$ are in a different coset of Y from $x_{it,jt}$ and $x_{it,4}$, we must have $w = x_{i,j} + x_{i,4} = x_{i',j'} + x_{i',4}$. Now, $x_{i,4} + x_{i',4} \in Z$. Hence, $x_{i,j} + x_{i',j'} \in Z$. But this is impossible, since $j = j'$. Thus, the 2-dimensional subspaces of the form ($x_{i,j}$, $x_{i,4}$) meet pairwise in 0. *A* similar argument shows that Thus, the 2-dimensional subspaces of the form (xi,j , $x_{4,j}$) meet pairwise in 0. Next suppose that $w \in (x_{i,j}, x_{i,4}) \cap (x_{i',j'}, x_{4,j'})$ and $w = 0$. Since $x_{i,j} + x_{i,4} \in Y$ and $x_{it,jt} + x_{4,jt} \in Z$ and $Y \cap Z = 0$, w must be one of $x_{i,j}$ and $x_{i,4}$ and one of $x_{it,jt}$ and $x_{4,j'}$. By our choice of subspaces, all four of these elements differ. Hence, the six 2-dimensional subspaces ($x_{1,3}$, $x_{1,4}$), ($x_{2,1}$, $x_{2,4}$), ($x_{3,2}$, $x_{3,4}$), ($x_{3,1}$, $x_{4,1}$), ($x_{1,2}$, $x_{4,2}$), and ($x_{2,3}$, $x_{4,3}$) meet our requirements.

Suppose now that Z + (t_3) does not contain one of our six 2-dimensional subspaces. Then, ($x_{2,4}$, $x_{3,4}$) must be one of them. If Y + (t_3) contains one of our six 2-dimensional subspaces, it must be ($x_{3,1}$, $x_{3,2}$). It is impossible to complete the selection of six 2-dimensional subspaces. The alternative is that Y + (t_2) and Y + (t_4) contain one each of our six 2-dimensional subspaces, the one in the former must be ($x_{2,1}$, $x_{2,3}$). So, ($x_{3,1}$, $x_{4,1}$) must also be one. As must ($x_{4,2}$, $x_{4,3}$) and ($x_{1,2}$, $x_{3,2}$).

We may verify that these six 2-dimensional subspaces meet pairwise in 0 exactly as for the previous case. They clearly meet *A* and *B* in 0. Hence, the six 2-dimensional subspaces ($x_{1,4}$, $x_{1,3}$), ($x_{2,1}$, $x_{2,3}$), ($x_{4,2}$, $x_{4,3}$), ($x_{4,1}$, $x_{3,1}$), ($x_{1,2}$, $x_{3,2}$), and ($x_{2,4}$, $x_{3,4}$) meet our requirements.

4 Determination of all sets of subs paces satisfying Hypothesis A

In this section we determine in V (5, 2) all sets of subspaces satisfying the conditions of Hypothesis *A* of section 3.

First we need a general lemma on subspaces of vector spaces.

Lemma 4.1 Let V be a vector space. Let V_1 and V_2 be non-zero subspaces of V such that $V = V_1 \oplus V_2$. Let W_1 and W_2 be subspaces of V such that dim W_1 = dim W_2, dim $W_1 \cap V_1$ = dim $W_2 \cap V_1$ and dim $W_1 \cap V_2$ = dim $W_2 \cap V_2$. Then there is a linear transformation $\theta \in GL(V)$ such that $V_1\theta = V_1$, $V_2\theta = V_2$ and $W_1\theta = W_2$.

Proof. For i = 1, 2, let $\pi i : V \rightarrow V_i$ be the projection of V on V_i. Let d = dim W_1, m_1 = dim $W_1 \cap V_1$ and m_2 = dim $W_1 \cap V_2$. For i = 1, 2, we write $W_i = W_i \cap V_1 \oplus$ dim $W_i \cap V_2 \oplus U_i$ where U_i is a suitably chosen subspace of W_i and dim $U_i = d - m_1 - m_2$.

Fix i,j ∈ 1, 2. The kernel $U'_{i,j}$of the linear mapping $\pi_j |U_i$ is a subspace of V_{3-j} . Hence, $U_{i,j} \subseteq (V_{3-i} \cap W_i) \cap U_i = 0$. So, π_j maps U_i bijectively to $U_i\pi_j$.

Let $v \in (W_i \cap V_j) \cap U_{i\pi j}$. Write $v = u\pi_j$. Since $u = u\pi_j + u\pi_{3-j}$ we get $u\pi_{3-j} = u - v \in W_i$. Hence, $u\pi_{3-j} \in W_i \cap V_{3-j}$. Since $0 = v + u\pi_{3-j} - u$, we get $v = 0$, $u\pi_{3-j} = 0$ and $u = 0$. Hence, $(W_i \cap V_j) \cap U_i\pi_j = 0$.

