Inconsistency of students' mental object of numbers with irrational numbers

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Abstract *:*This study focuses on the Iranian students' notions on irrational numbers. Since there was not a clear view on student's notions to lead our future investigations, the general question "what is an irrational number?" was asked from 30 students in second year of high school in Tehran/Iran. Three categories were extracted from the answers: representation, operation and inclusion. In the second phase, through a questioner of 8 questions answered by 50 students in the same grade, their professed knowledge and performance on quadratic irrational numbers was determined. After analyzing the gathered data, in order to have more validity on our considerations, two more questions on quadratic irrationals was asked from the 30 students of the same group. Finally, ten students' were interviewed. The results showed that in addition to students' deficiencies in formal knowledge, in three aspects, students' mental object of a number in general is not consistent with their conceptions of irrational numbers: closure in representation, the relationship of the number and its correspondent point on the real number line and the basic function of a number.

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1. Introduction

Irrationality of numbers is an important concept that appears at the end of a process of conceptual development of numbers (natural, whole, integers and rational numbers) in mathematics curriculum in Iran. The concept of irrational numbers is not introduced until the early years of high school, when the students have already learned more or less all arithmetical operations like addition, subtraction, multiplication, division and taking square roots of positive rational.

In school text books of high school in Iran, the types of irrational numbers are almost restricted to quadratic irrationals and pi. As researchers and mathematics teacher, authors have noticed that there are some conflicts and ambiguities in students' understanding of irrational numbers. It consequently awoke the question where these ambiguities and conflicts have root in. From analyzing the gathered data in a preliminary study, two sub questions were formed. First, whether an irrational quantity imply the same meaning of a number at whole in its common setting; and in a narrower sense, whether students' notion of irrational numbers is consistent with their mental object of real numbers. This, somehow, relates to students' intuition about irrational numbers and the way that a mental object of a concept (number) is formed.

Encapsulation of a process into a mental object has been a major subject, in different studies. Tall and Gray make distinction between a perceived object and a conceived object.((Tall D., 1999) A conceived object is formed when there is reflection on perceptions and actions, where the focus is on actions/process upon physical manifestations . From this point of view, a mental object could be designated to a number, say number 5. Other scholars have also discussed mental objects of mathematical concepts. Piaget, for instance, discuses about "empirical abstraction" and "pseudo- empirical abstraction" which in the latter, the knowledge is derived from the process which the individual performs on the objects (Tall D., 1999) Tall, Gray and Simpson indicate that Pseudo- empirical abstraction constructs a conceived object which in many settings takes the form of a "procept". "procept" is a notion invented by Gray and Tall which allocates dual nature of "process" and "concept" to a symbol. (Tall & Gray, 1991)

What we mean by "mental object" in this paper, is more or less the same as "conceived mental object" or the object obtained by "pseudo- empirical abstraction". Mental object of a number relates to some properties, which for the case of students seems to relate to their intuition of a number. In consistency to the findings of Zazcis and Sirotic, we have also noticed that representation is a main source for students' misconception of irrational numbers (Zazkis & Sirotic, Making Sensa of Irrational Numbers: Focusing on Representation, 2004) (Zazkis & Sirotic, Representing and Defining Irrational Numbers: Exposing Missing Link, 2010); but in a broader sense, students' natural tendency to "closure" in representation is affected by their mental object of a number.The notion of "closure" in representation, was focused in our investigation; a term which was first defined by Kevin Collis. ((Collis, 1975). In his development of levels of closure, he identified the lowest level as when a student feels that a problem requires a unique result – a single number that replaces two other numbers connected by an operation. This is typically the way students work out answers in arithmetical problems, such as $3 + 6 = x$. In this problem $x = 9$. If students are only exposed to problems of the type $a + b = c$, when they encounter a problem such as $3 + 6 = \square + 2$, they want to find the unique solution to 3 + 6, rather than explore the relationship between the two equations. Alternatively some students might give the answer '9' to this problem – they find a unique solution by adding all available numbers. At the second level students can work with combined elements without actually replacing the elements with a unique answer. At the third level, students can work with formulae as a whole object and the fourth and final level occurs when a student can deal not only with variables in a formula, but is able to discuss what the effect of changing one variable would have on other variables in the formula without having to substitute or work with actual numbers. (Collis, 1975). Intuitively, a number is a mathematical object for students, an entity with a closed form of representation; in the same time the sign of $\sqrt{\ }$ in quadratic irrationals is assumed by students as an arithmetical operation rather than a symbol of a numeral. So they look for "answer" i.e. the value for square root of a rational, whose representation has an extended form, then this "answer" with its non-ending decimals makes conflict with students' intuition of a number as a closed package.

