Peristaltic flow of a magnetohydrodynamic Oldroyd 4-constant fluid in a planar channel

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Abstract. The purpose of this paper is to generalize the flow problem considered in [21] for magnetohydrodynamic fluid. The problem is first formulated in the form of a differential equation by taking into account the effects of magnetic field and then solved numerically for various values of material parameters. Here, in addition to the non-Newtonian parameters, non-dimensional Hartman number also comes into play. The effects of Hartman number on longitudinal velocity, stream function, longitudinal pressure gradient and pressure rise per wavelength are discussed with the help of graphs.

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Key Words. Odroyd 4-constant fluid, channel flow, numerical solution, iterative methods.

1. Introduction.

Flow of fluid induced by propagation of waves along the flexible walls of the channel, also known as peristaltic flow is of vital importance and subject of recent interest due to its occurrence in physiology and industry. Specifically, in physiology peristaltic flows occur in transport of urine from kidney to the bladder, movement of chyme in gastro-intestinal tract, vasomotion of small blood vessels and the flows in many glandular ducts. In industry peristaltic mechanism is exploited for transport of corrosive fluids and in manufacturing of peristaltic pumps. Bio-medical instruments such as heart-lung machine also operate according to peristaltic mechanism.

Pioneering works on peristaltic flows were made by Latham [1], Shapiro et al. [2], Fung and Yih [3] and many others [4]-[6]. In all these studies the considered fluid obeys the Newton's law of viscosity. However, it is well known that many physiological and industrial fluids are non-Newtonian in nature and cannot be understood using Newton's law of viscosity. Raju and Devanathan [8] first time analyzed the peristaltic flow by considering fluid to be non-Newtonian. Later on several researchers investigated interaction of peristalsis with rheologically complex fluids [7]-[17].

Amongst many non-Newtonian fluids Oldroyd-B fluid is quite popular. This model is capable of predicting viscoelastic effects such as stress relaxation and retardation. Flows of Oldroyd-B fluids were studied extensively in the literature [18]-[20]. However, this fluid model does not exhibit viscoelastic effects when peristaltic flow under long wavelength approximation is considered. The simplest non-Newtonian model which can predict rheological effects under long wavelength assumption is Oldroyd 4-constant model. Ali et al. [21] discussed the peristaltic motion of Oldroyd 4-consatant fluid in a planner channel. However, there analysis is only valid for hydrodynamic fluid. The study of peristaltic flow with magnetohydrodynamic (MHD) effects fall in the area of biomagnetic fluid dynamics (BFD). Flows of MHD biological fluids are quite important in bioengineering and medical sciences. These fluids are extensively found in living creatures and there flows are greatly influenced by magnetic field. Blood, urine, chyme etc. are examples of biofluids.Further, MHD peristaltic flows of biofluids are useful in problems of conductive physiological fluids for example the blood and blood pump machines and peristaltic MHD compressor. Motivated by these facts the purpose of this paper is to extended the analysis of Ali et al. [21] for a magnetohydrodynamic fluid.

2. Flow equations.

Neglecting the body forces, the governing equations of the incompressible fluid are given by

$$\operatorname{div}\bar{V}=0, \quad (1)$$

$$\rho\frac{d\bar{V}}{d\bar{t}}=-\nabla\bar{p}+\operatorname{div}\bar{S}, \quad (2)$$

where $\bar{V}$ is the velocity, ρ is the density, $\frac{d}{d\bar{t}}$ is the material derivative, $\bar{p}$ is the pressure and $\bar{S}$ is the extra stress tensor. The extra stress tensor in an Oldroyd 4-constant fluid satifies the following equations.

The constitutive equation for the Oldroyd 4-constant fluid is given by

$$\mathbf{S}+\lambda_1\frac{\mathbf{DS}}{\mathbf{D}t}+\lambda_3\operatorname{tr}(\mathbf{S})\mathbf{A}_1=\mu(1+\lambda_2\frac{\mathbf{D}}{\mathbf{D}t})\mathbf{A}_1, \quad (3)$$

where λ_1 and λ_3 are the relaxation times,

λ_2 is the retardation time, $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor, defined by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \tag{4}$$

$\mathbf{L}$ is the velocity gradient and

$$\frac{\mathbf{DS}}{\mathbf{D}t} = \frac{\mathbf{dS}}{\mathbf{d}t} - \mathbf{LS} - \mathbf{SL}^T, \tag{5}$$

is the upper-convected time derivative. It should be noted that the model (3) includes the Oldroyd 3-consatnt model (for $\lambda_3 = 0$), the Maxwell model (for $\lambda_2 = \lambda_3 = 0$) , the viscous fluid model (for $\lambda_1 = \lambda_2 = \lambda_3 = 0$(and the second grade fluid (if $\lambda_1 = \lambda_3 = 0$) as the limiting cases.

2. Problem formulation.

Consider a two dimensional channel of uniform thickness $2a$. Let it be filled with a homogenous incompressible Oldroyd 4-constant fluid. The walls of the channel are assumed flexible. Assume two symmetric infinite wave trains traveling with velocity c along the walls. If $\overline{X}$ and $\overline{Y}$ are the longitudinal and transverse coordinates, respectively, The fluid considered here is electrically conducting. A uniform magnetic field $\overline{\mathbf{B}}_0$ is applied perpendicular to the flow. The total magnetic field is

$$\overline{\mathbf{B}} = \overline{\mathbf{B}}_0 + \overline{\mathbf{b}}, \tag{6}$$

where $\overline{\mathbf{b}}$ is induced magnetic field. However, under the assumption of small magnetic Reynold number the induced magnetic field can be neglected.then the wall surface is mathematically defined as

$$\overline{h}(\overline{X},\overline{t}) = a + b\cos\left[\frac{2\pi}{\lambda}\left(\overline{X} - c\overline{t}\right)\right]. \tag{7}$$

Here b is the wave amplitude, λ is the wavelength and $\overline{t}$ is the time. A further assumption is that there is no motion of the wall in the longitudinal direction. This assumption implies that for the no-slip condition i.e., longitudinal velocity is zero at the wall.

For the flow under consideration the velocity field is given by

$$\overline{\mathbf{V}} = \left[\overline{U}(\overline{X},\overline{Y},\overline{t}), \overline{V}(\overline{X},\overline{Y},\overline{t}), 0\right], \tag{8}$$

where $\overline{U}$ and $\overline{V}$ are the longitudinal and transverse velocity components, respectively.