For $i, j \in \{1, 2\}$, choose bases $A_{i,j}$ of $W_i \cap V_j$ and bases B_i of U_i. Let $\sigma : B_1 \rightarrow B_2$ be a bijection. From the preceding remarks, $B_i\pi_j$ are linearly independent sets of size $d - m_1 - m_2$ and $A_{i,j} \cup B_i\pi_j$ are linearly independent sets in V_j of size $d - m_{3-j}$. We extend $A_{i,j} \cup B_i\pi_j$ to a basis $C_{i,j}$ of V_j .

We define the desired linear transformation as follows. $D_i = C_{i,1} \cup C_{i,2}$ is a basis of V for i = 1, 2. The transformation is obtained by mapping D_1 bijectively to D_2 so that $C_{1,j}$ is mapped bijectively to $C_{2,j}$ for j = 1, 2. In mapping $C_{1,j}$ to $C_{2,j}$, we map $v\pi_j$ to $v\sigma\pi_j$ for all $v \in B_1$ and $A_{1,j}$ bijectively to $A_{2,j}$ and assign the rest of the basis $C_{1,j}$ to the rest of the basis $C_{2,j}$ arbitrarily.

The only point to note is that since $v\pi_j$ maps to $v\sigma\pi_j$ for all $v \in B_1$ and j = 1, 2, $v = v\pi_1 + v\pi_2$ maps to $v\sigma\pi_1 + v\sigma\pi_2 = v\sigma$ for all $v \in B_1$. Hence, the linear transformation induces a non-singular linear transformation $U_1 \rightarrow U_2$.

Corollary 4.2 Let $V = V(5, 2)$. Let X, Y, Z be subspaces of V with dim$X$ = 3, dim$Y$ = dim$Z$ = 2 and any two of the three subspaces meet only in the zero vector. If X', Y', Z' are three subspaces with the analogous properties in V , then there exists a non-singular linear transformation mapping X, Y, Z onto X', Y', Z', respectively.

Using this corollary, a computer search determined all sets of subspaces satisfying Hypothesis *A* of section 2. In view of the above corollary, we can take the 3-dimensional subspace to be generated by the vectors (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0) and take two the 2−dimensional subspaces to be the one generated by (0, 0, 0, 0, 1), (0, 0, 0, 1, 0) and the other generated by (0, 0, 1, 0, 1), (0, 1, 0, 1, 0).

We list below the sets of eight 2-dimensional subspaces satisfying Hypothesis *A* and these extra constraints, giving their non-zero elements.

subspace set 1:
[0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 0, 1, 1]
[0, 0, 1, 0, 1], [0, 1, 0, 1, 0], [0, 1, 1, 1, 1]
[0, 0, 1, 1, 0], [1, 0, 0, 1, 1], [1, 0, 1, 0, 1]
[0, 0, 1, 1, 1], [1, 1, 0, 0, 1], [1, 1, 1, 1, 0]
[0, 1, 0, 0, 1], [1, 0, 0, 1, 0], [1, 1, 0, 1, 1]
[0, 1, 0, 1, 1], [1, 0, 1, 1, 0], [1, 1, 1, 0, 1]
[0, 1, 1, 0, 1], [1, 0, 1, 1, 1], [1, 1, 0, 1, 0]
[0, 1, 1, 1, 0], [1, 0, 0, 0, 1], [1, 1, 1, 1, 1]

subspace set 3:
[0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 0, 1, 1]
[0, 0, 1, 0, 1], [0, 1, 0, 1, 1], [0, 1, 1, 1, 0]
[0, 0, 1, 1, 0], [1, 0, 0, 0, 1], [1, 0, 1, 1, 1]
[0, 0, 1, 1, 1], [1, 1, 0, 1, 0], [1, 1, 1, 0, 1]
[0, 1, 0, 0, 1], [1, 0, 0, 1, 0], [1, 1, 0, 1, 1]
[0, 1, 0, 1, 0], [1, 0, 1, 0, 1], [1, 1, 1, 1, 1]
[0, 1, 1, 0, 1], [1, 0, 0, 1, 1], [1, 1, 1, 1, 0]
[0, 1, 1, 1, 1], [1, 0, 1, 1, 0], [1, 1, 0, 0, 1]