Tall et al, also discuss the twofold nature of$\sqrt{n}$. While students try to find the value for the square root of number n by repeated approximation, they focus on the procedural or operational aspect of $\sqrt{n}$. Tall et al then hypothesize that students by reflecting on their actions become aware of the objective aspect of $\sqrt{n}$. Through repeated re-presentations $\sqrt{n}$ becomes a symbol of operation which is embedded in the figurative material. (Tall D., 1999) .The second sub-question was provoked by focusing on how students had defined an irrational number. Deficiencies in students' basic knowledge of irrational numbers, were partly because they did not simply know, -or have forgotten- the formal definitions which is in consistency to the findings of Fischbein et al (Fisschbein, Jehiam, & Cohen, 1995); moreover, our study revealed that the approach of school text books in Iran in introducing and discussing irrational numbers might be the reason for some of the misconceptions.

Purpose of the study

The purpose of this study is to determine the reasons for students' misconception of quadratic irrational numbers in Iran. Specifically, this study is undertaken to determine the conflicts in students' conception of irrational numbers and its relationship with students' mental object of numbers.

Literature review

There have been indirect approaches for understanding of irrational numbers in parts of research literature on limits and infinity, however, studies that focus directly on the comprehension and didactical approach of the irrationals, are not extensive indeed. Ficshbein et al. , made the hypothesis that the possible obstacles for misconception of irrational numbers are the intuitive difficulties appeared also in the history of mathematics for the discovery of them that is the incommensurability of irrational numbers and the continuity of the set of real numbers. (i.e. despite that Q is an everywhere dense set, it cannot cover all the points of a given interval). Their finding did not confirm the hypothesis but they suggest that these intuitive difficulties ought to be projected rather than to be ignored. (Fisschbein, Jehiam, & Cohen, 1995)

The study of Peled and Hershkovits on 70 prospective teachers focused on the difficulties that prevent student teachers from integrating various knowledge pieces into a flexible whole (Peled & Hershkovits, 1999). Contrary to Fischbein and et al(1995), they found that in spite of knowing the definitions and characteristics of irrational numbers, student teachers failed to make relationships between different representations. They concluded that the main source of misconceptions was limit process. In their study on a group of pre-service secondary mathematics teachers, Zazkis and Sirotic focused on the role that representation play in concluding rationality or irrationality of a number in the framework of "opaque" and "transparent" representation introduced by Lesh, Behr and Post (1987) (Zazkis & Sirotic, Making Sensa of Irrational Numbers: Focusing on Representation, 2004) (Zazkis & Sirotic, Representing and Defining Irrational Numbers: Exposing Missing Link, 2010). They found that usually, transparent features of the decimal representation of irrational numbers are not attended or recognized. They suggest that the misunderstanding of irrational numbers is rooted in misunderstanding of rational numbers, that is the understanding of when and how the division of whole numbers gives rise to repeating decimals and conversely, that every repeating decimal can be represented as a ratio of two integers. In another research, again on secondary mathematics teachers, Zazkis and Sirotic investigated the participants' understanding of irrational numbers in a conceptual framework of "dimension of knowledge" suggested by Torish et al. (1998) (Sirotic & Zazkis, 2007). They addressed three issues: richness and density of numbers, the fitness of rational and irrational numbers on the real number line, and the operations amongst the elements of the two sets. The results indicated that there were inconsistencies between the participants' intuition and their formal and algorithmic knowledge. The main source for misconceptions relating to the mentioned issues was again related to the over reliance on infinite non-repeating decimal representation of irrational numbers.

Method

Our pervious experiences and literature review suggested us that the misconceptions in irrational numbers might be caused by students' intuitive difficulties, lack of sufficient mathematical knowledge or both of them. We had applied an interpretive / descriptive approach for our research, so collecting and analysis of the data were interwoven and occurred simultaneously. Since we had no clear idea of students' notions on irrational numbers, in the preliminary stage of our research, we asked 30 students in the second year of high school to define irrational numbers. After analyzing the self-constructed definitions of the students and based on our findings, we designed a questioner of 8 open answered tasks which was distributed among 50 students in the second year of high school in Tehran.

Four questions evaluated the students' professed knowledge and definition of irrational numbers. Four other questions were some tasks about quadratic irrationals. Through analyzing the answers to latter 4 questions we seek for students' notions indirectly, in order to reveal the reflection of their intuitive knowledge on their mathematical performance. In the (first four) questions related to students' professed knowledge and definitions, three issues were addressed: representation, operation and inclusion. In the designed tasks (second 4 questions) our focus was on the representational and operational nature of the symbol of quadratic irrational numbers and the representation of irrational numbers on the real number line. In both, students had difficulty to relate an "irrational number" to their general mental image of a "number". After few months of distributing the first questioner, another questioner of 2 questions were designed and distributed among 30 students of the same group. This was done again to confirm our considerations on the role of "closure" in students' conceptions.Semi-structured interviews were done with ten of the students to have a deeper view on the matter. Besides, we looked for more indications of the inconsistencies between students' mental object of numbers in general and irrational numbers.