Substituting Eq. (8) in Eqs. (1) and (2) yield the following scalar equations. The governing equations, taking into account the effect of magnetic field in the laboratory frame are

$$\frac{\partial \overline{U}}{\partial \overline{X}} + \frac{\partial \overline{V}}{\partial \overline{Y}} = 0, \tag{9}$$

$$\rho\left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\overline{U} = -\frac{\partial \overline{p}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{X}\overline{X}}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{X}\overline{Y}}}{\partial \overline{Y}} + (\mathbf{J}\times\mathbf{B})_{\overline{X}}, \tag{10}$$

$$\rho\left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\overline{V} = -\frac{\partial \overline{p}}{\partial \overline{Y}} + \frac{\partial \overline{S}_{\overline{X}\overline{Y}}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{Y}\overline{Y}}}{\partial \overline{Y}} + (\mathbf{J}\times\mathbf{B})_{\overline{Y}}. \tag{11}$$

Now we proceed to calculate the components of stress tensor using the relation (5).

$$\overline{\mathbf{L}} = \begin{pmatrix} \frac{\partial \overline{U}}{\partial \overline{X}} & \frac{\partial \overline{U}}{\partial \overline{Y}} \\ \frac{\partial \overline{V}}{\partial \overline{X}} & \frac{\partial \overline{V}}{\partial \overline{Y}} \end{pmatrix}, \tag{12}$$

and

$$\overline{\mathbf{L}}^T = \begin{pmatrix} \frac{\partial \overline{U}}{\partial \overline{X}} & \frac{\partial \overline{V}}{\partial \overline{X}} \\ \frac{\partial \overline{U}}{\partial \overline{Y}} & \frac{\partial \overline{V}}{\partial \overline{Y}} \end{pmatrix}. \tag{13}$$

Then

$$\overline{\mathbf{A}}_1 = \overline{\mathbf{L}} + \overline{\mathbf{L}}^T = \begin{pmatrix} 2\frac{\partial \overline{U}}{\partial \overline{X}} & \frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}} \\ \frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}} & 2\frac{\partial \overline{V}}{\partial \overline{Y}} \end{pmatrix}, \tag{14}$$

for two-dimensional flow we can define

$$\overline{\mathbf{S}} = \begin{pmatrix} \overline{S}_{\overline{X}\overline{X}} & \overline{S}_{\overline{X}\overline{Y}} \\ \overline{S}_{\overline{X}\overline{Y}} & \overline{S}_{\overline{Y}\overline{Y}} \end{pmatrix}. \tag{15}$$

With the help of Eq. (15) we have

$$\frac{D\overline{\mathbf{S}}}{D\overline{t}} = \begin{pmatrix} \left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\overline{S}_{\overline{X}\overline{X}} - 2\overline{S}_{\overline{X}\overline{X}}\frac{\partial \overline{U}}{\partial \overline{X}} - 2\overline{S}_{\overline{X}\overline{Y}}\frac{\partial \overline{U}}{\partial \overline{Y}} & \left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\overline{S}_{\overline{X}\overline{Y}} - \overline{S}_{\overline{X}\overline{X}}\frac{\partial \overline{V}}{\partial \overline{X}} - \overline{S}_{\overline{Y}\overline{Y}}\frac{\partial \overline{U}}{\partial \overline{Y}} \\ \left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\overline{S}_{\overline{X}\overline{Y}} - \overline{S}_{\overline{X}\overline{X}}\frac{\partial \overline{V}}{\partial \overline{X}} - \overline{S}_{\overline{Y}\overline{Y}}\frac{\partial \overline{U}}{\partial \overline{Y}} & \left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\overline{S}_{\overline{Y}\overline{Y}} - 2\overline{S}_{\overline{X}\overline{Y}}\frac{\partial \overline{V}}{\partial \overline{X}} - 2\overline{S}_{\overline{Y}\overline{Y}}\frac{\partial \overline{V}}{\partial \overline{Y}} \end{pmatrix}. \tag{16}$$

Similarly

$$\frac{D\overline{\mathbf{A}}_1}{D\overline{t}} = \begin{pmatrix} 2\left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\frac{\partial \overline{U}}{\partial \overline{X}} - 4\left(\frac{\partial \overline{U}}{\partial \overline{X}}\right)^2 - 2\frac{\partial \overline{U}}{\partial \overline{Y}}\left(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}\right) & \left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\left(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}\right) - 2\frac{\partial \overline{U}}{\partial \overline{X}}\frac{\partial \overline{V}}{\partial \overline{X}} - 2\frac{\partial \overline{U}}{\partial \overline{Y}}\frac{\partial \overline{V}}{\partial \overline{Y}} \\ \left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\left(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}\right) - 2\frac{\partial \overline{U}}{\partial \overline{X}}\frac{\partial \overline{V}}{\partial \overline{X}} - 2\frac{\partial \overline{U}}{\partial \overline{Y}}\frac{\partial \overline{V}}{\partial \overline{Y}} & 2\left(\frac{\partial}{\partial \overline{t}} + \overline{U}\frac{\partial}{\partial \overline{X}} + \overline{V}\frac{\partial}{\partial \overline{Y}}\right)\frac{\partial \overline{V}}{\partial \overline{Y}} - 4\left(\frac{\partial \overline{V}}{\partial \overline{Y}}\right)^2 - 2\frac{\partial \overline{V}}{\partial \overline{Y}}\left(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}\right) \end{pmatrix}, \tag{17}$$

and

$$tr(\overline{\mathbf{S}})\overline{\mathbf{A}}_1 = \begin{pmatrix} 2(\overline{S}_{\overline{X}\overline{X}} + \overline{S}_{\overline{Y}\overline{Y}})\frac{\partial \overline{U}}{\partial \overline{X}} & (\overline{S}_{\overline{X}\overline{X}} + \overline{S}_{\overline{Y}\overline{Y}})\left(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}\right) \\ (\overline{S}_{\overline{X}\overline{X}} + \overline{S}_{\overline{Y}\overline{Y}})\left(\frac{\partial \overline{U}}{\partial \overline{Y}} + \frac{\partial \overline{V}}{\partial \overline{X}}\right) & 2(\overline{S}_{\overline{X}\overline{X}} + \overline{S}_{\overline{Y}\overline{Y}})\frac{\partial \overline{V}}{\partial \overline{Y}} \end{pmatrix}. \tag{18}$$

