The Effect of Employing Paradoxes in Teaching and Learning Mathematics in Correcting Students' Common Mistakes in Solving Equation

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Abstract: Paradoxes create a special challenge to learners and the incoherence involved causes great problems in their mind. Students' facing paradoxes provide an invaluable chance for the teachers to direct their attention towards the point and help them resolve the problem of concern. Therefore, because of the importance of paradoxes, the present paper aims at studying the effect of paradoxes in teaching and learning mathematics, specially, in cases in which students make frequent mistakes which are here referred as "common mistakes". To do this, 150 students 14-15 years old, participated in the study to investigate the efficiency of paradoxes in two most frequent errors: "simplification in equations" and "eliminating the radical from square expressions". The study lasted about four months during which the researcher taught relevant concepts in relation to the problems through paradoxes. The analysis of the data collected from a pre-test and a post-test showed that employing paradoxes is quite efficient in teaching and learning mathematics as for correcting "common mistakes.[Ahmad Shahvarani, Ali Barahmand , Asghar Seif, **The Effect of Employing Paradoxes in Teaching and Learning Mathematics in Correcting Students' Common Mistakes in Solving Equation**. *Life Sci J* 2012;9(4):5789-5792] (ISSN:1097-8135). http://www.lifesciencesite.com. 862

Key words: paradox; learning mathematics; teaching mathematics; common mistakes in solving equation

1. Introduction

Mathematics teachers and educators all over the words look for means for integrating of contents with teaching of psychological and pedagogical issues [7]. A popular trend in mathematics education advocates connecting mathematical instructions to students' intuitions and prior experiences [6]. For example, the Principles and Standards for School Mathematics (NCTM 2000) suggests that ''a pattern of building new learning on prior learning and experience is established early and repeated'', and that ''students of all ages have a considerable knowledge base on which to build, including ideas developed in prior school instruction and those acquired through everyday experience''. [8]

Paradoxes create a special challenge to learners and the incoherence involved causes great problems in their mind. Students' facing paradoxes provide an invaluable chance for the teachers to direct their attention towards the point and help them resolve the problem of concern.

2. The relationship between paradoxes and learning

Cognitive conflicts have long been a part of psychological theories of cognitive change. According to [6] engaging learners in resolving paradoxes could trigger a state of cognitive conflict which, for some learners, resulted in the construction of new cognitive structures. Cognitive conflict is regarded as a state in which learners become aware of inconsistent or competing ideas. It was found the role of cognitive conflict in enhancing a change from instrumental to relational understanding of mathematics [10]. Based on the [7] a cognitive conflict is strongly related to paradoxes. Using paradoxes, in their simplest forms, as a puzzle, can cause learning by incoherence. A special challenge is created between what a person knows and what they see, such that they cannot discover the reason of duality. The person may even doubt his/her knowledge. And it is at this moment when the learner comes to gain more knowledge by improving incomplete knowledge. Some researchers believe that paradoxes have helped forward mathematics in the history. For example, the Zeno's paradox of infinity in the 5th century B.C., caused the concept of limitation and calculus to appear in the 17 th century.

3. Research Context

Usually, mathematics teachers, based on their experiences, know that many students have problems in some areas. These highly frequent problems are referred to here as "common mistakes". Teachers always seek methods for decreasing these problems such as using different teaching methods or providing more examples, or even predicting and introducing the mistakes the students are supposed to do. Nevertheless, a number of research report that some students still have difficulty with these concepts and these problems cause other problems in cases too. So far, however there has been little discussion on the effect of paradoxes in teaching mathematics for improving the mistakes.

4. Methodology

4.1. Purpose

In this paper, accepting the relationship between paradoxes and learning, the effect of paradoxes in improving and correcting two common mistakes by students in relation to "simplification in equations" and "eliminating the radical from square expressions", will be studied.

4.2. Participants

The research subjects were 150 students 14-15 years old from a public school who had already been taught solving quadratic equations by formula, and the study lasted about four months.