Phase 1:

1. Professed knowledge and definitions

In the first question, it was asked that "what is an irrational number?". This question was repeated again in order to confirm our categorization (representation, operation, inclusion) in the preliminary stage of the research, to consider whether any reduction is necessary in the categories or any new one should be added. Almost all of the students focused on quadratic irrationals in their answers (most of them had brought examples that showed their point of views) and this convinced us once more that our focus on quadratic irrational numbers was reasonable. Although this emphasis by students shows the deficiencies in their knowledge or/and in the educational approaches, we had limited ourselves to investigate the conflicts and misconceptions in the domain of students' notions of irrational numbers. From the answers to the few first questions, (such as "what is an irrational number?"/ "is an irrational, a real number?"), (again) three main categories of students' views on irrationals were extracted: a) representational b) operational c) inclusion. Some students had mentioned only one of the above aspects in their answers and some of them more. As an example, the answers of three students have been translated below:

"*A number, whose exact value is unknown or uncountable, it could not be written as a fraction, its decimal part continues infinitely"*

"It does not have complete square root, its result is inexact, it could be calculated just approximately"

"They are ambiguous, unknown numbers; they do not have exact values."

In the first answer, we interpret the phrases "*exact value*" and "*uncountable*" as an indication for operational aspect. "*could not been written as a fraction*" and "*its decimal part continues infinitely*" , indicated the representational aspect. The phrases "*incomplete square root*", "*inexact result*", "*calculated approximately*" in second and third answers indicated operational aspect and the phrases "*ambiguous*" and "*unknown numbers*".

Representational aspect

Most of the previous research on irrational numbers has emphasized the role of representation in students' understanding of and performance on irrational numbers. Studies showed participants had difficulties in making relationship between decimal and non-decimal representation of an irrational number like $\sqrt{2}$; mostly appeared while finding the place of an irrational on real number line. Some of the reasons were mentioned as: misconception of rational numbers; of when and how the division of two whole numbers give rise to repeating decimals and conversely (Zazkis & Sirotic, Making Sensa of Irrational Numbers: Focusing on Representation, 2004) (Zazkis & Sirotic, Representing and Defining Irrational Numbers: Exposing Missing Link, 2010); misconception of limit process (Peled & Hershkovits, 1999); over reliance on infinite non-repeating decimal representation of irrational numbers (Sirotic & Zazkis, 2007). In first stage of our study, "representation" has been noticed as an element of students' thought in defining irrational numbers. So, over reliance on decimal representation has been considered – not only as a reason for students' misconception – but mainly as a result of their mental tendency to "closure" in representing any kind of "number". In two ways this tendency to closure has been revealed: first in student's self constructed definition of irrational numbers and second in their manipulations, for example adding two irrational $\sqrt{2}$ and $\sqrt{3}$ or finding the place of $\sqrt{2}+\sqrt{3}$ on the number line; these will be discussed later.In their definition of irrationals, students used phrases like:

> "*ambiguous, incomplete numbers …*"
> "*incomplete unknown numbers with never ending decimals …*"
> "*inexact numbers with never ending decimals …*"

distinguish and categorize these *unknown* numbers as a certain set of numbers. This relates to the issue of *inclusion*. Some students wrote "irrational numbers are those that are not rational". This also relates to the issue of *inclusion;* the numbers were included in a specific set of numbers, say irrationals and were not included in rational numbers sets.

The students' answers to these questions have been brought in table 1.

Representational	**Inclusion**			**Operational**		
Lack of Closure	Undefined/ ambiguous	Real numbers/ not rational	Not real numbers	No exact answer/ Approximate answer	No operation between them	do not have perfect square root
12	10	3	35	12	4	14

This feeling of "suspicion" and "waiting for an end" shows students' uneasiness with non-repeating decimal representation of irrational numbers that has been revealed in their definitions. The next question was "is an irrational a real number?"; 35 out of 50 students answered "no" to this question. At first sight, it may seem that similar to the findings of Fishbein et al, students' lack of formal knowledge has caused such a mistake; but looking deeply to their explanation shows that students could not properly make a relationship between an "irrational number" and their general "mental object" of a "number":

> "*it is not a real number, because it has not an exact value …",*
> *"no, because it is incomplete …*".