Inserting Eqs. (12)-(18) into relation (3) and equating the corresponding components on the both sides yield the following equations

$$\bar{S}_{\bar{X}\bar{X}}+\bar{\lambda}_1\left[\left(\frac{\partial}{\partial \bar{t}}+\bar{U}\frac{\partial}{\partial \bar{X}}+\bar{V}\frac{\partial}{\partial \bar{Y}}\right)\bar{S}_{\bar{X}\bar{X}}-2\frac{\partial \bar{U}}{\partial \bar{X}}\bar{S}_{\bar{X}\bar{X}}-2\frac{\partial \bar{U}}{\partial \bar{Y}}\bar{S}_{\bar{X}\bar{Y}}\right]+2\bar{\lambda}_3(\bar{S}_{\bar{X}\bar{X}}+\bar{S}_{\bar{Y}\bar{Y}})\frac{\partial \bar{U}}{\partial \bar{X}}$$
$$=2\mu\frac{\partial \bar{U}}{\partial \bar{X}}+2\mu\bar{\lambda}_2\left[\left(\frac{\partial}{\partial \bar{t}}+\bar{U}\frac{\partial}{\partial \bar{X}}+\bar{V}\frac{\partial}{\partial \bar{Y}}\right)\frac{\partial \bar{U}}{\partial \bar{X}}-2\left(\frac{\partial \bar{U}}{\partial \bar{X}}\right)^2-\frac{\partial \bar{U}}{\partial \bar{Y}}\left(\frac{\partial \bar{U}}{\partial \bar{Y}}+\frac{\partial \bar{V}}{\partial \bar{X}}\right)\right], \quad (19)$$

$$\bar{S}_{\bar{X}\bar{Y}}+\bar{\lambda}_1\left[\left(\frac{\partial}{\partial \bar{t}}+\bar{U}\frac{\partial}{\partial \bar{X}}+\bar{V}\frac{\partial}{\partial \bar{Y}}\right)\bar{S}_{\bar{X}\bar{Y}}-2\frac{\partial \bar{V}}{\partial \bar{X}}\bar{S}_{\bar{X}\bar{X}}-2\frac{\partial \bar{U}}{\partial \bar{Y}}\bar{S}_{\bar{Y}\bar{Y}}+\bar{\lambda}_3(\bar{S}_{\bar{X}\bar{X}}+\bar{S}_{\bar{Y}\bar{Y}})\left(\frac{\partial \bar{U}}{\partial \bar{Y}}+\frac{\partial \bar{V}}{\partial \bar{X}}\right)\right]$$
$$=\mu\left(\frac{\partial \bar{U}}{\partial \bar{Y}}+\frac{\partial \bar{V}}{\partial \bar{X}}\right)+\mu\bar{\lambda}_2\left[\left(\frac{\partial}{\partial \bar{t}}+\bar{U}\frac{\partial}{\partial \bar{X}}+\bar{V}\frac{\partial}{\partial \bar{Y}}\right)\left(\frac{\partial \bar{U}}{\partial \bar{Y}}+\frac{\partial \bar{V}}{\partial \bar{X}}\right)-2\left(\frac{\partial \bar{U}}{\partial \bar{X}}\frac{\partial \bar{V}}{\partial \bar{X}}+\frac{\partial \bar{U}}{\partial \bar{Y}}\frac{\partial \bar{V}}{\partial \bar{Y}}\right)\right], \quad (20)$$

$$\bar{S}_{\bar{Y}\bar{Y}}+\bar{\lambda}_1\left[\left(\frac{\partial}{\partial \bar{t}}+\bar{U}\frac{\partial}{\partial \bar{X}}+\bar{V}\frac{\partial}{\partial \bar{Y}}\right)\bar{S}_{\bar{Y}\bar{Y}}-2\frac{\partial \bar{V}}{\partial \bar{X}}\bar{S}_{\bar{X}\bar{X}}-2\frac{\partial \bar{V}}{\partial \bar{Y}}\bar{S}_{\bar{Y}\bar{Y}}\right]+2\bar{\lambda}_3(\bar{S}_{\bar{X}\bar{X}}+\bar{S}_{\bar{Y}\bar{Y}})\frac{\partial \bar{V}}{\partial \bar{Y}}$$
$$=2\mu\frac{\partial \bar{V}}{\partial \bar{Y}}+2\mu\bar{\lambda}_2\left[\left(\frac{\partial}{\partial \bar{t}}+\bar{U}\frac{\partial}{\partial \bar{X}}+\bar{V}\frac{\partial}{\partial \bar{Y}}\right)\frac{\partial \bar{V}}{\partial \bar{Y}}-2\left(\frac{\partial \bar{V}}{\partial \bar{Y}}\right)^2-\frac{\partial \bar{V}}{\partial \bar{X}}\left(\frac{\partial \bar{U}}{\partial \bar{Y}}+\frac{\partial \bar{V}}{\partial \bar{X}}\right)\right], \quad (21)$$

In the laboratory frame ($\bar{X},\bar{Y}$), the flow in a channel is unsteady. However, it can be treated as steady in a coordinates system ($\bar{x},\bar{y}$) moving at the wave speed (wave frame). The transformations relating coordinates and velocities in two frames are given by

$$\bar{x}=\bar{X}-c\bar{t},\quad \bar{y}=\bar{Y},\quad \bar{u}=\bar{U}-c,\quad \bar{v}=\bar{V}, \quad (22)$$

where $\bar{u}$ and $\bar{v}$ are respectively the dimensional velocity components parallel to $\bar{x}$ and $\bar{y}$ in the wave frame. We make use of Maxwell equations with generalized Ohm's law and find that

$$\mathbf{J}\times\mathbf{B}=[-\sigma B_0^2\bar{U},-\sigma B_0^2\bar{V}]. \quad (23)$$