4.3. Procedure

First, a pre-test consisting a set of three carefully chosen questions based on research literature and authors' teaching experience in mathematics classes (see figure 1) was performed and after analyzing the mistakes in the students' responses, the researcher began to teach the concepts problematic through paradoxes. Then, the researcher designed a post-test based on the concepts, consisting five questions (see figure 2), for investigating the influence of the method.

4.3.1 The Pre-Test

The researcher conducted a pre-test (see figure 1), for recognition of the problems.

Is this simplification true?

$$1)x^2 = x \quad \Rightarrow \quad x = 1$$

Are these conclusions true?

$$2)y^2 = x \quad \Rightarrow \quad y = \sqrt{x}$$

$$3)(x+1)^2 = 4x^2 \quad \Rightarrow \quad x + 1 = 2x$$

Figure 1: Pre-Test

4.3.2 The Paradox' Method

According to [6] paradoxes provide educators with an important instructional tool that can help bridge the gap between mathematics and education by provoking discussion and controversy. Thus two important points are worth mentioning here:

1- Students should be given enough time for thinking, and
2- Students should be taught step by step via dialogue and discussion.

4.3.3 Simplification in equation

According to [5] some students simplified the variable *m* from both sides of the equation *4m=2m* and concluded that this equation does not have any answer. Regarding [1], many students facing the equation *(n-1).(n-2)=(n-1).(n-3)*, immediately simplified the same expressions, before anything.

The paradox used in this section was as following, (Quoted from [11]):

Look at my solution at the board carefully. Please tell me if you see any mistakes:

Let x=0 , so this relation is true , $x^2 = x^2 + x$

$$\Rightarrow \quad x^2 = x(x+1)$$

$$\Rightarrow \quad x = x + 1$$

$$\Rightarrow \quad 0 = 1$$

The students were surprised by the solution. Some students thought the mistake was due to the decomposition and the rest attributed it to the end of the solution. This was a suitable time for thinking, and the author asked to the students to write the question and think about the reason for the mistake. After 50 minutes, none of them could find any solutions and they were eager for understanding the reason for the paradox. This was suitable time for teaching via guidance as follows:

Please listen to my questions and answer carefully:

1- *Is this conclusion true?*

$$2x = 2y \quad \Rightarrow \quad x = y$$

Most answers were positive.

2- *If I put each number instead of 2, is the conclusion true again? For example:*

$$7x = 7y \quad \rightarrow \quad x = y$$

$$-5x = -5y \quad \rightarrow \quad x = y$$

$$\vdots$$

$$cx = cy \quad \rightarrow \quad x = y$$

Again, the answer was positive.

3- *If I put 0 instead of 2, what happens? Is the conclusion true?*

$$0 \times x = 0 \times y \quad \rightarrow \quad x = y$$

Still, some of the students thought the relation was true and could not realize the problem.

4- *This relation is true:* $\underbrace{0 \times 2}_{0} = \underbrace{0 \times 3}_{0}$. *If I simplify zero from both sides, I will have* $2 = 3$ *. This equality is obviously wrong. But, where is the source of the mistake?*

This time, some of the subjects pointed to the particular property of zero and in response to the reason of the paradox, they could understand the source of the mistake. Then, the author stressed the "zero factor" which functions as zero number, such as:

(n-1).(n-2)=(n-1).(n-3).

4.3.4. Eliminating the radical from square expressions

This problem is a common mistake too. The results of the pre-test showed that 73% of the subjects solved the equation $(x+1)^2 = 4x^2$ as follows:

$(x+1)^2 = 4x^2 \quad \rightarrow \quad x+1 = 2x \quad \rightarrow \quad x = 1$

In this case, firstly, it was necessary for the students to understand their mistake. So, they were asked to check if $x = -\frac{1}{3}$ was the answer to the equation. It was known that $x = -\frac{1}{3}$ was an answer. Then, the author asked them "why could your method not find this number as an answer?".