The formal definition of irrational numbers had been taught to the students before, but their mental image of a "number" in general was dominant, such that they forget about the formal definition. In our opinion, one important reason for misconception of irrational numbers is the conflict between the characteristic of "closure" in students' mental object of a "number" on one hand, and never ending digits in decimal representation of irrational numbers on the other hand. This characteristic and the related mental conflict may be explained in such words: "any number has a closed, unique representation; a complete and ended one and because an irrational number is incomplete, inexact, with never ending digits in decimal part (accepted as a definition), so it is not a number at.

1.2.Inclusion

It was important for us that students put irrational numbers in which class or set of numbers. There were three notions in students' explanations: belonging to real number set- not belonging to real number set- no inclusion at all (mostly indicated by words like ambiguous, unknown, …) In their word explanations, many of students used phrases like "ambiguous", "unknown", "undefined", "inexact"; 35 out of 50 students answered "no" to the direct question "is an irrational number a real number?". In the answers to the question: "what is an irrational number?" Only 3 students defined irrational numbers as "real numbers which are not rational". In other question, students were asked to tell their agreement or disagreement to the statements: "any point on the number line is correspondent to a real number and vice versa" and "any irrational number is correspondent to a point on the real number line" ; 33

of students agreed to former and 27 students agreed to the latter statement.

Inclusion	An irrational is not a real number.	35
	Any point on the real number line corresponds to real number and vice versa.	33
	Any irrational number corresponds to a point on the real number line.	27

The conflicts between students intuitive and formal knowledge on classification of irrational numbers is obvious. Intuitively, most of them had not assumed irrational numbers as real numbers, because there are "incomplete", "inexact" and "ambiguous". May be they meant that these "unknown numbers" (as they had indicated in their statements", are not "really numbers" instead of saying "real numbers". But talking about the real number line, many of them agreed that any irrational number corresponds to a point on the real number line. This shows also that students' intuitive knowledge of numbers is independent of their conception of "real number line" and they have difficulties in making relationship between their formal and intuitive knowledge. The following statement is one of the students' opinions about irrationals:

"*numbers which are only meaningful on the number line, otherwise nonsense!*"

It seems that in the case of irrational numbers, "real number line" is not included by students' general mental object of a number. Formally, students accept the correspondence between an irrational number and a point on the real number line, but intuitively they do not accept the inclusion of irrational numbers by real number set. The exact wording of the school text book in the first year of high school in Iran is as following: "each point on the real number line, if does not correspond to a rational number, then corresponds to an irrational one; these two sets of numbers, together make the larger real number set". This ambiguous definition, of course, causes some problems in students' understandings. The most obvious contradiction is that "real number set has been defined by the aid of irrational number set which itself has been defined by the aid of real number line!".

1.3.Operational aspect

The third issue, addressed in students' answers to the question -what is an irrational number?- was operational aspect. From operational point of view, students addressed two issues in their self- constructed definitions of irrational number. First, they defined an irrational number as the square root of numbers like 2, 3, 5… which are not square numbers. Many before that students learn about irrationals in their lessons, they learn the techniques for taking square root of numbers. So $\sqrt{\ }$ is more considered as a sign for arithmetical operation - something like addition (+) or subtraction (-) - rather than as a part of the symbol of a quadratic irrational number such as $\sqrt{2}$. This was obvious in the ways that students had defined irrational numbers:

"a number which has not a proper result …"

"their result could be approximately determined …"

"numbers which have not complete square root, the result is inexact …"

Students had used several times the word "*result*" in their explanations. These kinds of notions indicate that students emphasized more on operational role of sign $\sqrt{\ }$ and the operation of taking square root. The dual nature of the sign $\sqrt{\ }$ could be a source of conflict in understanding quadratic irrational numbers: " $\sqrt{2}$ could be assumed as a *numeral*, a closed package, an encapsulated entity with a unique symbol, a *number* which belongs to a certain set of numbers called irrationals, so $\sqrt{\ }$ is part of its symbol; on the other hand, it could be assumed as a sign for an *operation* called *taking root square*; the result of such operation on some rational numbers is a number with ended decimals, so it is correspondent to an exact point on the number line, for some other rational numbers the answer is a number with non-ending decimals, so the answer is not exactly but approximately a rational number." (this conflict including both rational and irrational numbers with non-ending decimals, makes the misconception that this kind of numbers are *inexact*) The other kind of operational aspect which students had addressed to, was the "arithmetical operation between irrational numbers".

"irrational numbers are those which there could be no arithmetical operations between them"

By "arithmetical operation" they more meant addition and minus. They also gave examples: "$\sqrt{2}+\sqrt{3}$". $\sqrt{2}+\sqrt{3}$ seems to be a "process", addition of two irrational numbers rather than an "object" or a number in their minds. Again, closure plays an important role in considering a number as an object or a single entity in mind. In next section, this would be discussed in more detail.