With the help of Eq. (22), Eqs. (9)-(11) and (19)-(21) becomes

$$\frac{\partial \bar{U}}{\partial \bar{X}}+\frac{\partial \bar{V}}{\partial \bar{Y}}=0, \quad (24)$$

$$\rho\left(\frac{\partial}{\partial \bar{t}}+\bar{U}\frac{\partial}{\partial \bar{X}}+\bar{V}\frac{\partial}{\partial \bar{Y}}\right)\bar{U}=-\frac{\partial \bar{p}}{\partial \bar{X}}+\frac{\partial \bar{S}_{\bar{X}\bar{X}}}{\partial \bar{X}}+\frac{\partial \bar{S}_{\bar{X}\bar{Y}}}{\partial \bar{Y}}-\sigma B_0^2\bar{U}, \quad (25)$$

$$\rho\left(\frac{\partial}{\partial \bar{t}}+\bar{U}\frac{\partial}{\partial \bar{X}}+\bar{V}\frac{\partial}{\partial \bar{Y}}\right)\bar{V}=-\frac{\partial \bar{p}}{\partial \bar{Y}}+\frac{\partial \bar{S}_{\bar{X}\bar{Y}}}{\partial \bar{X}}+\frac{\partial \bar{S}_{\bar{Y}\bar{Y}}}{\partial \bar{Y}}-\sigma B_0^2\bar{V}. \quad (26)$$

Upon making use of the transformation (22), Eqs. (24)-(26) can be casted in the wave frame as

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}=0, \quad (27)$$

$$\rho\left(\bar{u}\frac{\partial}{\partial \bar{x}}+\bar{v}\frac{\partial}{\partial \bar{y}}\right)\bar{u}=-\frac{\partial \bar{p}}{\partial \bar{x}}+\frac{\partial \bar{S}_{\bar{x}\bar{x}}}{\partial \bar{x}}+\frac{\partial \bar{S}_{\bar{x}\bar{y}}}{\partial \bar{y}}-\sigma B_0^2(\bar{u}+c), \quad (28)$$

$$\rho\left(\bar{u}\frac{\partial}{\partial \bar{x}}+\bar{v}\frac{\partial}{\partial \bar{y}}\right)\bar{v}=-\frac{\partial \bar{p}}{\partial \bar{y}}+\frac{\partial \bar{S}_{\bar{x}\bar{y}}}{\partial \bar{x}}+\frac{\partial \bar{S}_{\bar{y}\bar{y}}}{\partial \bar{y}}-\sigma B_0^2\bar{v}. \quad (29)$$

$$\bar{S}_{\bar{x}\bar{x}}+\bar{\lambda}_1\left[\left(\bar{u}\frac{\partial}{\partial \bar{x}}+\bar{v}\frac{\partial}{\partial \bar{y}}\right)\bar{S}_{\bar{x}\bar{x}}-2\frac{\partial \bar{u}}{\partial \bar{x}}\bar{S}_{\bar{x}\bar{x}}-2\frac{\partial \bar{u}}{\partial \bar{y}}\bar{S}_{\bar{x}\bar{y}}\right]+2\bar{\lambda}_3(\bar{S}_{\bar{x}\bar{x}}+\bar{S}_{\bar{y}\bar{y}})\frac{\partial \bar{u}}{\partial \bar{x}} \quad (30)$$
$$=2\mu\frac{\partial \bar{u}}{\partial \bar{x}}+2\mu\bar{\lambda}_2\left[\left(\bar{u}\frac{\partial}{\partial \bar{x}}+\bar{v}\frac{\partial}{\partial \bar{y}}\right)\frac{\partial \bar{u}}{\partial \bar{x}}-2\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2-\frac{\partial \bar{u}}{\partial \bar{y}}\left(\frac{\partial \bar{u}}{\partial \bar{y}}+\frac{\partial \bar{v}}{\partial \bar{x}}\right)\right],$$

$$\bar{S}_{\bar{x}\bar{y}}+\bar{\lambda}_1\left[\left(\bar{u}\frac{\partial}{\partial \bar{x}}+\bar{v}\frac{\partial}{\partial \bar{y}}\right)\bar{S}_{\bar{x}\bar{y}}-2\frac{\partial \bar{v}}{\partial \bar{x}}\bar{S}_{\bar{x}\bar{x}}-2\frac{\partial \bar{u}}{\partial \bar{y}}\bar{S}_{\bar{y}\bar{y}}+\bar{\lambda}_3(\bar{S}_{\bar{x}\bar{x}}+\bar{S}_{\bar{y}\bar{y}})\left(\frac{\partial \bar{u}}{\partial \bar{y}}+\frac{\partial \bar{v}}{\partial \bar{x}}\right)\right] \quad (31)$$
$$=\mu\left(\frac{\partial \bar{u}}{\partial \bar{y}}+\frac{\partial \bar{v}}{\partial \bar{x}}\right)+\mu\bar{\lambda}_2\left[\left(\bar{u}\frac{\partial}{\partial \bar{x}}+\bar{v}\frac{\partial}{\partial \bar{y}}\right)\left(\frac{\partial \bar{u}}{\partial \bar{y}}+\frac{\partial \bar{v}}{\partial \bar{x}}\right)-2\left(\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{v}}{\partial \bar{x}}+\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial \bar{v}}{\partial \bar{y}}\right)\right],$$

$$\bar{S}_{\bar{y}\bar{y}}+\bar{\lambda}_1\left[\left(\bar{u}\frac{\partial}{\partial \bar{x}}+\bar{v}\frac{\partial}{\partial \bar{y}}\right)\bar{S}_{\bar{y}\bar{y}}-2\frac{\partial \bar{v}}{\partial \bar{x}}\bar{S}_{\bar{x}\bar{y}}-2\frac{\partial \bar{v}}{\partial \bar{y}}\bar{S}_{\bar{y}\bar{y}}\right]+2\bar{\lambda}_3(\bar{S}_{\bar{x}\bar{x}}+\bar{S}_{\bar{y}\bar{y}})\frac{\partial \bar{v}}{\partial \bar{y}} \quad (32)$$
$$=2\mu\frac{\partial \bar{v}}{\partial \bar{y}}+2\mu\bar{\lambda}_2\left[\left(\bar{u}\frac{\partial}{\partial \bar{x}}+\bar{v}\frac{\partial}{\partial \bar{y}}\right)\frac{\partial \bar{v}}{\partial \bar{y}}-2\left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2-\frac{\partial \bar{v}}{\partial \bar{x}}\left(\frac{\partial \bar{v}}{\partial \bar{x}}+\frac{\partial \bar{u}}{\partial \bar{y}}\right)\right]$$

