Topological and Singular Solitons of *B(m, n)* Equation with Generalized Evolution

Bouthina S. Ahmed[1] and Anjan Biswas[2, 3]

[1]Department of Mathematics, College of Girls, Ain Shams University, P. O. Box.11566, Cairo, Egypt
[2]Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA
[3]Department of Mathematics, Faculty of Science King Abdul-Aziz University, Jeddah, Saudi Arabia
Biswas.Anjan@gmail.com, Ahmed_Bouthina@live.com

Abstract: This paper studies the *B(m, n)* equation with generalized evolution. The ansatz method is applied to extract the topological as well as singular soliton solutions to the equation. It will be observed that specific choice of pairs parameter values, both of these solitons will exist. This will lead to four exhaustive cases and all of these cases are analyzed for the existence of soliton solutions.

[Bouthina S. Ahmed and Anjan Biswas. **Topological and Singular Solitons of *B(m, n)* Equation with Generalized Evolution.** *Life Sci J* 2013; 10(1):3002-3005]. (ISSN: 1097-8135). http://www.lifesciencesite.com. 368

Keywords: Evolution equation, Integrability, Solitons.

1. Introduction

The theory of nonlinear evolution equations (NLEEs) has made a remarkable progress in the past few decades [1-10]. There has been an overwhelming advancement that was observed, especially with regards to the integrability aspect of these NLEEs. These NLEEs appear in various areas of research. A few of them are fluid dynamics, nuclear physics, plasma physics and nonlinear optics. Recently there has been a need for NLEEs in the Area of biological sciences. There are several kinds of NLEEs that are studied in this area of research. The nonlinear Schrodinger's equation describes the dynamics of solitons in alpha-helix proteins. A lot of results have been obtained and reported in this direction [2-7]. Very recently, Boussinesq equation (BE) has gained a lot of importance in mathematical biology. The coupled BE has been studied in the context of aneurysm [9]. Another area where BE with dual nonlinearity is studied is neurosciences. The BE in this context describes the dynamics of solitons in biomembranes [3, 8]. Thus BE is gaining profound popularity in the area of mathematical biosciences. Hence it is imperative to take a deeper look at generalized form of BE that will shed some light on the applicable areas especially in the context of Life sciences. This form of BE is referred to as the *B(m, n)* equation with generalized evolution. The *B(m, n)* equation with generalized evolution term that is going to be studied in this paper is given by [2]

$$(q^l)_{tt}+a(q^m)_{xx}-b(q^n)_{xxxx}=0 \qquad (1)$$

Here, the first term is the generalized evolution term, while the second term represents the nonlinear term and the third term is the dispersion term. Also, $a, b \in R$ and are constants, while *l*, *m* and $n \in Z^+$. This equation is the generalized form of Boussinesq equation, where, in particular, The Case *l=m=n=1* leads to the Boussinesq equation. In this paper, the general case for $l > 1$ will be studied. Incidentally, this equation, for the special case with *l=1* was studied before by many authors [6].

2. Topological soliton solution

We assume the soliton solution to be of the form

$$q(x,t)=A\tanh^p B(x-vt) \qquad (2)$$

where *A*, *B* are free parameters and *v* represents the velocity of the soliton. The exponent *p* will be determined as a function of *l, m* and *n*.

Substituting (2) into (1) yields,

$$\begin{aligned}&lpv^2A^lB^2\{(lp-1)\tanh^{lp-2}\tau-2lp\tanh^{lp}\tau+(lp+1)\tanh^{lp+2}\tau\\&+ampA^mB^2\{(mp-1)\tanh^{mp-2}\tau-2mp\tanh^{mp}\tau+(mp+1)\tanh^{mp+2}\tau\}\\&-bnpA^nB^4\{(np-1)(np-2)(np-3)\tanh^{np-4}\tau\\&+(np+1)(np+2)(np+3)\tanh^{np+4}\tau-2(n^2p^2+(np-2)^2)(np-1)\tanh^{np-2}\tau\\&-2(n^2p^2+(np+2)^2)(np+1)\tanh^{np+2}\tau\\&+(4n^3p^3+(np-1)^2(np-2)+(np+1)^2(np+2)\tanh^{np}\tau\}=0\end{aligned} \qquad (3)$$

The analysis of this equation will be split into the following four cases depending on the structure of the equality of the exponents