2. Task analysis

The second group of the questions consists of some tasks about quadratic irrational numbers. In the first task, students were asked to find the precise amount of "$\sqrt{2}+\sqrt{5}$", "$\sqrt{2}+1$", "$\sqrt{2}+3\sqrt{2}$", if possible. Students answered the question in two ways; by decimal representation and by radical form of representation. The following table shows students' answers.

Find the precise amount			
Answers	$\sqrt{2}+\sqrt{5}$	$\sqrt{2}+1$	$\sqrt{2}+3\sqrt{2}$
Decimal form	27	21	11
Radical form	4	10	25

Some students did not answered the task or used both form, for example for $\sqrt{2}+3\sqrt{2}$, they had answered $4\sqrt{2}$ and then written "4×1.2" It seems that extended form of the number $\sqrt{2}+\sqrt{5}$ has caused many students to apply decimal representation; despite that they were asked to find the "exact" amount, only four of them wrote $\sqrt{2}+\sqrt{5}$ as the precise result. On the other hand, 25 students wrote $4\sqrt{2}$ as the result for $\sqrt{2}+3\sqrt{2}$ without replacing $4\sqrt{2}$ by decimal representation.

The above text is one of the students' answers to the afore-mentioned question who has replaced $\sqrt{2}+3\sqrt{2}$ by $4\sqrt{2}$ and decimal form of representation for two other expressions. In the following case, the student has first replaced all radical forms of the numbers by decimal form, but then she has noticed that they are approximations not "precise" amounts, so she has decided to put $4\sqrt{2}$ for first expression. But for two other expressions, she has obliged herself to apply decimal representation.

In above text, student has realized that the decimal representation is an approximation for quadratic irrational numbers (she has not applied it). Although she has applied the extended form of $\sqrt{2}+1$ as an exact amount of the number itself, for $\sqrt{2}+\sqrt{5}$ she has explained that "it's impossible". This happens because of the good extend of the examples of the form of

$\sqrt{m}+n, m, n \in N$ in their text books. The number $\sqrt{2}+1$, although with an extended form of representation, seems to be known for the student as a package, a unique number or an object. But for her "it's impossible" either to put decimal form for exact amount of the number $\sqrt{2}+\sqrt{5}$, or to accept the number itself, as an object or unique number.

In next question, students were asked to determine "which of the phrases is a (an exact or unique) number?" The number of students who agreed with the statement has been brought in following table.

Which one is a number?		
$\sqrt{2}+\sqrt{5}$	$\sqrt{2}+1$	$\sqrt{2}+3\sqrt{2}$
12	19	23

In according to our opinion, more students agreed that $\sqrt{2}+3\sqrt{2}$ is an exact number, because they could write it in a closed form of $4\sqrt{2}$. On the other hand, only 12 students accept that $\sqrt{2}+\sqrt{5}$ is a unique number and has an exact value; we think it is because of the extended form of it. Some students had not answered this question. The other questions and the table of the students' answers are as following: "Which of the above phrases is correspondent to a point on the real number line?" Students answered this question in two ways: "it is exactly correspondent to a point" or "it is approximately correspondent to a point".

	$\sqrt{2}+\sqrt{5}$	$\sqrt{2}+1$	$\sqrt{2}+3\sqrt{2}$
Correspondent exactly to a point	16	23	23
Correspondent approximately to a point	21	8	12

For the same reason that more students supposed $\sqrt{2}+\sqrt{5}$ is not an exact or unique number, they believed that it is approximately (not precisely) correspondent to a point on the real number line. "Find the place of each of the above phrases on the number line."Again, students had answered the question in two ways: by applying decimal form of representation and finding approximately a point on the real number line and by geometrical construction. (Pythagorean relation)

	$\sqrt{2}+\sqrt{5}$	$\sqrt{2}+1$	$\sqrt{2}+3\sqrt{2}$
By decimal representation	27	15	24
By geometrical construction	5	29	14

The following example of the student's answers shows that how the extended form of a number makes separate mental images for each part.

She has explained: "we can show them on the number line but *separately*, but if take them out of the radical, then we can calculate the sum and show it as a point on the number line. Irrational numbers on the number line and out of it, As it was explained before, there is a mismatching between students' conception of irrational numbers and the place of quadratic irrational numbers on the number line. Irrational numbers on the number line and out of it seems to be two different things for some of the students.

In above example, student has explained "only the first expression i.e. $\sqrt{2}+3\sqrt{2}$ could be replaced precisely on the number line". She has not applied decimal representation for it but in two other expressions she has used decimal form. However, when she wants to replace $\sqrt{2}+3\sqrt{2}$ on the number line, she has threatened it as two separate objects. In the first case, a segment of the length of $4\sqrt{2}$ has been constructed by the student but it has not been showed on the number line. There is a mismatching between the students' conceptions of the irrational number, length of a segment and a corresponding point on the number line; and as the example shows this does not happen only because of decimal representation.

phase 2:

1. Process or object

In this stage and after few months, we asked two more questions from 30 of the same group of students. The first question was *"what is* $\sqrt{7}+1$*?"* The answers of the students were categorized in two main categories: "object" and "process".