In order to non-dimensionlize the governing Eqs. (27)-(32), we introduce the following variables and parameters.

$$\bar{x}=\frac{\lambda x}{2\pi},\quad \bar{y}=ay,\quad \bar{u}=cu,\quad \bar{v}=cv,\quad \mathrm{Re}=\frac{\rho ca}{\mu},$$
$$\bar{S}=\frac{\mu c}{a}S,\quad \bar{p}=\frac{\lambda\mu c}{2\pi a^2}p,\quad \bar{h}=ah,\quad \delta=\frac{2\pi a}{\lambda}, \quad (33)$$
$$\lambda_1=\frac{\bar{\lambda}_1 c}{a},\quad \lambda_2=\frac{\bar{\lambda}_2 c}{a},\quad \lambda_3=\frac{\bar{\lambda}_3 c}{a}.$$

Using these variables and parameters and defining the stream function by the relation

$$u=\frac{\partial\psi}{\partial y},\qquad v=-\delta\frac{\partial\psi}{\partial x},\tag{34}$$

With the help of dimensionless parameters defined by Eq. (33) and stream function given by Eq. (34), we obtain the following dimensionless equations, the continuity equation (27) is identically satisfied and the Eqs. (28)-(32) take the following form

$$\delta\mathrm{Re}\left[\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\psi}{\partial y}\right]=-\frac{\partial p}{\partial x}+\delta\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xy}}{\partial y}-M^2\left(\frac{\partial\psi}{\partial y}+1\right),\tag{35}$$

$$-\delta^3\mathrm{Re}\left[\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial\psi}{\partial x}\right]=-\frac{\partial p}{\partial y}+\delta^2\frac{\partial S_{xy}}{\partial x}+\delta\frac{\partial S_{yy}}{\partial y}-M^2c\delta^2\frac{\partial\psi}{\partial x},\tag{36}$$

$$\delta\mathrm{Re}\left[\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2\psi}{\partial y^2}+\delta^2\frac{\partial^2\psi}{\partial x^2}\right)\right]=\delta\frac{\partial^2(S_{xx}-S_{yy})}{\partial x\partial y}+\left(\frac{\partial^2}{\partial y^2}-\delta^2\frac{\partial^2}{\partial x^2}\right)S_{xy}-M^2\left[\frac{\partial\psi}{\partial y}\left(\frac{\partial\psi}{\partial y}+1\right)\right]+M^2c\delta\frac{\partial^2\psi}{\partial y^2}.\tag{37}$$

$$S_{xx}+\lambda_1\left[\delta\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)S_{xx}-2\delta\frac{\partial^2\psi}{\partial x\partial y}S_{xx}-2\frac{\partial^2\psi}{\partial y^2}S_{xy}\right]+2\delta\lambda_3(S_{xx}+S_{yy})\frac{\partial^2\psi}{\partial x\partial y}$$
$$=2\delta\frac{\partial^2\psi}{\partial x\partial y}+2\lambda_2\left[\delta^2\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial^2\psi}{\partial x\partial y}-2\delta^2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2-\frac{\partial^2\psi}{\partial y^2}\left(\frac{\partial^2}{\partial y^2}-\delta^2\frac{\partial^2}{\partial x^2}\right)\right],\tag{38}$$

$$S_{xy}+\lambda_1\left[\delta\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)S_{xy}+\delta^2\frac{\partial^2\psi}{\partial x^2}S_{xx}-\frac{\partial^2\psi}{\partial y^2}S_{yy}\right]+\lambda_3(S_{xx}+S_{yy})\left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)$$
$$=\left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)+\lambda_2\left[\delta\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)+2\delta\left(\delta^2\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\psi}{\partial x\partial y}\right)\right],\tag{39}$$

$$S_{yy}+\lambda_1\left[\delta\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)S_{yy}+2\delta^2\frac{\partial^2\psi}{\partial x^2}S_{xy}+2\delta\frac{\partial^2\psi}{\partial x\partial y}S_{yy}\right]-2\delta\lambda_3(S_{xx}+S_{yy})\frac{\partial^2\psi}{\partial x\partial y}$$
$$=-2\delta\frac{\partial^2\psi}{\partial x\partial y}+2\lambda_2\left[-\delta^2\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\frac{\partial^2\psi}{\partial x\partial y}-2\delta^2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2+\delta^2\frac{\partial^2\psi}{\partial x^2}\left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)\right].\tag{40}$$

In above equations *Re* is the Reynolds number, δ is the Wave number and λ_{1-3} are the Wessenberg numbers.

It is formidable task to solve the Eqs. (35)-(40) in their current form. Fortunately, many physiological processes, where peristalsis is involved, the wavelength of the wave is large as compared to the radius of the vessel or organ. This assumption amount to assume that [illegible] □0 and known in the literature as long wavelength approximation. Therefore, under the long wavelength and low Reynolds number assumptions, Eqs. (35)-(40) reduces to

$$\frac{\partial S_{xy}}{\partial y}=\frac{\partial p}{\partial x}+M^2\left(\frac{\partial\psi}{\partial y}+1\right),\tag{41}$$

$$\frac{\partial p}{\partial y}=0,\tag{42}$$

$$\frac{\partial^2 S_{xy}}{\partial y^2}=-M^2\left[\frac{\partial\psi}{\partial y}\left(\frac{\partial\psi}{\partial y}+1\right)\right].\tag{43}$$

Solving Eq. (38) and Eq. (39) for S_{xy} when δ □0 we get

$$\frac{\partial}{\partial y}\left[\left\{\frac{1+2\alpha_1\left(\frac{\partial^2\psi}{\partial y^2}\right)^2}{1+2\alpha_2\left(\frac{\partial^2\psi}{\partial y^2}\right)^2}\right\}\frac{\partial^2\psi}{\partial y^2}\right]=\frac{dp}{dx}+M^2\left(\frac{\partial\psi}{\partial y}+1\right),\tag{44}$$

and

$$\frac{\partial^2}{\partial y^2}\left[\left\{\frac{1+2\alpha_1\left(\frac{\partial^2\psi}{\partial y^2}\right)^2}{1+2\alpha_2\left(\frac{\partial^2\psi}{\partial y^2}\right)^2}\right\}\frac{\partial^2\psi}{\partial y^2}\right]-M^2\frac{\partial^2\psi}{\partial y^2}=0.\tag{45}$$

It is to be noted that when $M=0$, Eqs. (44) and (45) reduce to corresponding equations of hydrodynamic fluid.