2.1. Case I: $l=n, \quad m \neq n$

In (3) equating the exponents of $mp+2$ and $np+4$, we get

$$p=\frac{2}{m-n} \qquad (4)$$

Equating the coefficients of the function pairs $\tanh^{mp+2}\tau$, $\tanh^{np+4}\tau$ and $\tanh^{mp}\tau$, $\tanh^{np+2}\tau$

$$an(np+1)A^m - np(np+1)(np+2)(np+3)A^nB^2 = 0 \quad (5)$$

$$n(np+1)A^n v^2 - 2an^2A^m - 2(n^2p^2 + (np+2)^2)(np+1)A^nB^2 = 0 \quad (6)$$

Solving (5) and (6) leads

$$A = \left[\frac{-v(m+n)}{4an}\right]^{\frac{1}{m-n}} \quad (7)$$

$$B = \frac{m-n}{2n}\sqrt{\frac{-v}{2b}} \quad (8)$$

Thus the solution of (1) is

$$q(x,t) = A\tanh^{\frac{2}{m-n}} B(x - vt) \quad (9)$$

Figure 1 describes the topological soliton solution of $q(x,t)$ with the constants $a = 1, b = -1$ and the parameter values are $m = 2, n = 1, v = 0.3$.

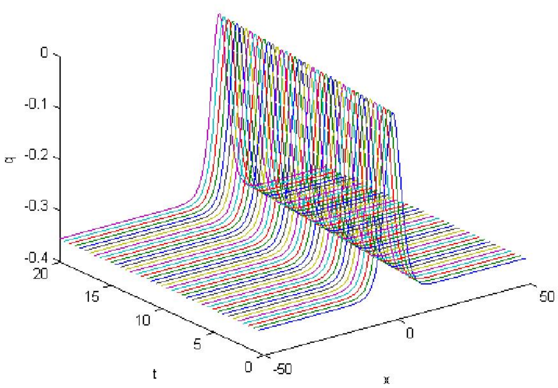

Figure 1: Topological 1-soliton solution with *m=2, n=1, a=1, b=-1, v=0.3*

2.2. Case II: $l \neq n, \quad m = n$

In (3) equating the exponents of $lp + 2$ and $np + 4$, implies

$$p = \frac{2}{l-n} \quad (10)$$

Equating the coefficients of the function pairs $\tanh^{lp+2}\tau$, $\tanh^{np+4}\tau$ and $\tanh^{lp}\tau$, $\tanh^{np+2}\tau$

$$l(lp+1)A^l v^2 - nb(np+1)(np+2)(np+3)A^nB^2 = 0 \quad (11)$$

$$-2lv^2A^l + an(np+1)A^l + 2nb\{n^2p^2 + (np+2)^2\}A^nB^2 = 0 \quad (12)$$

Solving (11) and (12) implies

$$A = \left[\frac{-4nv}{a(l+n)}\right]^{\frac{1}{l-n}} \quad (13)$$

$$B = \frac{l-n}{2n}\sqrt{\frac{-a}{2b}} \quad (14)$$

which imposes *ab<0*

Thus the topological soliton solution of (1) is

$$q(x,t) = A\tanh^{\frac{2}{l-n}} B(x - vt) \quad (15)$$

Figure 2 describes the topological 1- soliton solution of $q(x,t)$ with the constants $a = 1, b = -1$ and the parameters $m = 2, n = 1, v = 0.3$.

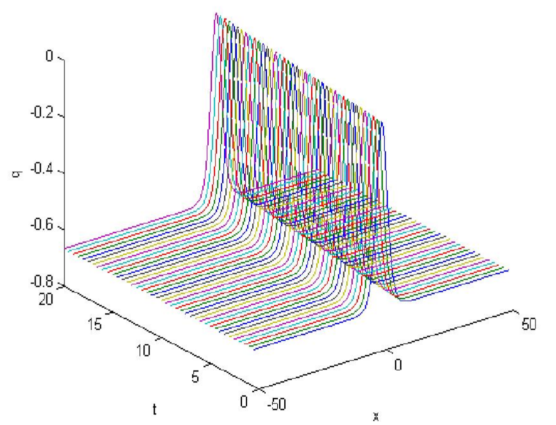

Figure 2: Topological 1-soliton solution with *l=2, m=2, n=1, a=1, b=-1, v=0.3*

2.3. Case III: $l = m, \ m \neq n$

In (3) equating the exponents of $lp + 2$ and $np + 4$, yields

$$p = \frac{2}{l-n} \quad (16)$$

Equating the coefficients of the function pairs $\tanh^{lp+2}\tau$, $\tanh^{np+4}\tau$ and $\tanh^{lp}\tau$, $\tanh^{np+2}\tau$