Students' answers in the "object" group were categorized in two sub-categories: "irrational number" and "a number between..,"

A few of the students answered that: "it is an irrational number". Some of them declared that "it is a number between 2 and 3, but it has not a precise amount." In both, the expression was threatened by students as an "object". The rest of the students, except for a few of them who had non-relevant notions, threatened the expression as a "process". The answers was such as: "It is the sum of an irrational number with a rational one" or "it is the root square of 7 plus 1, which has not a precise value." or "it's the sum of two numbers". The results have been brought in following table:

Process	Object	
The sum of two numbers	Irrational number	A number between…
19	3	4

Again the results shows that for most of the students (19) "process" aspect of the irrational number in "extended" form is dominant. Considering irrational quantities as "object" or "numbers" by Muslim mathematicians led to the introduction of positive real numbers. (Berggren, 1986; Katz, 1998) One reason for such a vision was dealing with equations and their irrational roots freely and carelessly about their nature. Such a thing motivated us to design the following question:

Which one of the following expression could be the root for an equation?(multiple choice is possible)

12 $\sqrt{2}$ $\sqrt{5}+2$

The results have been brought in the table below.

12	$\sqrt{2}$	$3\sqrt{2}+5\sqrt{6}-\frac{\sqrt{10}-\sqrt{\sqrt{3}}}{3}$	$\sqrt{5}+2$
15	14	7	13

1. Interviews

Semi-structured interviews were conducted with 10 of the students in order to gain a deeper understanding of the students' notions on irrational numbers. The interviews began with the question "what is a *number* in your opinion?"

Most of the students mentioned the application of numbers in daily life (counting, measuring, order) , but some of them pointed to the application of numbers in developing mathematical concepts as well as its usage in other branches of science. Some also considered it as a "symbol" which contains a concept and acts like the letters in a language. In brief, students' opinions about number could be classified in two groups: Numbers as instruments (in daily life or in other branches of science) Numbers as a part of mathematics language (symbols that carry a content or a concept). The interviews were continued to reach to the concept of irrational numbers. Some the of students defined irrationals as the square root of a number. (not mentioning the prime numbers) Some defined it as *number* between two (rational) numbers which its value could not be told precisely. This was not seen or noticed before, in the first stage of our investigations and in students' written answers to the similar question. Majority of the students had difficulty in recognizing the relationship between an irrational number and its correspondent point on the number line (and vice versa). The tradition of constructing numbers on the number line makes a habit for students: each set of number develops to make the next number set which contains the former set. N develops to make W, W develops to make Z, Z develops to make Q, and each set is the subset of the new-made number set. In all of the mentioned numbers, the (mental) real number line matches the line in reality for students, i.e. the (segment) length, the place on the number line and the number itself, all match one another. Irrational numbers does not obey this tradition; some of them could be constructed separately (by Pythagorean relation) and some of them could be placed approximately between two rational numbers on the number line. The concept of length, place on the number line (a point) and the irrational number mismatches each other for students. A part of Tania's interview has been brought in following: (I:interviewer, T: Tania)

I: well, what is an irrational number?

T: aw…, fractions, they are fractions * between 1 and 2 * too much of them, *well, *3*, radical numbers which have not complete square roots.

I: Fractions like 1/3 or 2/5?

T:aw… these are not irrationals, the numbers which are between two numbers, * several fractions there could be.

I: could you give an example:

T: For example between 2 and 3. If we take the denominator 5, fractions like 1/5, 2/5 and many of them. We can take the denominator 1000.

I: Are these irrationals?

T: well? *

I: Is $\sqrt{2}$ rational or irrational?

T: $\sqrt{2}$? * irrational, aw…yes irrational.

I: could we cut a piece of wood with the length of $\sqrt{2}$?

T: $\sqrt{2}$? Aw…

I: Its length is $\sqrt{2}$

T: no, not possible * because it's an irrational number, its decimal part is too much

I: well this a real number line (I draw the real number axis)

T: ok

I: how can we show $\sqrt{2}$ on the real number line?

T: *3* aw… from here * then * here * this is $\sqrt{2}$ (she makes a right angle triangle)

I: you by Pythagorean relation?

T: yes

I: well, where is $\sqrt{2}$?

T: we use compasses, * like this (shows it by hand on the number line)

I: "The points on the number line * are correspondent to both * rational* and irrational numbers" do you agree?

T: aw.. no.

I: no? *why?

T: no, only rational numbers.