The paradox used in this case, was taken from [6] as the following:

"Look at the board carefully, if you notice any mistakes at any stage, please tell me:

$-8 = -8$

$\Rightarrow \quad 16 - 24 = 4 - 12$

$\Rightarrow \quad 16 - 24 + 9 = 4 - 12 + 9$

$\Rightarrow \quad 4^2 - 2 \times 4 \times 3 + 3^2 = \quad 2^2 - 2 \times 2 \times 3 + 3^2$

$\Rightarrow \quad (4-3)^2 = (2-3)^2$

$\Rightarrow \quad 4 - 3 = 2 - 3$

$\Rightarrow \quad 4 = 2$

This paradox was very surprising for all. Like the previous one, the students were providing with enough time for thinking and the source of the mistake was found via discussion step by step by themselves. Finally, after resolution, they achieved $x = -\frac{1}{3}$ which was again surprise for them.

4.3.5.The Pos-Test

After the treatment the effects of teaching mathematics using paradoxes were tested by administrating a carefully designed post-test (see figure 5 below).

Are these conclusions true? Please explain your reason in each case.

$2)(x^2+1)x^2 = (x^2+1)(2x-1) \quad \Rightarrow \quad x^2 = (2x-1)$

$1)(2x+1)^3 = 2x+1 \quad \Rightarrow \quad (2x+1)^2 = 1$

$3)(x+2)^2 = 9x^2 \quad \Rightarrow \quad x+2 = 3x$

Solve these equations and write your solution please.

$4)x^4 = 16$

$5)(x+3)^2 = (x+2)^2$

Figure 2: Post-Test

5. Data Analysis

Analyzing the data from the pre-test reveals that a great number of the students have problems and they commit the mentioned mistakes. The following table shows the percentage of the students' mistakes concerning each question.

Table 1: the Pre-Test Results

Question	Incorrect Response
Q 1	69%
Q 2	73%
Q 3	75%

Even, a brief look at the results revealed the necessity of emphasizing the points. Following the treatment and analyzing the data obtained through the post-test pointed out the significant role of using paradoxes to improve students' common mistakes (see table 2 below).

Table 2: the Post-Test Results

Question	Incorrect Response
Q 1	19%
Q 2	22%
Q 3	28%
Q 4	21%
Q 5	24%

According to Kolmogorov-Smirnov test the data were not normal, thus, Wilcoxon test were used. Because of z=*-7.02* and p-value was less than *0.0005*, the effect of the method is significant with error $\alpha = 0.05$(see table 3)

Table 3: Wilcoxon Signed Ranks Test

		N	Mean Rank
posttest – pretest	Negative Ranks	9[a]	23.50
	Positive Ranks	72[b]	43.19
	Ties	19[c]	
	Total	100	

a. posttest < pretest
b. posttest > pretest
c. posttest = pretest

Moreover, according to Mean Rank presented in table 4, the post-test is more successful than pre-test.

Table 4: Test Statistics[b]

	posttest – pretest
Z	-7.023[a]
Asymp. Sig. (2-tailed)	.000

a. Based on negative ranks.
b. Wilcoxon Signed Ranks Test

6. Discussion

An aha-moment can occur when students face a paradox. This state can be effectively used for teaching mathematical concepts related to the essence of the paradox [4]. A desirable outcome is to make the learner to apply the concepts to resolve the paradox. A whole lesson can be designed to lead the discoverer to her goal [4].In this study we attempted to show that mathematical paradoxes carry an impornant message, thinking about which provides the learner with an opportunity to refine their understanding.

7. Conclusion

The most obvious finding emerging from this study seems to be that employing paradoxes is effective for teaching and learning mathematics in highlighting and correcting common mistakes. Teaching through this method is attractive and because the students themselves participate in the learning process, it is important from educational viewpoint as well.

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