subspace set 5:
[0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 0, 1, 1]
[0, 0, 1, 0, 1], [1, 0, 0, 1, 1], [1, 0, 1, 1, 0]
[0, 0, 1, 1, 0], [0, 1, 0, 1, 1], [0, 1, 1, 0, 1]
[0, 0, 1, 1, 1], [1, 1, 0, 0, 1], [1, 1, 1, 1, 0]
[0, 1, 0, 0, 1], [1, 0, 0, 1, 0], [1, 1, 0, 1, 1]
[0, 1, 0, 1, 0], [1, 0, 1, 1, 1], [1, 1, 1, 0, 1]
[0, 1, 1, 1, 0], [1, 0, 0, 0, 1], [1, 1, 1, 1, 1]
[0, 1, 1, 1, 1], [1, 0, 1, 0, 1], [1, 1, 0, 1, 0]

subspace set 7:
[0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 0, 1, 1]
[0, 0, 1, 0, 1], [1, 1, 0, 1, 0], [1, 1, 1, 1, 1]
[0, 0, 1, 1, 0], [1, 0, 0, 0, 1], [1, 0, 1, 1, 1]
[0, 0, 1, 1, 1], [0, 1, 0, 1, 0], [0, 1, 1, 0, 1]
[0, 1, 0, 0, 1], [1, 0, 0, 1, 0], [1, 1, 0, 1, 1]
[0, 1, 0, 1, 1], [1, 0, 1, 0, 1], [1, 1, 1, 1, 0]
[0, 1, 1, 1, 0], [1, 0, 0, 1, 1], [1, 1, 1, 0, 1]
[0, 1, 1, 1, 1], [1, 0, 1, 1, 0], [1, 1, 0, 0, 1]

subspace set 2:
[0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 0, 1, 1]
[0, 0, 1, 0, 1], [0, 1, 0, 1, 0], [0, 1, 1, 1, 1]
[0, 0, 1, 1, 0], [1, 1, 0, 0, 1], [1, 1, 1, 1, 1]
[0, 0, 1, 1, 1], [1, 0, 0, 0, 1], [1, 0, 1, 1, 0]
[0, 1, 0, 0, 1], [1, 0, 0, 1, 0], [1, 1, 0, 1, 1]
[0, 1, 0, 1, 1], [1, 0, 1, 0, 1], [1, 1, 1, 1, 0]
[0, 1, 1, 0, 1], [1, 0, 1, 1, 1], [1, 1, 0, 1, 0]
[0, 1, 1, 1, 0], [1, 0, 0, 1, 1], [1, 1, 1, 0, 1]

subspace set 4:
[0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 0, 1, 1]
[0, 0, 1, 0, 1], [0, 1, 0, 1, 1], [0, 1, 1, 1, 0]
[0, 0, 1, 1, 0], [1, 1, 0, 0, 1], [1, 1, 1, 1, 1]
[0, 0, 1, 1, 1], [1, 0, 0, 0, 1], [1, 0, 1, 1, 0]
[0, 1, 0, 0, 1], [1, 0, 0, 1, 0], [1, 1, 0, 1, 1]
[0, 1, 0, 1, 0], [1, 0, 1, 1, 1], [1, 1, 1, 0, 1]
[0, 1, 1, 0, 1], [1, 0, 0, 1, 1], [1, 1, 1, 1, 0]
[0, 1, 1, 1, 1], [1, 0, 1, 0, 1], [1, 1, 0, 1, 0]

subspace set 6:
[0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 0, 1, 1]
[0, 0, 1, 0, 1], [1, 0, 0, 1, 1], [1, 0, 1, 1, 0]
[0, 0, 1, 1, 0], [0, 1, 0, 1, 1], [0, 1, 1, 0, 1]
[0, 0, 1, 1, 1], [1, 1, 0, 1, 0], [1, 1, 1, 0, 1]
[0, 1, 0, 0, 1], [1, 0, 0, 1, 0], [1, 1, 0, 1, 1]
[0, 1, 0, 1, 0], [1, 0, 1, 0, 1], [1, 1, 1, 1, 1]
[0, 1, 1, 1, 0], [1, 0, 1, 1, 1], [1, 1, 0, 0, 1]
[0, 1, 1, 1, 1], [1, 0, 0, 0, 1], [1, 1, 1, 1, 0]

subspace set 8:
[0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 0, 1, 1]
[0, 0, 1, 0, 1], [1, 1, 0, 1, 0], [1, 1, 1, 1, 1]
[0, 0, 1, 1, 0], [1, 0, 0, 1, 1], [1, 0, 1, 0, 1]
[0, 0, 1, 1, 1], [0, 1, 0, 1, 0], [0, 1, 1, 0, 1]
[0, 1, 0, 0, 1], [1, 0, 0, 1, 0], [1, 1, 0, 1, 1]
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[0, 1, 1, 1, 0], [1, 0, 1, 1, 1], [1, 1, 0, 0, 1]
[0,1, 1, 1, 1],[1, 0, 0, 0,1], [1, 1, 1,1,0]

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