I: why not irrationals?

T: because we don't know how much they are, * we cannot show them on the number line.

I: but you showed $\sqrt{2}$ on the number line before, it is irrational?!

T: (laughing) well, no, *2* I was wrong. * before?

I: Well, is it possible or not?

T: yes, * it's possible * possible

Geometrical construction of numbers of the type $\sqrt{n}$, is only a formal knowledge which has no usage in measurement for Tania. It is not even related to the correspondent point of an irrational number. This procedural knowledge has not helped Tania in improving the concept of irrationality.

The same thing happed for Hoora. She showed $\sqrt{2}$ on the number line by the same way.

I: Is any irrational number correspondent to an exactly one point on the real number line?

T: no * because it is not distinguished.

I: So, there is not a point for an irrational number on the real number line?

T: no, there is not

I: but showed it before? ($\sqrt{2}$),(she made it geometrically)

T:We show irrational numbers in an area, a place between two numbers *, approximately

I: Irrational number is not correspondent to point? Just around it

T: no

I: you mean the place of the irrational point is empty? An empty point?

T: Around that point, aw… it is not a number at all * it consists of many decimals * it's not really a number * if it were we didn't call it "irrational"

I: Is not really a number?

T: well, awe… how can I say * it is, but it is very complicated

I: *

T: (laughing) I don't know!

Discussion

In our study two types of questions were designed to determine the students' understandings of irrational numbers. First, we examined students' professed knowledge through the questions about the nature of the irrational numbers. Three main categories were extracted from their answers: representation, operation and inclusion. In all of the three categories, "closure" played a key role in students' conceptions of irrational numbers. Students' explanations made us to think that the conflicts between "extended representation" and "closed representation", "process" and "object" and "numeral" and "operation" might be the sources of students' misconceptions of quadratic irrationals. In the second phase of our study, two types of questions were asked through a questionnaire. The first type of questions, again examined the students professed knowledge through their self-constructed definitions of irrational numbers. Similarly, the three aforementioned aspects of their notions were extracted which confirmed our preliminary study (the results are shown in tables 1, 2,3,4,5 & 6).

The second type of questions was some tasks to determine the three aspects, indirectly. In this category, four questions were asked: "*Find the precise amount of following phrases*.", "*Which of the following phrases is a number?*","*Which of the following phrases is correspondent to a point on the real number line?*", "Find the place of each of the following phrases on the real number line."

There seemed to be a relationship between students' answers to these four questions. Many of the students, who replaced radical form of the number by decimal representation for $\sqrt{2}+\sqrt{5}$, believed that it is not an exact number, it does not correspond to an exact point on the real number line and they did not applied Pythagorean relation to construct $\sqrt{2}+\sqrt{5}$. In the same time, many of them did not replace radical form of the number by decimal representation for $\sqrt{2}+3\sqrt{2}$; instead they wrote $4\sqrt{2}$ as the result. They claimed it is an exact number and it corresponds to an exact point on the real number line.

There seems to be a relationship between students' conceptions:

Extended representation $\longrightarrow$ *tendency to closure decimal representation* $\longrightarrow$ *non-ending decimal part* $\longrightarrow$ *not being an exact number (process rather than object)* $\longrightarrow$ *Correspondent approximately to a point on the real number line*

Although non- ending decimals make a feeling of "waiting for an end" and "suspicious" which consequently leads to students' misconceptions, the tendency to apply decimal representation is not a general habit in all cases. The tendency to "closure" for representing a number might be a reason for applying decimal representation, hence, in the case of $\sqrt{2}+3\sqrt{2}$, many of students did not applied decimal representation. We believe that replacing decimal representation is an answer to the tendency for "closure", which make the irrational number more close to the general mental image of a number, but the non-ending decimals of a number, itself, causes some conflicts and misconceptions such as the feeling of inexactness.

For the case $\sqrt{2}+1$, a conflict was seen between students' intuitive and formal knowledge; many students replaced the radical form by decimal representation to find the "precise amount" and many of them believed that it corresponds approximately (not exactly) to a point on the real number line. In the same time, they constructed the number geometrically (by Pythagorean relation) and not by replacing decimals to find the place of the number on the real number line. This happened because there are good amount of examples in textbooks of high school that ask students to find the corresponded point of numbers such as $\sqrt{m+n}, m, n \in N$ on the real number by geometrical ways. So students were more or less familiar to $\sqrt{2}+1$ as a number which could be constructed by geometrical methods on the real number line. The following, is an example of such a conception in students.