$$l(lp+1)v^2A^l + al(lp+1)A^l - nb(np+1)(np+2)(np+3)A^nB^2 = 0 \quad (17)$$

$$l^2pv^2A^l + al^2pA^l + 2nb\{n^2p^2 + (np+2)^2\}A^nB^2 = 0 \quad (18)$$

Solving (17) and (18) there exists a zero solution only, that is

$$q(x,t) = 0 \quad (19)$$

2.4. Case IV: $l \neq m \neq n$

In (3) equating the exponents of $lp + 2$ and $np + 4$, we get

$$p = \frac{2}{l-n} \tag{20}$$

where

$$l \neq n \tag{21}$$

In (3) equating the exponents of $mp + 2$ and $np + 4$, leads to

$$p = \frac{2}{m-n} \tag{22}$$

where

$$m \neq n \tag{23}$$

From (20) and (22) it follows that

$$l = m \tag{24}$$

From (21) and (23) it follows that

$$l \neq m \neq n \tag{25}$$

Thus (24) and (25) are contradiction and there is no solution in this case.

3. Singular soliton solution

We assume the singular soliton solution to be of the form

$$q(x,t) = A\,\text{csch}^{p} B(x - vt) \tag{26}$$

where A And B Are free parameters of the soliton And v represents the velocity of the soliton. The exponent p will be determined as a function of l, m and n. Substituting this hypothesis into (1) gives

$$\begin{aligned}
&lpv^2A^lB^2\{(lp+1)\text{csch}^{lp+2}\tau + lp\,\text{csch}^{lp}\tau\} \\
&+ampA^mB^2\{(mp+1)\text{csch}^{mp+2}\tau + mp\,\text{csch}^{mp}\tau\} \\
&-bnpA^nB^4\{(np+1)(np+2)(np+3)\text{csch}^{np+4}\tau \\
&+(n^2p^2+(np+2)^2)\text{csch}^{np+2}\tau + n^3p^3\text{csch}^{np}\tau\} = 0
\end{aligned} \tag{27}$$

3.1. Case I: $l = n,\ \ m \neq n$

In (27) equating the exponents of $mp + 2$ and $np + 4$, leads to

$$p = \frac{2}{m-n} \tag{28}$$

Equating the coefficients of the function pairs $\text{csch}^{mp+2}\tau$, $\text{csch}^{np+4}\tau$ and $\text{csch}^{mp}\tau$, $\text{csch}^{np+2}\tau$ we get

$$m(mp+1)aA^m - np(np+1)(np+2)(np+3)bA^nB^2 = 0 \tag{29}$$

$$n(np+1)A^mv^2 + m^2paA^m + nb(np+1)(n^2p^2+(np+2)^2)A^nB^2 = 0 \tag{30}$$

Solving (29) and (30) leads to

$$A = \left[\frac{v(m+n)}{2an}\right]^{\frac{1}{m-n}} \tag{31}$$

$$B = \frac{(m-n)v}{2n\sqrt{b}} \tag{32}$$

Thus the singular 1-soliton solution of (1) is

$$q(x,t) = A\,\text{csch}^{\frac{2}{m-n}} B(x - vt) \tag{33}$$

3.2. Case II: $l \neq n,\ \ m = n$

In (27) equating the exponents of $lp + 2$ and $np + 4$ yields

$$p = \frac{2}{l-n} \tag{34}$$

Equating the coefficients of the function pairs $\text{csch}^{lp+2}\tau$, $\text{csch}^{np+4}\tau$ and $\text{csch}^{lp}\tau$, $\text{csch}^{np+2}\tau$