3. Rate of volume flow and boundary conditions

The instantaneous volume flow rate in the fixed frame is given by

$$Q=\int_0^{\bar h}\bar U(\bar X,\bar Y,\bar t)d\bar Y, \tag{46}$$

where $\bar h$ is a function of $\bar X$ and $\bar t$.
The rate of volume flow in the wave frame is given by

$$q=\int_0^{\bar h}\bar u(\bar x,\bar y)d\bar y, \tag{47}$$

where $\bar h$ is a function of $\bar x$ alone. If we substitute Eq. (22) into Eq. (46) and make use of Eq. (33), we find that the two rates of volume flow are related by

$$Q=q+c\bar h. \tag{48}$$

The time mean flow over a period T at a fixed position $\bar X$ is defined as

$$\bar Q=\frac{1}{T}\int_0^T Qdt. \tag{49}$$

Substituting Eq. (48) into Eq. (49) and integrating we get

$$\bar Q=q+ac. \tag{50}$$

defining the dimensionless time mean flows Θ and F respectively in the fixed and wave frame as

$$\Theta=\frac{\bar Q}{ac},\ \text{and}\ F=\frac{\bar F}{ac}, \tag{51}$$

one finds that Eq. (50) can be written as

$$\Theta=F+1, \tag{52}$$

where

$$F=\int_0^h\frac{\partial\psi}{\partial y}dy=\psi(h)-\psi(0) \tag{53}$$

and h represents the dimensionless form of the surface of the peristaltic wall, i.e.

$$h(x)=1+\phi\cos x. \tag{54}$$

Here $\phi=b/a$ is the amplitude ratio or the occlusion.
If we select the zero value of the streamline at the centerline $(y=0)$. we have

$$\psi(0)=0, \tag{55}$$

then the wall $(y=h)$ is a streamline of value

$$\psi(h)=F. \tag{56}$$

The appropriate boundary conditions for the dimensionless stream function in the wave frame

$$\psi=0,\quad \frac{\partial^2\psi}{\partial y^2}=0,\quad \text{at } y=0,$$
$$\psi=F,\quad \frac{\partial\psi}{\partial y}=-1,\quad \text{at } y=1+\phi\cos x. \tag{57}$$

4. Numerical solution

In this section we proceed to find direct numerical solution of the differential Eq. (45) subject to boundary conditions (57) by means of a suitable numerical technique. The differential Eq. (45) is nonlinear in ψ and cannot be solved by the direct finite difference method. In solving such a nonlinear equations, iterative methods are commonly used. We can now construct an iterative procedure in the following form

$$\left(\frac{1+2\alpha_1\left(\frac{\partial^2\psi^{(n)}}{\partial y^2}\right)^2}{1+2\alpha_2\left(\frac{\partial^2\psi^{(n)}}{\partial y^2}\right)^2}\right)\frac{\partial^4\psi^{(n+1)}}{\partial y^4}+2\frac{\partial}{\partial y}\left(\frac{1+2\alpha_1\left(\frac{\partial^2\psi^{(n)}}{\partial y^2}\right)^2}{1+2\alpha_2\left(\frac{\partial^2\psi^{(n)}}{\partial y^2}\right)^2}\right)\frac{\partial^3\psi^{(n+1)}}{\partial y^3}$$
$$+\frac{\partial^2}{\partial y^2}\left(\frac{1+2\alpha_1\left(\frac{\partial^2\psi^{(n)}}{\partial y^2}\right)^2}{1+2\alpha_2\left(\frac{\partial^2\psi^{(n)}}{\partial y^2}\right)^2}\right)\frac{\partial^2\psi^{(n+1)}}{\partial y^2}-M^2\frac{\partial^2\psi^{(n+1)}}{\partial y^2}=0,$$

(58)

$$\psi=0,\quad \frac{\partial^2\psi^{(n+1)}}{\partial y^2}=0,\quad \text{at } y=0,$$
$$\psi=F,\quad \frac{\partial\psi^{(n+1)}}{\partial y}=-1,\quad \text{at } y=h. \tag{59}$$

where the index (n) indicates the iterative step. It is easy to confirm that if the indices (n) and $(n+1)$ are withdrawn, the Eq. (58) is consistent with the original differential Eq. (45). Equation (58) and the boundary conditions (59) define a linear differential boundary value problem for $\psi^{(n+1)}$. By means of the finite difference method a linear algebraic equation system can be deduced and solved for each iterative step $(n+1)$. Therefore, a sequence of functions $\psi^{(0)}(x,y),\psi^{(1)}(x,y),\psi^{(2)}(x,y),\ldots$ is determined in the following manner: if an initial estimated $\psi^{(0)}(x,y)$ is given, then $\psi^{(1)}(x,y),\psi^{(2)}(x,y),\ldots$ are calculated successively as the solutions of the boundary value problem (58) and (59). Unfortunately, such an iteration is often divergent, especially when the initial estimated $\psi^{(0)}(x, y)$ is not given carefully and suitably. Usually, in order to achieve a better convergence, the so called method of successive under-relaxtion is used. We solve the boundary value problem (58) and (59) for the iterative step $(n+1)$ to obtain an estimated value of $\psi^{(n+1)}$ and $\tilde{\psi}^{(n+1)}$, then $\psi^{(n+1)}$ is defined by the formula

$$\psi^{(n+1)} = \psi^{(n)} + \omega\left(\tilde{\psi}^{(n+1)} - \psi^{(n)}\right), \quad \omega \in (0,1] \qquad (60)$$

where ω is under-relaxation parameter. We should choose ω so small that convergent iteration is reached. In our simulation we choose an initial guess of $\psi^{(0)}(x, y) = Fy / h$ which fulfils the first and third boundary conditions in (57). Of course, some other choices are also possible. The iteration should be carried out until the relative differences of the computed $\psi^{(n+1)}$ and $\psi^{(n)}$ between two iterative steps are smaller than a given error chosen to be 10^{-8}.