In first line, student has applied decimal representation, since in the second line she has constructed the number by geometrical method. Our study confirmed that students' misconceptions of quadratic irrational numbers relate to both their intuitions of numbers and to their formal knowledge. Students' intuitive difficulties relate to the conflicts between their mental object of real numbers as figurative materials or objects and irrational quantities as process or operation. On the other hand, students' deficiencies in their basic formal knowledge of irrationals are obvious in their professed definitions which have been brought in table 2. 35 of 50 students professed that irrational numbers are not real numbers (or they are not really numbers!). Although we suppose that even this deficiency in students' knowledge is because of the power of their intuitions of numbers at all which make them to forget about what they had learned before. Interviews also revealed some other aspects of students' misconceptions. Mismatching between an irrational number, its correspondent point on the number line and the length of a segment was a reason for students' misconceptions. Measurement was one of the characteristics of numbers which was claimed by students, but was not included by irrational numbers. This is one of the aspects of inconsistencies between mental object of a number in general and an irrational number. Mismatching of (mental) real number line and the line in reality was another source of inconsistency. The sources of students' misconception of irrational number and the inconsistencies with their mental object of numbers in general could be listed as following:

1. Lack of formal knowledge (consistent with the finding of Fischbein et al) (Fisschbein, Jehiam, & Cohen, 1995)
2. Tendency in "Closure" in representation
Mismatching between number line and the line in reality
3.Lack of characteristics of numbers like "measurement", by irrational numbers in students' opinions

For the case of closure, we suggest that various and appropriate examples and tasks in which quadratic irrationals have been used (with radical sign) may help students to have a proper mental object of irrationals. As good examples from history of Islamic mathematics we could point to the works of Abu-Kamil and Al-Karaji who applied arithmetic operations to irrational quantities. (Katz, 1998) Actually, they did not give any definition of "number" but just dealt with the various surd quantities using numerical rather than geometrical techniques. Khayyam also, considered the ratio of the diagonal of a square to the side ($\sqrt{2}$), or the ratio of a circle to its diameter (π) as a new kinds of numbers.(objects rather than process) (Berggren, 1986). As it was seen in our study, students had difficulty in assuming a quantity like $\sqrt{m}+\sqrt{n}$ or even $\sqrt{m+n}$ as an object, a whole or a "number" rather they considered it as a process or an operation of addition.(Tables 4,8and 9) Tendency to closure caused them to replace decimal representation which consequently made the procedural aspect of the quadratic irrational number even stronger. For ancient Greek, irrationals were not known as "numbers", rather they know them as incommensurables, or magnitudes. Evidence shows that Muslim mathematicians dealt with irrationals as "numbers" or better to say "objects" rather than process. For example, consider the following problem solved by Abu-Kamil: (Katz, 1998)

"*if one says that 10 is divided into two parts, and one part is multiplied by itself and the other by the root of 8, and subtract the quantity of the product of one part times the root of 8 from … the product of the other part multiplied by itself, it gives 40.*"

The equation in this case is " $(10-x)(10-x)-x\sqrt{8}=40$. After rewriting the equation in the form $x^2+60=20x+\sqrt{8x^2}(=20+\sqrt{8})x)$, Abu-Kamil carried out his solution to conclude that $x=10+\sqrt{2}-\sqrt{42+\sqrt{800}}$ and that 10-x, the "other part" is equal to $\sqrt{42+\sqrt{800}}-\sqrt{2}$, Or the following formulas developed by Abu Kamil:

$$\sqrt{A+B}=\sqrt{\frac{A+\sqrt{A^2-B^2}}{2}}+\sqrt{\frac{A-\sqrt{A^2-B^2}}{2}}$$

$$\sqrt[3]{A}+\sqrt[3]{B}=\sqrt[3]{3\sqrt[3]{A^2B}+3\sqrt[3]{AB^2}+A+B}$$

Al- karaji also applied arithmetic operations to irrational numbers. He interpreted the various classes of incommensurables in *Elements X* as classes of "numbers", on which the various operations of arithmetic were defined.

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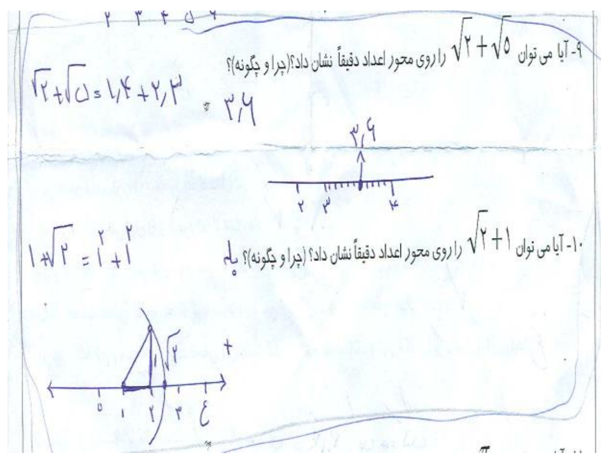

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