$$l(lp+1)v^2A^l - n(np+1)(np+2)(np+3)bA^nB^2 = 0 \tag{35}$$

$$l^2pv^2A^l + n(np+1)aA^n - n(np+1)\{n^2p^2+(np+2)^2\}bA^nB^2 = 0 \tag{36}$$

Solving (35) and (36) implies

$$A = \left[\frac{a(l+n)}{2nv^2}\right]^{\frac{1}{n-l}}, \tag{37}$$

$$B = \frac{l-n}{2n}\sqrt{\frac{a}{b}} \tag{38}$$

Thus the singular 1-soliton solution of (1) is

$$q(x,t) = A\,\text{csch}^{\frac{2}{l-n}} B(x - vt) \tag{39}$$

3.3. Case III: $l = m,\ \ m \neq n$

In (27) equating the exponents of $lp + 2$ and $np + 4$, we get

$$p = \frac{2}{l-n} \tag{40}$$

Equating the coefficients of the function pairs $\text{csch}^{lp+2}\tau$, $\text{csch}^{np+4}\tau$ and $\text{csch}^{lp}\tau$, $\text{csch}^{np+2}\tau$

$$l(lp+1)v^2A^l + l(lp+1)aA^l - n(np+1)(np+2)(np+3)bA^nB^2 = 0 \quad (41)$$

$$l^2pv^2A^l + l^2paA^l - n(np+1)\{n^2p^2 + (np+2)^2\}bA^nB^2 = 0 \quad (42)$$

Solving (41) and (42) there exists a zero solution only

$$q(x,t) = 0 \quad (43)$$

3.4. Case IV: $l \neq m \neq n$

There is no solution in this case as in the case of the topological soliton solution.

4. Conclusions

This paper addressed the *B(m, n)* equation with generalized evolution in the context of topological and singular soliton solutions. The results in this paper, although eerie similar to a previously published paper, will make sense in several other areas of study such as life sciences or other branches of biological or clinical sciences [2]. The numerical simulations, of this paper, support the analytical results that are obtained and hence the numeric makes a lot of sense.

These results will be further analyzed in future. For example *B(m, n)* equation with time-dependent coefficients or rather stochastic coefficients will be dealt with. These situations will be a much closer to reality, especially in the context of biosciences. Furthermore, several perturbation terms will be added and the corresponding perturbed *B(m, n)* equation will be studied from an integrability stand point. These just form a foot in the door.

Corresponding author
Bouthina S. Ahmed
Department of Mathematics, College of Girls, Ain Shams University, P. O. Box 11566, Cairo, Egypt
Ahmed_Bouthina@live.com

References

[1] A. Bhrawy and Mustafa Obaid ."New exact solutions for the Zhiber-Shabat equation using the extended F-expansion Method". *Life Science Journal*, Volume 9, Issue 4, 1154-1162. (2012).
[2] A. Biswas ."1-soliton solution of the *B(m, n)* equation with generalized evolution". *Communications In Nonlinear Science And Numerical Simulation*.Volume 14, Issue 8, August 2009, Pages 3226-3229.
[3] A. Biswas, A. H. Kara, M. Savescu, A. Bokhari & F. Z. Zaman. "Solitons and conservation laws in neurosciences". Submitted.
[4] A. Biswas, A. Moran, D. Milovic, F. Majid & K. Biswas. "An exact solution of the modified nonlinear Schrodinger's equation for Davydov solitons In alpha-helix proteins". *Mathematical Biosciences*. Volume 227, Issue 1, 68-71. (2010).
[5] A. Biswas, Chaudry Masood Khalique. "Stationary solutions of the modified nonlinear Schrodinger's equation in alpha-helix proteins". *Advanced Studies In Biology*, Volume 2, Number 3, 99-103. (2010)
[6] A. Biswas, D. Milovic & D. Milic. "Solitons in alpha-helix proteins by He's variational principle". *International Journal Of Biomathematics*. Volume 4, Number 4, 423-429. (2011).
[7] A. Biswas, P. Suarez, J. Gillispie & F. M "Simulations of solitons in alpha-helix proteins". *Annals of Biological Research*. Volume 2, Issue 1, 156-161. (2011).
[8] B. Lautrup, R. Appali, A. D. Jackson & T. Heimburg. "The stability of solitons in biomembranes and nerves". *The European Physical Journal E*. Volume 34, Issue 6, Article-57. (2011).
[9] S. Noubissie, R. A. Kraenkel & P. Woafo. "Disturbance and repair of solitary waves in blood vessel with aneurysm". *Communications in Nonlinear Science and Numerical Simulation*. Volume 14, Issue 1, 51-60. (2009).
[10] Y. Shang. "New exact special solutions with solitary patterns For Boussiensq-like *B(m, n)* equations with fully nonlinear dispersion". *Applied Mathematics and Computation* 2006;173: 1137–1148.

2/2/2013