5. Results and discussion

To see the effect of Hartman number on various features of the peristaltic motion we have plotted Figs. 1-4.

In Fig. 1(a) the longitudinal velocity u is plotted against y for different values of α_1 at a fixed position $x = -\pi$ with non-zero values of M .Fig. 1(b) is made to see the effects of α_2 on u for MHD fluid. The profiles of stream function for MHD fluid for different values of α_1 and α_2 are shown in Figs. 1(c) and 1(d). We observe from these figures the similar behavior as observed for hydrodynamics fluid. However, Figs. 2(a)-(d) reveal some interesting results. These are summarized below.

- An increase in M increases the velocity near the boundary. However, near the centerline the situation is reversed.
- The values of stream function decreases in going from hydrodynamic to magneto hydrodynamic fluid.

The variation of longitudinal pressure gradient dp / dx over one wavelength for different values of M is shown in Figs. 3(a)-(d). In Figs. 3(a) and 3(b) $\Theta = 0.8$, while in Figs. 3(c) and 3(d) $\Theta = 0.2$. The following results are worth mentioning.

- The magnitude of longitudinal pressure gradient increases with an increase in M.
- The longitudinal pressure gradient resists/assist the flow in the narrow/wider of the channel for the small values of Θ. For the large values of Θ longitudinal pressure gradient become favorable over the whole width of the channel.

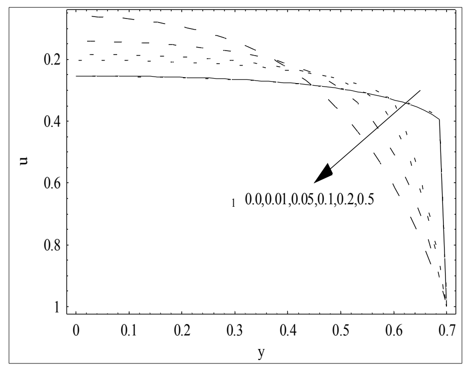

Fig. 1(a): Plot of the longitudinal velocity u for various values of $\alpha_1(\alpha_2 = 0.5)$. The other parameters chosen are $M = 5$, $F = -0.2$ and $\phi = 0.3$.

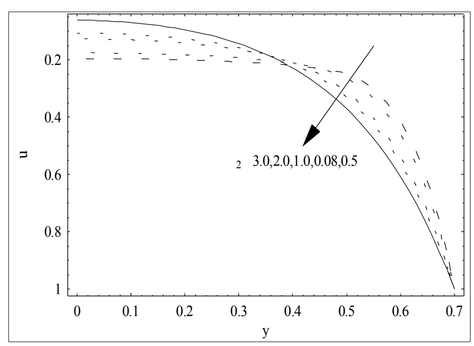

Fig. 1(b): Plot of the longitudinal velocity u for various values of $\alpha_2(\alpha_1 = 0.5)$. The other parameters chosen are $M = 5$, $F = -0.2$ and $\phi = 0.3$.

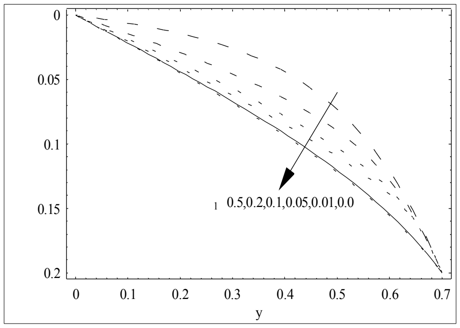

Fig. 1(c): Plot of the stream function ψ for various values of $\beta_1 (\beta_2 = 0.5)$. The other parameters chosen are $M = 5, F = -0.2$ and $\Omega = 0.3$.

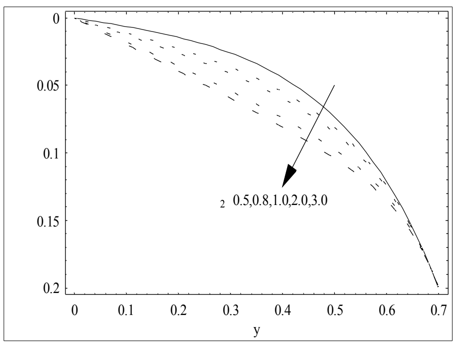

Fig. 1(d): Plot of the stream function ψ for various values of $\beta_2 (\beta_1 = 0.5)$. The other parameters chosen are $M = 5$, $F = -0.2$ and $\Omega = 0.3$.

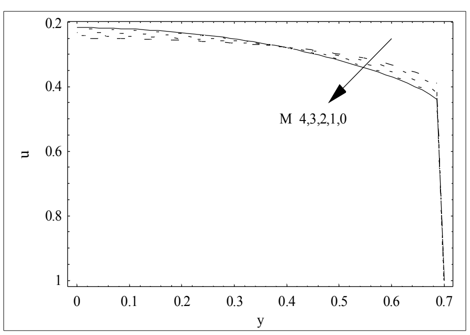

Fig. 2(a): Plot of the longitudinal velocity u for various values of M. The other parameters chosen are $\beta_1 = 0, \beta_2 = 0.5, F = -0.2$ and $\Omega = 0.3$.

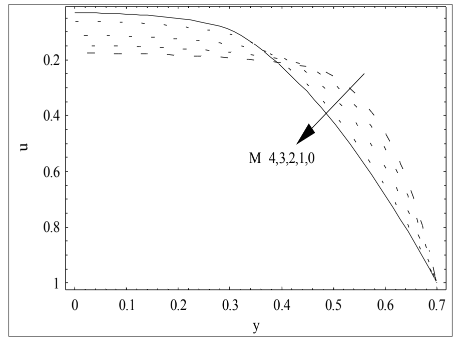

Fig. 2(b): Plot of the longitudinal velocity u for various values of M. The other parameters chosen are $\beta_1 = 0.5, \beta_2 = 3, F = -0.2$ and $\Omega = 0.3$.

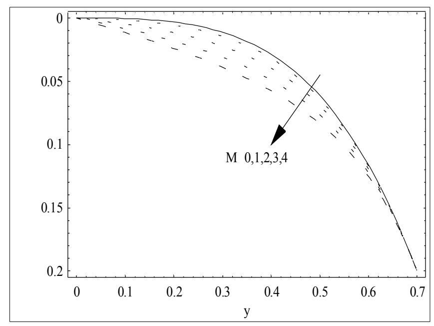

Fig. 2(c): Plot of the stream function ψ for various values of M. The other parameters chosen are $\beta_1 = 0.5, \beta_2 = 0.5, F = -0.2$ and $\Omega = 0.3$.

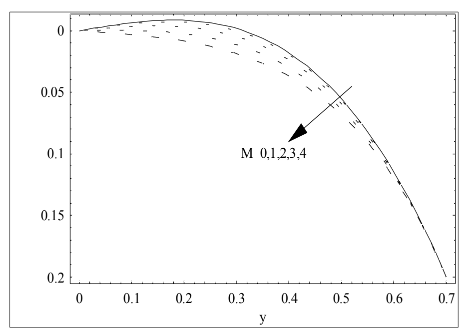

Fig. 2(d): Plot of the stream function ψ for various values of M. The other parameters chosen are $\beta_1 = 0.5, \beta_2 = 1, F = -0.2$ and $\Omega = 0.3$.

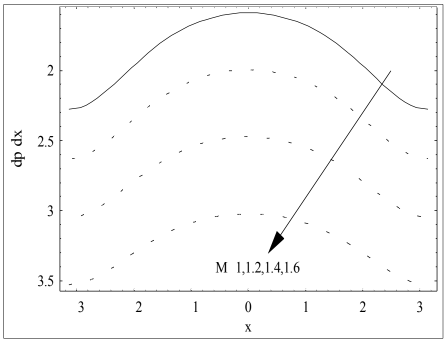

Fig. 3(a). Plot of the longitudinal pressure gradient dp/dx for various values of M. The other parameters chosen are $\alpha_1 = 0.1, \alpha_2 = 0.5, \Theta = 0.8$ and $\phi = 0.3$.

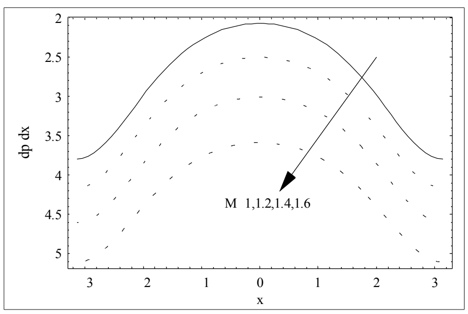

Fig. 3(b). Plot of the longitudinal pressure gradient dp/dx for various values of M. The other parameters chosen are $\alpha_1 = 0.5, \alpha_2 = 0.8, \Theta = 0.8$ and $\phi = 0.3$.

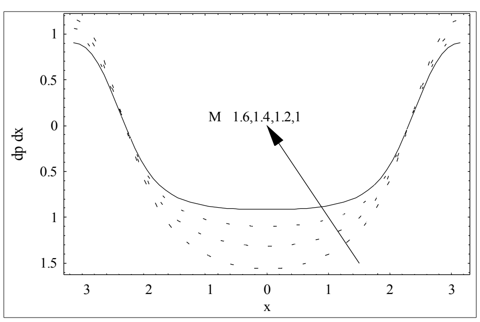

Fig. 3(c). Plot of the longitudinal pressure gradient dp/dx for various values of M. The other parameters chosen are $\alpha_1 = 0.1, \alpha_2 = 0.5, \Theta = 0.2$ and $\phi = 0.3$.

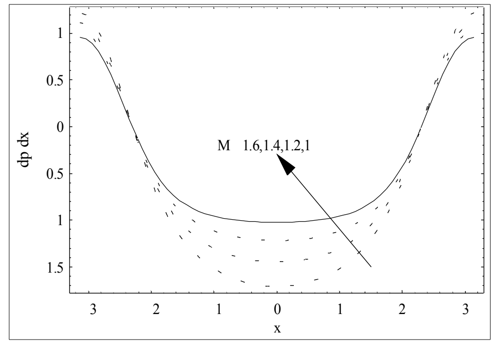

Fig. 3(d). Plot of the longitudinal pressure gradient dp/dx for various values of M. The other parameters chosen are $\alpha_1 = 0.5, \alpha_2 = 0.8, \Theta = 0.2$ and $\phi = 0.3$.

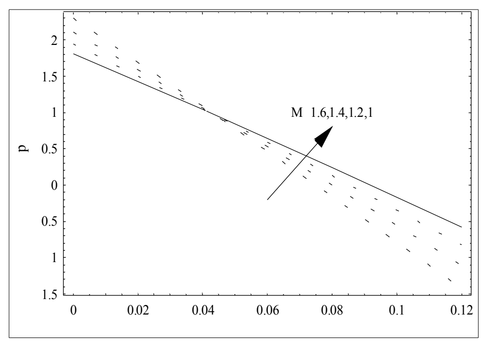

Fig. 4. Profile of pressure rise per wavelength Δp versus flow rate Θ for various values of M. The other parameters chosen are $\alpha_1 = 0, 0.1, 0.2, 0.4, 0.5, \alpha_2 = 0.5$ and $\phi = 0.3$.

Table 1: Values of Δp for different values of M and Θ.

M	$\Theta = 0$	$\Theta = 0.04$	$\Theta = 0.08$	$\Theta = 0.12$	$\Theta = 0.16$	$\Theta = 0.2$	$\Theta = 0.24$
0	0.88432	0.55901	0.212003	-0.15553	-0.56624	-1.06249	-1.66624
1	1.80726	1.04089	0.246788	-0.58384	-1.46769	-2.42249	-3.43287
1.2	1.94645	1.05173	0.130933	-0.82452	-1.83173	-2.91073	-4.04737
1.4	2.11167	1.06628	-0.00318	-1.10535	-2.25792	-3.48343	-4.76919
1.6	2.30321	1.08524	-0.15463	-1.42541	-2.74548	-4.13974	-5.59735

Figure 4 and Table 1 illustrates the relation between pressure rise per wavelength ΔP and flow rate Θ for the various values of M. It is observed that P_0 increases by increasing $M.$ This means that the peristalsis has to do work against greater pressure rise for MHD fluid as compared to hydrodynamic fluid. Moreover, in co-pumping the pumping rate decreases for large values of $M.$

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